INFLATION OF THE TOROIDAL MEMBRANE*

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1. Introduction. The purpose of this paper is to discuss the existence, uniqueness, and asymptotic behavior of tensile solutions for the nonlinear toroidal membrane inflated by a constant normal pressure P. The theory to be adopted is that suggested by Bromberg and Stoker in [1].

The equations governing the axially symmetric deformation of a membrane are a combination of three sets of equations—(1) the strain-displacement equations, (2) Hooke's law, and (3) the equilibrium equations. For the toroidal membrane these equations have the form

$$e_\theta = (\mu' + \omega) + \frac{\Gamma}{2}(\omega' - \mu)^2 \qquad (' = d/d\theta) \tag{1.1a}$$

$$e_\phi = \lambda \frac{\mu \cos\theta + \omega \sin\theta}{\chi} \tag{1.1b}$$

$$\alpha = \frac{1}{1-\nu^2}(e_\theta + \nu e_\phi) \tag{1.2a}$$

$$\beta = \frac{1}{1-\nu^2}(e_\phi + \nu e_\theta) \tag{1.2b}$$

$$(\chi\alpha)' = \lambda \cos\theta\beta - \Gamma\chi\alpha(\omega' - \mu) \tag{1.3a}$$

$$\Gamma[\chi\alpha(\omega' - \mu)]' = \lambda \sin\theta\beta + \chi\alpha - \chi \tag{1.3b}$$

where E and ν are the Young's modulus and Poisson ratio and Γ is the dimensionless normal pressure, i.e. $\Gamma = Pb/Eh$ where h is the thickness of the elastic surface. The quantities α and β are the dimensionless stresses in the θ and ϕ directions (cf. Fig. 1), i.e. $\alpha = \sigma_\theta/\Gamma E$ and $\beta = \sigma_\phi/\Gamma E$, e_θ and e_ϕ are related to the strains ε_θ and ε_ϕ by $e_\theta = \varepsilon_\theta/\Gamma$ and $e_\phi = \varepsilon_\phi/\Gamma$, and ω and μ are related to the normal and tangential displacements by $\omega = w/\Gamma b$ and $\mu = u/\Gamma b$. The geometric constant $\lambda = b/a < 1$ and

$$\chi = 1 + \lambda \sin\theta. \tag{1.4}$$

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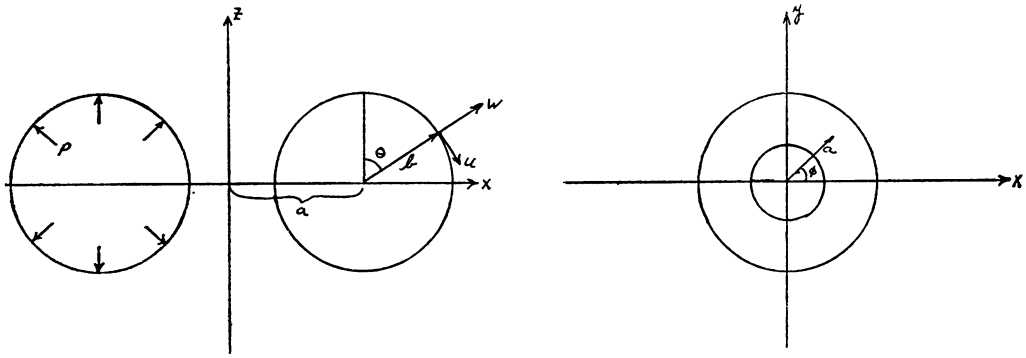

FIG. 1.

In this paper it will be assumed that the deformation of the toroidal surface is symmetric about the x-y plane. It follows that ω and μ must satisfy the boundary conditions

$$\omega'(-\pi/2) = \omega'(\pi/2) = 0, \tag{1.5a}$$

$$\mu(-\pi/2) = \mu(\pi/2) = 0. \tag{1.5b}$$

Assuming that ω and μ satisfy the conditions (1.5), Eq. (1.3a) implies that

$$\alpha'(-\pi/2) = \alpha'(\pi/2) = 0. \tag{1.6}$$

The problem of the deformation of a toroidal membrane by internal normal pressure has been studied by a number of authors (cf. the bibliography in [2, 3]), using both asymptotic techniques and numerical approximations. Unfortunately, none of these techniques have succeeded in completely resolving the difficulties associated with the toroidal membrane. Thus the system of equations (1.1), (1.2), and (1.3) with boundary conditions (1.5) and (1.6) have not been solved numerically, although an approximate linear equation has been successfully integrated (cf. [2]). Moreover, asymptotic results have of necessity been purely formal, since the existence of a solution has not been proved nor the necessary estimates obtained.

The difficulty in treating the above system of equations is a familiar one in nonlinear membrane theory, i.e. a straightforward linearization of the equations leads to a system of equations of reduced order. In the case of the toroidal membrane, this fact evidently implies the existence of an internal boundary layer occurring at $\theta = 0$ (cf. [3]). Thus the linearized equations yield smooth solutions for α and β, but the corresponding solutions for ω and μ are singular at $\theta = 0$.

In this paper it will be shown that the system of equations (1.1), (1.2), and (1.3) have a solution satisfying the boundary conditions (1.5) and (1.6). In particular, the following theorem will be proved.

THEOREM (1.1): *Let Γ_0 be an arbitrary but fixed number such that $\Gamma_0 > 0$. There exists an interval $0 \leq \lambda < \lambda^*(\Gamma_0) \leq 1$ such that Eqs.* (1.1), (1.2), *and* (1.3) *have a unique tensile solution* ($\alpha > 0$) *for all $\Gamma \geq \Gamma_0$ satisfying the boundary conditions* (1.5) *and* (1.6).

The proof of this theorem is constructive, yields bounds on the solution, and implies a procedure for the numerical integration of the equations. However, no numerical computations have been attempted. As a consequence of the proof of Theorem (1.1) we also obtain results on the asymptotic behavior of the solutions as $\lambda \to 0$ ($b/a \to 0$)

and as $\Gamma \to \infty$ ($P \to \infty$). In order to simplify the notation, introduce

$$||\cdot|| = \max_{-\pi/2 \le \theta \le \pi/2} |\cdot|.$$

The results on asymptotic behavior are

THEOREM (1.2): *Let Γ be any number such that $\Gamma > 0$.*

$$\lim_{\lambda \to 0} ||\alpha(\theta; \lambda, \Gamma) - 1|| = 0, \tag{1.7a}$$

$$\lim_{\lambda \to 0} ||\beta(\theta; \lambda, \Gamma) - \tfrac{1}{2}|| = 0, \tag{1.7b}$$

$$\lim_{\lambda \to 0} \left|\left|\omega(\theta; \lambda, \Gamma) - \frac{1 - 2\nu}{\lambda} \sin \theta - \frac{2 - \nu}{2}\right|\right| = 0, \tag{1.7c}$$

$$\lim_{\lambda \to 0} \left|\left|\mu(\theta; \lambda, \Gamma) - \frac{1 - 2\nu}{\lambda} \cos \theta\right|\right| = 0. \tag{1.7d}$$

Although Theorem 1.2 implies that $\omega \to \infty$ and $\mu \to \infty$ as $\lambda \to 0$ it does not follow that w and u are unbounded as $\lambda \to 0$. In fact Theorem 1.2 implies that

$$w \to \frac{Pb}{Eh}\left[(1 - 2\nu)a \sin \theta + \frac{2 - \nu}{2} b\right], \qquad u \to \frac{Pba}{Eh} \cos \theta$$

as $\lambda \to 0$.

THEOREM (1.3): *There exists a function $\boldsymbol{\alpha}(\theta, \lambda)$—a solution of a linear equation—and functions $\boldsymbol{\beta}(\theta, \lambda)$, $\boldsymbol{\omega}(\theta, \lambda)$, and $\boldsymbol{\mu}(\theta, \lambda)$, each uniquely determined by $\boldsymbol{\alpha}(\theta, \lambda)$, such that*

$$\lim_{\Gamma \to \infty} ||\alpha(\theta; \lambda, \Gamma) - \boldsymbol{\alpha}(\theta; \lambda)|| = 0, \tag{1.8a}$$

$$\lim_{\Gamma \to \infty} ||\beta(\theta; \lambda, \Gamma) - \boldsymbol{\beta}(\theta; \lambda)|| = 0, \tag{1.8b}$$

$$\lim_{\Gamma \to \infty} ||\omega(\theta; \lambda, \Gamma) - \boldsymbol{\omega}(\theta; \lambda)|| = 0, \tag{1.8c}$$

$$\lim_{\Gamma \to \infty} ||\mu(\theta; \lambda, \Gamma) - \boldsymbol{\mu}(\theta; \lambda)|| = 0 \tag{1.8d}$$

for λ in some interval $0 \le \lambda < \lambda^* \le 1$. The function $\boldsymbol{\alpha}(\theta; \lambda)$ in Theorem 1.3 is a solution of the equation

$$\left[\chi^{1+2\nu}\left(\frac{\chi^{1-\nu}}{\cos \theta} \boldsymbol{\alpha}\right)'\right]' - \lambda^2(1 - \nu^2)\chi^{\nu} \cos \theta \boldsymbol{\alpha} = f(\theta; \lambda, A). \tag{1.9}$$

The inhomogeneous term in Eq. (1.9) is defined by

$$f(\theta; \lambda, A) = \frac{\chi^{1+\nu} \cos^2 \theta + \chi^{\nu}(1 - A + \sin \theta - \lambda^2 \cos^2 \theta/2)(\sin \theta + \lambda(1 + \nu) \cos^2 \theta)}{\cos^3 \theta} \tag{1.10}$$

where A is a parameter determined by an algebraic equation, but in any case A satisfies the condition $2 > A > 0$. Although Eq. (1.9) is singular, at $\theta = \pm\pi/2$, it follows from the discussion in this paper that the equation has a unique solution satisfying the boundary conditions (1.6) for each A ($2 > A > 0$). However, the question of the qualitative behavior of solutions to Eq. (1.9) satisfying the boundary conditions (1.6) has not been treated.

In Sec. 2 of this paper it is indicated how to reduce the system of equations (1.1), (1.2) and (1.3) to a single equation for α depending on a parameter A (the actual details of the calculation are in the appendix (Sec. 5) at the end of this paper). In Sec. 3 it is proved that this equation has a solution for a certain range of the parameter A, and in Sec. 4 it is proved possible to choose A in such a way as to yield nonsingular solutions for the displacements ω and μ.

2. Formulation. The stress equations. In this section the three pairs of equations (1.1), (1.2), and (1.3) will be reduced to a single equation for the dimensionless stress α. The function β is eliminated from Eqs. (1.3a) and (1.3b) by multiplying (1.3a) by $\sin\theta$ and (1.3b) by $\cos\theta$ and subtracting. The resulting differential equation can be written

$$(g\chi\alpha)' = \chi\cos\theta \tag{2.1}$$

where

$$g = \sin\theta - \Gamma\cos\theta\psi, \quad \psi = \omega' - \mu. \tag{2.2, 2.3}$$

It follows from (2.1) that

$$\psi = \frac{1}{\Gamma\cos\theta}\left(\sin\theta - \frac{F(\theta;\lambda) - A}{\chi\alpha}\right). \tag{2.4}$$

A is a constant of integration and

$$F(\theta;\lambda) = \int_{-\pi/2}^{\theta} \chi\cos\eta\, d\eta = 1 + \sin\theta - \frac{\lambda^2}{2}\cos^2\theta. \tag{2.5}$$

If the function ψ is to be regular at $\theta = \pm\pi/2$ Eq. (2.4) requires that

$$\alpha(-\pi/2) = A/(1-\lambda), \quad \alpha(\pi/2) = (2-A)/(1+\lambda). \tag{2.6a, b}$$

Thus $\alpha(-\pi/2)$ and $\alpha(\pi/2)$ must satisfy the condition

$$(1+\lambda)\alpha(\pi/2) + (1-\lambda)\alpha(-\pi/2) = 2. \tag{2.7}$$

Note also that if the membrane is in tension throughout, as is to be expected, the conditions (2.6) imply that

$$2 > A > 0. \tag{2.8}$$

In order to insure the regularity of ψ it will be assumed for the moment that α satisfies (2.6). It then follows from L'Hospital's rule that

$$\psi(-\pi/2) = \psi(\pi/2) = 0, \tag{2.9}$$

a condition already implied by the boundary conditions (1.5).

Eqs. (1.1) and (1.2) can be reduced to a single equation for β as a function of α and ψ. For this purpose note that the Eqs. (1.1) imply that

$$(\chi e_\phi)' - \lambda\cos\theta e_\theta = \lambda\sin\theta\psi - \frac{\Gamma\lambda\cos\theta}{2}\psi^2 \tag{2.10}$$

or, using (1.2),

$$(\chi^{1+\nu}\beta)' = \nu(\chi^{1+\nu}\alpha)' + (1-\nu^2)\lambda\cos\theta\chi^\nu\alpha + \chi^\nu\left(\lambda\sin\theta\psi - \frac{\Gamma\lambda\cos\theta}{2}\psi^2\right). \tag{2.11}$$

Eq. (2.11) can be solved explicitly for β as a function of α and ψ. Combining this result with (2.4) and (1.3a), we obtain an equation for α (cf. Sec. 5). In fact we find that the function α satisfying the boundary condition (1.6) must be a solution of the non-linear integral equation

$$\alpha(\theta; \lambda, \Gamma) = f_1(\theta; \lambda, \Gamma) - \frac{\lambda^2(1-\nu^2)}{\chi^{1-\nu}} \cos\theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos\eta\alpha \, d\eta + \frac{\lambda^2 \cos\theta}{2\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda) \frac{(F(\eta; \lambda) - A)^2}{\chi^{2-\nu} \cos\eta\alpha^2} \, d\eta \tag{2.12}$$

where $F(\theta; \lambda)$ is given by (2.5) and

$$\begin{aligned} G(\theta, \eta; \lambda) &= \frac{h(\eta; \lambda)(h(\pi/2; \lambda) - h(\theta; \lambda))}{h(\pi/2; \lambda)}, \qquad -\pi/2 \le \eta \le \theta, \\ &= \frac{h(\theta; \lambda)(h(\pi/2; \lambda) - h(\eta; \lambda))}{h(\pi/2; \lambda)}, \qquad \theta \le \eta \le \pi/2, \end{aligned} \tag{2.13}$$

$$h(\theta; \lambda) = \int_{-\pi/2}^{\theta} \frac{d\eta}{\chi^{1+2\nu}}. \tag{2.14}$$

The function $f_1(\theta; \lambda, \Gamma)$ in (2.12) is given by

$$\begin{aligned} f_1(\theta; \lambda, \Gamma) = {} & \frac{(F(\theta; \lambda) - A)\sin\theta}{\chi} \\ & + \frac{\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} H(\theta, \eta; \lambda) \sin\eta \left(\chi^{1-\nu} - \lambda\nu \frac{(F(\eta; \lambda) - A)}{\chi^{1+\nu}}\right) d\eta \\ & - \frac{\lambda^2 \cos\theta}{2\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \frac{\sin^2\eta}{\cos\eta} \, d\eta \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} H(\theta, \eta; \lambda) &= \frac{h(\theta; \lambda) - h(\pi/2; \lambda)}{h(\pi/2; \lambda)}, \qquad -\pi/2 \le \eta < \theta, \\ &= \frac{h(\theta; \lambda)}{h(\pi/2; \lambda)}, \qquad \theta < \eta \le \pi/2. \end{aligned} \tag{2.16}$$

It is also possible to express β in terms of α. Thus it is found that (cf. Sec. 5)

$$\beta(\theta; \lambda, \Gamma) = f_2(\theta; \lambda, \Gamma) + \nu\alpha(\theta; \lambda, \Gamma) + \frac{\lambda(1-\nu^2)}{\chi^{1+\nu}} \int_{-\pi/2}^{\pi/2} H(\eta, \theta; \lambda)\chi^\nu \cos\eta\alpha \, d\eta - \frac{\lambda}{2\Gamma\chi^{1+\nu}} \int_{-\pi/2}^{\pi/2} H(\eta, \theta; \lambda) \frac{(F(\eta; \lambda) - A)^2}{\chi^{2-\nu} \cos\eta\alpha^2} \, d\eta \tag{2.17}$$

where $H(\eta, \theta; \lambda)$ is simply the adjoint of $H(\theta, \eta; \lambda)$, i.e.

$$\begin{aligned} H(\eta, \theta; \lambda) &= \frac{h(\eta; \lambda)}{h(\pi/2; \lambda)}, \qquad -\pi/2 \le \eta < \theta, \\ &= \frac{h(\eta; \lambda) - h(\pi/2; \lambda)}{h(\pi/2; \lambda)}, \qquad \theta < \eta \le \pi/2, \end{aligned} \tag{2.18}$$

and

$$f_2(\theta; \lambda, \Gamma) = \frac{\lambda}{2\Gamma\chi^{1+\nu}} \int_{-\pi/2}^{\pi/2} H(\eta, \theta; \lambda) \frac{\chi^{\nu} \sin^2 \eta}{\cos \eta} \, d\eta$$
$$+ \frac{1}{\lambda h(\pi/2; \lambda)\chi^{1+\nu}} \int_{-\pi/2}^{\pi/2} \sin \eta \left(\chi^{1+\nu} - \lambda\nu \frac{(F(\eta; \lambda) - A)}{\chi^{1+\nu}}\right) d\eta. \tag{2.19}$$

The solution of (2.12) (if it exists) will determine α up to the constant A, and in addition (2.17) will determine β up to the constant A.

3. Existence of solutions to the stress equation. In this section it will be shown that under certain conditions Eq. (2.12) has, for each value of A in a certain interval, a unique solution.

LEMMA (3.1): *Let ϵ and Γ_0 be arbitrary but fixed numbers such that $0 < \epsilon < 1$ and $\Gamma_0 > 0$. If $2 - \epsilon \geq A \geq \epsilon$ there exists an interval $0 \leq \lambda < \lambda_1(\Gamma_0, \epsilon) \leq 1$ such that $f_1(\theta; \lambda, \Gamma) > 0$ for $-\pi/2 \leq \theta \leq \pi/2$ and $\Gamma \geq \Gamma_0$.*

Proof: The function $f_1(\theta; \lambda, \Gamma_0)$ is a continuous function of λ for $0 \leq \lambda < 1$. At $\lambda = 0$ (cf. (2.15))

$$f_1(\theta; 0, \Gamma_0) = (1 - A + \sin \theta) \sin \theta + \cos \theta \int_{-\pi/2}^{\pi/2} H(\theta, \eta; 0) \sin \eta \, d\eta \tag{3.1}$$

where (cf. (2.16))

$$\begin{aligned} H(\theta, \eta; 0) &= (\theta - \pi/2)/\pi, \qquad -\pi/2 < \eta < \theta, \\ &= (\theta + \pi/2)/\pi, \qquad \theta < \eta < \pi/2, \end{aligned} \tag{3.2}$$

from which it follows that

$$f_1(\theta; 0, \Gamma_0) = 1 + (1 - A) \sin \theta. \tag{3.3}$$

The minimum of $f_1(\theta; 0, \Gamma_0)$ is achieved at either $\theta = -\pi/2$ or $\theta = \pi/2$, i.e.

$$\min_{-\pi/2 \leq \theta \leq \pi/2} f_1(\theta; 0, \Gamma_0) = \min (A, 2 - A) \geq \epsilon > 0. \tag{3.4}$$

Since $f_1(\theta; \lambda, \Gamma_0)$ is a continuous function of λ, we conclude that $f_1(\theta; \lambda, \Gamma_0)$ is positive in some interval $0 \leq \lambda < \lambda_1 \leq 1$. Moreover, $f_1(\theta; \lambda, \Gamma)$ is an increasing function of Γ (cf. (2.15)) so that $f_1(\theta; \lambda, \Gamma) > 0$ for $0 \leq \lambda < \lambda_1 \leq 1$ and $\Gamma \geq \Gamma_0$. Q.E.D.

LEMMA (3.2): *There exists an interval $0 \leq \lambda \leq \lambda_2 \leq 1$ such that the linear equation*

$$V = Q(\theta) - \frac{\lambda^2(1 - \nu^2)}{\chi^{1-\nu}} \cos \theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^{\nu} \cos \eta V \, d\eta \tag{3.5}$$

with $Q(\theta)$ continuous $(-\pi/2 \leq \theta \leq \pi/2)$ has a unique solution. If $Q(\theta) > 0$ there exists an interval $0 \leq \lambda < \lambda_3 \leq 1$ such that $V(\theta) > 0$.

Proof: Eq. (3.5) is a linear Fredholm integral equation with a continuous kernel. The existence of a unique solution when λ is sufficiently small follows from the contraction mapping theorem. In order to prove the second part of the lemma, it suffices to show that

$$V \leq Q \tag{3.6}$$

since in this case (3.5) implies

$$V \geq Q(\theta) - \frac{\lambda^2(1-\nu^2)}{\chi^{1-\nu}} \cos\theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos\eta Q(\eta)\, d\eta > 0 \tag{3.7}$$

when λ is sufficiently small. To prove (3.7) rewrite (3.5) in the form

$$V - Q = -\frac{\lambda^2(1-\nu^2)}{\chi^{1-\nu}} \cos\theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos\eta(V - Q)\, d\eta$$
$$- \frac{\lambda^2(1-\nu^2)}{\chi^{1-\nu}} \cos\theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos\eta Q\, d\eta. \tag{3.8}$$

It follows from (3.8) that

$$\left[\chi^{1+2\nu}\left(\frac{\chi^{1-\nu}}{\cos\theta}\right)(Q - V)'\right]' - \lambda^2(1-\nu^2)\chi^\nu \cos\theta(Q - V) = -\chi^\nu \cos\theta Q$$
$$(-\pi/2 < \theta < \pi/2) \tag{3.9}$$

and

$$\lim_{\theta\to\pm\pi/2} \frac{\chi^{1-\nu}(Q - V)}{\cos\theta} = 0, \qquad \lim_{\theta\to\pm\pi/2} \left|\left(\frac{\chi^{1-\nu}(Q - V)}{\cos\theta}\right)'\right| < \infty. \tag{3.10a, b}$$

Assume $Q - V < 0$ in some interval $-\pi/2 \leq \theta_1 < \theta \leq \theta_2 \leq \pi/2$ where $Q - V = 0$ at $\theta = \theta_1$ and $\theta = \theta_2$. Multiplying (3.9) by $\chi^{1-\nu}(Q - V)/\cos\theta$ and integrating from θ_1 to θ_2 we find that

$$\int_{\theta_1}^{\theta_2} \left\{\chi^{1+2\nu}\left(\left[\frac{\chi^{1-\nu}}{\cos\theta}(Q - V)\right]'\right)^2 + \lambda^2(1-\nu^2)\chi(Q - V)^2\right\} d\theta = \int_{\theta_1}^{\theta_2} \chi Q(Q - V)\, d\theta. \tag{3.11}$$

The right side of (3.11) is negative, and the left side is positive; this contradiction proves the lemma. Q.E.D.

LEMMA (3.3): *If V is a solution of*

$$V = \frac{\lambda^2 \cos\theta}{2\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda) \frac{Q(\eta)}{\cos\eta}\, d\eta - \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos\eta V\, d\eta \tag{3.12}$$

with $Q(\theta) \geq 0$ and continuous $(-\pi/2 \leq \theta \leq \pi/2)$ then $V \geq 0$ for $-\pi/2 \leq \theta \leq \pi/2$.

Proof: Any solution of (3.12) is twice continuously differentiable in the interval $-\pi/2 < \theta < \pi/2$ and in fact must satisfy the differential equation

$$\left[\chi^{1+2\nu}\left(\frac{\chi^{1-\nu}}{\cos\theta} V\right)'\right]' - \lambda^2(1-\nu^2)\chi^\nu \cos\theta V = -\frac{\lambda^2}{2\Gamma}\frac{Q}{\cos\theta}. \tag{3.13}$$

In addition it follows from (3.12) that

$$\lim_{\theta\to\pm\pi/2} \left(\frac{\chi^{1-\nu} V}{\cos\theta}\right) = 0$$

and

$$\lim_{\theta\to-\pi/2}\left(\frac{\chi^{1-\nu}}{\cos\theta}V\right)' = \begin{cases}+\infty \text{ or}\\ \text{finite}\end{cases}$$

$$\lim_{\theta\to\pi/2}\left(\frac{\chi^{1-\nu}}{\cos\theta}V\right)' = \begin{cases}-\infty \text{ or}\\ \text{finite}\end{cases}$$

where the value of $(\chi^{1-\nu}V/\cos\theta)'$ at $\theta = \pm\pi/2$ depends on the behavior of $Q(\theta)$ at $\theta = \pm\pi/2$. Assume $V(\theta)$ is negative in some portion of the interval $-\pi/2 < \theta < \pi/2$, i.e. assume there exist two points θ_1 and θ_2 such that $V(\theta_1) = V(\theta_2) = 0$ and $V(\theta) < 0$ for $\theta_1 < \theta < \theta_2$. If $\lim_{\theta\to\pm\pi/2}(\chi^{1-\nu}V/\cos\theta)' = \mp\infty$ then $-\pi/2 < \theta_1 < \theta_2 < \pi/2$. On the other hand if $\lim_{\theta\to\pm\pi/2}(\chi^{1-\nu}V/\cos\theta)'$ is finite then $-\pi/2 \le \theta_1 < \theta_2 \le \pi/2$. In either case the fact that solutions of (3.13) must satisfy the identity

$$\int_{\theta_1}^{\theta_2}\left\{\chi^{1+2\nu}\left(\left[\frac{\chi^{1-\nu}}{\cos\theta}V\right]'\right)^2 + \lambda^2(1-\nu^2)\chi V^2\right\}d\theta = \frac{\lambda^2}{2\Gamma}\int_{\theta_1}^{\theta_2}\frac{Q\chi^{1-\nu}V}{\cos^2\theta}\,d\theta \tag{3.14}$$

leads to a contradiction. The situation in which $(\chi^{1-\nu}V/\cos\theta)'$ is finite at one end and infinite at the other leads to a similar contradiction. Q.E.D.

Lemmas (3.1), (3.2) and (3.3) will enable us to prove the existence of a solution to Eq. (2.12). For this purpose define the following iteration scheme:

$$\alpha_{n+1} = f_1(\theta;\lambda,\Gamma) - \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1-\nu}}\int_{-\pi/2}^{\pi/2}G(\theta,\eta;\lambda)\chi^{\nu}\cos\eta\alpha_{n+1}\,d\eta$$
$$+\frac{\lambda^2\cos\theta}{2\Gamma\chi^{1+\nu}}\int_{-\pi/2}^{\pi/2}G(\theta,\eta;\lambda)\frac{(F(\eta;\lambda)-A)^2}{\chi^{2-\nu}\cos\eta\alpha_n^2}\,d\eta \tag{3.15}$$

where α_0 is a solution of

$$\alpha_0 = f_1(\theta;\lambda,\Gamma) - \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1-\nu}}\int_{-\pi/2}^{\pi/2}G(\theta,\eta;\lambda)\chi^{\nu}\cos\eta\alpha_0\,d\eta \tag{3.16}$$

Specify ϵ and Γ_0 as in Lemma (3.1), so that $f_1(\theta;\lambda,\Gamma) \ge C > 0$ for $\Gamma \ge \Gamma_0$ and $2-\epsilon \ge A \ge \epsilon$ if λ is in the interval $0 \le \lambda \le \lambda_1(\Gamma_0,\epsilon)/2$. It thus follows from Lemma (3.2) that there is an interval $0 \le \lambda < \lambda_5 \le \lambda_1(\Gamma_0,\epsilon)/2$ such that Eq. (3.16) has a unique solution $\alpha_0(\theta;\lambda,\Gamma)$ for each $A(2-\epsilon \ge A \ge \epsilon)$ and $\Gamma \ge \Gamma_0$ and that

$$\alpha_0(\theta;\lambda,\Gamma) > 0. \tag{3.17}$$

Moreover, Lemma (3.2) implies that (3.15) has a unique solution for α_1 in the interval $0 \le \lambda < \lambda_5$. The function $\alpha_1 - \alpha_0$ is a solution of the equation

$$\alpha_1 - \alpha_0 = \frac{\lambda^2\cos\theta}{2\Gamma\chi^{1-\nu}}\int_{-\pi/2}^{\pi/2}G(\theta,\eta;\lambda)\frac{(F(\eta;\lambda)-A)^2}{\chi^{2-\nu}\cos\eta\alpha_0^2}\,d\eta$$
$$-\frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1-\nu}}\cos\theta\int_{-\pi/2}^{\pi/2}G(\theta,\eta;\lambda)\chi^{\nu}(\alpha_1-\alpha_0)\,d\eta \tag{3.18}$$

Lemma (3.3) implies that the solution of this equation is positive, i.e.

$$\alpha_1 \ge \alpha_0\,. \tag{3.19}$$

Similarly,

$$\alpha_2 - \alpha_0 = \frac{\lambda^2 \cos\theta}{2\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda) \frac{(F(\eta; \lambda) - A)^2}{\chi^{2-\nu} \cos \eta\alpha_1^2} d\eta$$
$$- \frac{\lambda^2(1 - \nu^2)}{\chi^{1-\nu}} \cos\theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos \eta(\alpha_2 - \alpha_0)\, d\eta \tag{3.20}$$

and

$$\alpha_1 - \alpha_2 = \frac{\lambda^2 \cos\theta}{2\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda) \frac{(F(\eta; \lambda) - A)^2}{\chi^{2-\nu} \cos \eta\alpha_1^2\alpha_2^2} (\alpha_1^2 - \alpha_0^2)\, d\eta$$
$$- \frac{\lambda^2(1 - \nu^2) \cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos \eta(\alpha_1 - \alpha_2)\, d\eta. \tag{3.21}$$

Eqs. (3.20) and (3.21) in conjunction with Lemma (3.3) implies that $\alpha_1 \geq \alpha_2 \geq \alpha_0$. By induction it is shown that

LEMMA (3.4): *Let ϵ and Γ_0 be arbitrary but fixed numbers such that $0 < \epsilon < 1$ and $\Gamma_0 \geq 0$. If $2 - \epsilon \geq A \geq \epsilon$ there exists an interval $0 \leq \lambda < \lambda_5(\Gamma_0, \epsilon) \leq 1$ such that the iterates defined by* (3.15) *satisfy*

$$\alpha_1 \geq \alpha_3 \geq \cdots \geq \alpha_{2m-1} \geq \alpha_m \geq \cdots \geq \alpha_2 \geq \alpha_0 \tag{3.22}$$

for $-\pi/2 \leq \theta \leq \pi/2$ and $\Gamma \geq \Gamma_0$.

A similar type of iterative behavior has been observed in treating circular plates [4] and circular membranes [5].

The iterates $\{\alpha_{2m}\}$ are a bounded monotone increasing sequence, and the iterates $\{\alpha_{2m+1}\}$ are a bounded monotone decreasing sequence. In addition, it follows from differentiation of (3.15) that the derivatives of α_{2m+1} and α_{2m} are uniformly bounded. Thus the sequence $\{\alpha_{2m+1}\}$ converges to a continuous function α_+ and $\{\alpha_{2m}\}$ converges to a continuous function α_- where

$$\alpha_+ \geq \alpha_- . \tag{3.23}$$

The functions α_+ and α_- are solutions of the pair of equations

$$\alpha_{+,-} = f_1(\theta; \lambda, \Gamma) + \frac{\lambda^2 \cos\theta}{2\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda) \frac{(F(\eta; \lambda) - A)^2}{\chi^{2-\nu} \cos \eta\alpha_{-,+}^2} d\eta$$
$$- \frac{\lambda^2(1 - \nu^2) \cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos \eta\alpha_{+,-}\, d\eta. \tag{3.24}$$

THEOREM (3.1): *Let ϵ and Γ_0 be arbitrary but fixed numbers such that $0 < \epsilon < 1$ and $\Gamma_0 > 0$. If $2 - \epsilon \geq A \geq \epsilon$ there exists an interval $0 \leq \lambda < \lambda^*(\Gamma_0, \epsilon) \leq 1$ such that Eq.* (2.12) *has a solution for all $\Gamma \geq \Gamma_0$. The solution is unique for each A.*

Proof: In order to prove the existence of the solution, it suffices to show that $\alpha_+ = \alpha_-$. When we note that $\alpha_1 \geq \alpha_+ \geq \alpha_- \geq \alpha_0$ Eqs. (3.24) imply that

$$||\alpha_+ - \alpha_-|| \leq \left\{\frac{\lambda^2(1 - \nu^2) \cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^\nu \cos \eta\, d\eta \right.$$
$$\left. + \frac{\lambda^2 \cos\theta}{\Gamma\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda) \frac{(F(\eta; \lambda) - A)^2}{\chi^{2-\nu} \cos \eta\alpha_0^3} d\eta\right\} ||\alpha_+ - \alpha_-||. \tag{3.25}$$

When λ is sufficiently small (3.25) yields $||\alpha_+ - \alpha_-|| = 0$, which implies that $\alpha_+ \equiv \alpha_-$. In order to prove the uniqueness, it suffices to note that, since every solution α of (2.12) satisfies the condition $\alpha \geq \alpha_0$, an estimate of the type (3.25) yields the uniqueness.

Q.E.D.

The preceding development also yields results on the asymptotic behavior of α and β as $\lambda \to 0$ and $\Gamma \to \infty$.

THEOREM (3.2): *Let A be any number such that $2 > A > 0$.*

$$\lim_{\lambda\to 0} ||\alpha(\theta; \lambda, \Gamma) - (1 + (1 - A) \sin \theta)|| = 0, \tag{3.26}$$

$$\lim_{\lambda\to 0} ||\beta(\theta; \lambda, \Gamma) - (\tfrac{1}{2} + \nu(1 - A) \sin \theta)|| = 0, \tag{3.27}$$

$$\lim_{\lambda\to 0} \left|\left|\psi(\theta; \lambda, \Gamma) - \frac{1}{\Gamma \cos \theta}\left[\sin \theta - \frac{1 - A + \sin \theta}{1 + (1 - A) \sin \theta}\right]\right|\right| = 0. \tag{3.28}$$

Proof: The function $\alpha(\theta; \lambda, \Gamma)$ satisfies the inequality $\alpha_1 \geq \alpha \geq \alpha_0$ and

$$\lim_{\lambda\to 0} \alpha_1(\theta; \lambda, \Gamma) = \lim_{\lambda\to 0} \alpha_0(\theta; \lambda, \Gamma) = 1 + (1 - A) \sin \theta \tag{3.29}$$

which implies (3.26). Eqs. (3.27) and (3.28) follow from (2.17), (3.26) and (2.4) and the fact that

$$\lim_{\lambda\to 0} \frac{1}{\lambda\chi^{1+\nu}h(\pi/2; \lambda)} \int_{-\pi/2}^{\pi/2} \sin \eta\left(\chi^{1-\nu} - \lambda\nu \frac{F(\eta; \lambda) - A}{\chi^{1-\nu}}\right) d\eta = \frac{1 - 2\nu}{2} \tag{3.30}$$

Q.E.D.

In order to discuss the asymptotic behavior of α and β as $\Gamma \to 0$ define two functions $\boldsymbol{\alpha}(\theta; \lambda)$ and $\boldsymbol{\beta}(\theta; \lambda)$ where $\boldsymbol{\alpha}$ is the solution of the linear integral equation

$$\boldsymbol{\alpha}(\theta; \lambda) = f_3(\theta; \lambda) - \frac{\lambda^2(1 - \nu^2)}{\chi^{1-\nu}} \cos \theta \int_{-\pi/2}^{\pi/2} G(\theta, \eta; \lambda)\chi^{\nu} \cos \eta\boldsymbol{\alpha}\, d\eta \tag{3.31}$$

where

$$f_3(\theta; \lambda) = \frac{(F(\theta; \lambda) - A) \sin \theta}{\chi} + \frac{\cos \theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} H(\theta, \eta; \lambda) \sin \eta\left(\chi^{1-\nu} - \lambda\nu \frac{F(\eta; \lambda) - A}{\chi^{1+\nu}}\right) d\eta \tag{3.32}$$

and

$$\boldsymbol{\beta}(\theta; \lambda) = f_4(\theta; \lambda) + \nu\boldsymbol{\alpha}(\theta; \lambda) + \frac{\lambda(1 - \nu^2)}{\chi^{1+\nu}} \int_{-\pi/2}^{\pi/2} H(\eta, \theta; \lambda)\chi^{\nu} \cos \eta\boldsymbol{\alpha}\, d\eta \tag{3.33}$$

where

$$f_4(\theta; \lambda) = \frac{1}{\lambda\chi^{1+\nu}h(\pi/2; \lambda)} \int_{-\pi/2}^{\pi/2} \sin \eta\left(\chi^{1-\nu} - \lambda\nu \frac{F(\eta; \lambda) - A}{\chi^{1+\nu}}\right) d\eta. \tag{3.34}$$

THEOREM (3.3): *Let ϵ and Γ_0 be arbitrary but fixed numbers such that $0 < \epsilon < 1$ and $\Gamma_0 > 0$. Assume λ is in the interval $0 \leq \lambda < \lambda^*(\Gamma_0, \epsilon) \leq 1$.*

$$\lim_{\Gamma\to\infty} ||\alpha(\theta; \lambda, \Gamma) - \boldsymbol{\alpha}(\theta; \lambda)|| = 0, \tag{3.35}$$

$$\lim_{\Gamma\to\infty} ||\beta(\theta; \lambda, \Gamma) - \beta(\theta; \lambda)|| = 0, \tag{3.36}$$

$$\lim_{\Gamma\to\infty} ||\psi(\theta; \lambda, \Gamma)|| = 0. \tag{3.37}$$

Proof: In the limit as $\Gamma \to \infty$ both α_0 and α_1 are solutions of (3.31). Since the solution of (3.31) is unique and $\alpha_1 \geq \alpha \geq \alpha_0$, Eq. (3.35) follows. Eq. (3.36) and (3.37) are obtained from (2.4) and (2.17) on letting $\Gamma \to \infty$. Q.E.D.

4. The displacement problem. The question of existence of a solution to the boundary value problem for the toroidal membrane is not completely settled by Theorem (3.1) since the solution contains an undetermined parameter A. In addition, it remains to decide whether the displacement equations have a solution.

Equations for ω and μ are given by (1.1a) and (2.3) where (1.1a) may be rewritten

$$\mu' + \omega = \alpha - \nu\beta - (\Gamma/2)\psi^2. \tag{4.1}$$

These two equations imply that ω must be a solution of

$$\omega'' + \omega = \alpha - \nu\beta + \psi' - (\Gamma/2)\psi^2 \tag{4.2}$$

and satisfy the boundary condition (1.5a). The homogeneous equation corresponding to (4.2) has a nontrivial solution $\sin\theta$ satisfying the boundary conditions. Thus the Fredholm alternative theorem implies that (4.2) has no solution satisfying the boundary conditions (1.5a) unless

$$\int_{-\pi/2}^{\pi/2} \left(\alpha - \nu\beta + \psi' - \frac{\Gamma}{2}\psi^2\right) \sin\theta\, d\theta = 0, \tag{4.3}$$

or, after integration by parts,

$$\int_{-\pi/2}^{\pi/2} (\alpha - \nu\beta) \sin\theta\, d\theta = \int_{-\pi/2}^{\pi/2} \left(\cos\theta\psi + \frac{\Gamma}{2}\sin\theta\psi^2\right) d\theta. \tag{4.4}$$

Eq. (4.4) is an algebraic equation for the determination of A.

THEOREM (4.1): *There exists an interval* $0 \leq \lambda < \lambda_6 \leq 1$ *such that Eq.* (4.4) *has a unique solution for* A *in the interval* $2 > A > 0$.

Proof: In order to simplify the notation, define

$$I_1(A, \lambda) = \int_{-\pi/2}^{\pi/2} (\alpha - \nu\beta) \sin\theta\, d\theta, \tag{4.5a}$$

$$I_2(A, \lambda) = \int_{-\pi/2}^{\pi/2} \left(\cos\theta\psi + \frac{\Gamma}{2}\sin\theta\psi^2\right) d\theta. \tag{4.5b}$$

Solutions of (4.4) for a fixed λ are determined by the intersections of the curves $I_1(A, \lambda)$ and $I_2(A, \lambda)$. In the case $\lambda = 0$ Eqs. (3.26), (3.27) and (3.28) imply that

$$I_1(A, 0) = \frac{(1 - \nu^2)(1 - A)\pi}{2}, \tag{4.6a}$$

$$I_2(A, 0) = \int_{-\pi/2}^{\pi/2} \left[\cos\theta\psi(\theta; 0, \Gamma) + \frac{\Gamma}{2}\sin\theta\psi^2(\theta; 0, \Gamma)\right] d\theta, \tag{4.6b}$$

where $\psi(\theta, 0, \Gamma)$ is given by (3.28). It is easily verified that, since $\psi(\theta; 0, \Gamma) = 0$ when $A = 1$,

$$I_1(1, 0) = I_2(1, 0) = 0. \tag{4.7}$$

Moreover, this is the only solution of $I_1(A, 0) = I_2(A, 0)$ for $2 > A > 0$, since

$$\frac{\partial}{\partial A} I_1(A, 0) = -\frac{(1-\nu^2)\pi}{2} < 0, \tag{4.8a}$$

$$\frac{\partial}{\partial A} I_2(A, 0) = \frac{1}{\Gamma} \int_{-\pi/2}^{\pi/2} \frac{\cos^2 \theta}{[1 + (1 - A) \sin \theta]^3} (1 + (1 - A) \sin \theta(\sin \theta - \cos \theta))\, d\theta > 0. \tag{4.8b}$$

The non-negativity of (4.8b) for $\Gamma > 0$ is guaranteed by the fact that the integrand is positive for $2 > A > 0$ and $-\pi/2 < \theta < \pi/2$. The theorem follows from the fact that $I_1(A, \lambda)$ and $I_2(A, \lambda)$ are continuous functions of λ for λ in a neighborhood of $\lambda = 0$. Q.E.D.

It also follows from the above discussion that if we consider the behavior of A as a function of λ and Γ, i.e. $A = A(\lambda, \Gamma)$, then

$$\lim_{\lambda \to 0} A(\lambda, \Gamma) = 1, \tag{4.9}$$

$$\lim_{\Gamma \to \infty} A(\lambda, \Gamma) = \mathbf{A}(\lambda) \tag{4.10}$$

where $\mathbf{A}$ is the solution of (4.4) in the limit as $\Gamma \to \infty$, i.e. (cf. (3.35), (3.36), (3.37))

$$\int_{-\pi/2}^{\pi/2} (\alpha - \nu\beta) \sin \theta \, d\theta = 0. \tag{4.11}$$

We summarize the results in the following corollary:

Corollary (4.1):

$$\lim_{\lambda \to 0} ||\alpha(\theta; \lambda, \Gamma) - 1|| = 0, \tag{4.12a}$$

$$\lim_{\lambda \to 0} ||\beta(\theta; \lambda, \Gamma) - \tfrac{1}{2}|| = 0, \tag{4.12b}$$

$$\lim_{\lambda \to 0} ||\psi(\theta; \lambda, \Gamma)|| = 0. \tag{4.12c}$$

The limiting behavior of α, β and ψ as $\Gamma \to \infty$ is given by (3.35), (3.36), (3.37) where $A = \mathbf{A}$ is the solution of (4.11).

Once A has been determined as a solution of Eq. (4.4), it follows that (4.2) has a solution, although this solution is determined only up to the nontrivial solution of the homogeneous equation. In fact, the most general solution of (4.2) satisfying the boundary conditions (1.5a) can be written

$$\omega(\theta; \lambda, \Gamma) = C \sin \theta + \int_{-\pi/2}^{\pi/2} G_1(\theta, \eta)\left(\alpha - \nu\beta + \psi' - \frac{\Gamma}{2} \psi^2\right) d\eta \tag{4.13}$$

where C is an arbitrary constant and $G_1(\theta, \eta)$ is a modified Green's function for (4.2) (cf. [6]). The function $G_1(\theta, \eta)$ can be found explicitly for (4.2) and can be written

$$\begin{aligned} G_1(\theta, \eta) &= \frac{\theta - \pi/2}{\pi} \cos \theta \sin \eta + \tfrac{1}{2} \sin \theta \cos \eta, \qquad -\pi/2 \le \eta \le \theta, \\ &= \frac{\theta + \pi/2}{\pi} \cos \theta \sin \eta - \tfrac{1}{2} \sin \theta \cos \eta, \qquad \theta \le \eta \le \pi/2. \end{aligned} \tag{4.14}$$

After an integration by parts, (4.13) becomes

$$\omega(\theta; \lambda, \Gamma) = C \sin \theta + \int_{-\pi/2}^{\pi/2} G_1(\theta, \eta)\left(\alpha - \nu\beta - \frac{\Gamma}{2}\psi^2\right) d\eta - \int_{-\pi/2}^{\pi/2} G_2(\theta, \eta)\psi \, d\eta \tag{4.15}$$

where

$$\begin{aligned} G_2(\theta, \eta) &= \frac{\theta - \pi/2}{\pi} \cos \theta \cos \eta - \tfrac{1}{2} \sin \theta \sin \eta, \qquad -\pi/2 \le \eta \le \theta, \\ &= \frac{\theta + \pi/2}{\pi} \cos \theta \cos \eta + \tfrac{1}{2} \sin \theta \sin \eta, \qquad \theta \le \eta \le \pi/2. \end{aligned} \tag{4.16}$$

Eq. (4.15) determines ω up to a multiple of $\sin \theta$. The function μ is given explicitly by (2.3), or after differentiating (4.15)

$$\mu(\theta; \lambda, \Gamma) = C \cos \theta - \int_{-\pi/2}^{\pi/2} G_3(\theta, \eta)\left(\alpha - \nu\beta - \frac{\Gamma}{2}\psi^2\right) d\eta + \int_{-\pi/2}^{\pi/2} G_4(\theta, \eta)\psi \, d\eta \tag{4.17}$$

where

$$\begin{aligned} G_3(\theta, \eta) &= \frac{\theta - \pi/2}{\pi} \sin \theta \sin \eta - \tfrac{1}{2} \cos \theta \cos \eta - \frac{1}{\pi} \cos \theta \sin \eta, \qquad -\pi/2 \le \eta \le \theta, \\ &= \frac{\theta + \pi/2}{\pi} \sin \theta \sin \eta + \tfrac{1}{2} \cos \theta \cos \eta - \frac{1}{\pi} \cos \theta \sin \eta, \qquad \theta \le \eta \le \pi/2, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} G_4(\theta, \eta) &= \frac{\theta - \pi/2}{\pi} \sin \theta \cos \eta + \tfrac{1}{2} \cos \theta \sin \eta + \frac{1}{\pi} \cos \theta \cos \eta, \qquad -\pi/2 \le \eta \le \theta, \\ &= \frac{\theta + \pi/2}{\pi} \sin \theta \cos \eta - \tfrac{1}{2} \cos \theta \sin \eta + \frac{1}{\pi} \cos \theta \cos \eta, \qquad \theta \le \eta \le \pi/2. \end{aligned} \tag{4.19}$$

The constant C occurs in (4.15) and (4.17) because of the fact that in obtaining the original integral equation for α, i.e. Eq. (2.12), we have differentiated (1.1b). Thus C is to be determined so that (1.1b) is satisfied. However, since this equation will be satisfied up to an additive constant, it suffices to choose C so that (1.1b) is satisfied for a single value of θ. Evaluating (1.1b) at $\theta = 0$ we find

$$(1/\lambda)(\beta(0; \lambda, \Gamma) - \nu\alpha(0; \lambda, \Gamma)) = \mu(0; \lambda, \Gamma). \tag{4.20}$$

Equivalently,

$$\begin{aligned} C = \frac{1}{\lambda}(\beta(0; \lambda, \Gamma) - \nu\alpha(0; \lambda, \Gamma)) \\ + \int_{-\pi/2}^{\pi/2} G_3(0, \eta)\left(\alpha - \nu\beta - \frac{\Gamma}{2}\psi^2\right) d\eta - \int_{-\pi/2}^{\pi/2} G_4(0, \eta)\psi \, d\eta. \end{aligned} \tag{4.21}$$

Eq. (4.21) completes the proof of Theorem (1.1). It is also possible to describe the asymptotic behavior of $\omega(\theta; \lambda, \Gamma)$ and $\mu(\theta; \lambda, \Gamma)$ as $\lambda \to 0$. It follows immediately that

$$\lim_{\Gamma \to \infty} C(\lambda, \Gamma) = \mathbf{C}(\lambda) = \frac{1}{\lambda}(\boldsymbol{\beta}(0; \lambda) - \nu\boldsymbol{\alpha}(0; \lambda)) + \int_{-\pi/2}^{\pi/2} G_3(0, \eta)(\boldsymbol{\alpha} - \nu\boldsymbol{\beta}) \, d\eta \tag{4.22}$$

so that

$$\lim_{\Gamma\to\infty} ||\omega(\theta; \lambda, \Gamma) - \omega(\theta, \lambda)|| = 0, \tag{4.23a}$$

$$\lim_{\Gamma\to\infty} ||\mu(\theta; \lambda, \Gamma) - \mathfrak{y}(\theta; \lambda)|| = 0, \tag{4.23b}$$

where

$$\omega(\theta; \lambda) = \mathbf{C} \sin \theta + \int_{-\pi/2}^{\pi/2} G_1(\theta, \eta)(\alpha - \nu\beta)\, d\eta, \tag{4.24a}$$

$$\mathfrak{y}(\theta; \lambda) = \mathbf{C} \cos \theta - \int_{-\pi/2}^{\pi/2} G_3(\theta, \eta)(\alpha - \nu\beta)\, d\eta. \tag{4.24b}$$

This completes the proof of Theorem (1.3). Eq. (1.9) is determined by differentiating (3.31).

In order to determine the asymptotic behavior of $C(\lambda, \Gamma)$ as $\lambda \to 0$, we note that (4.21) in conjunction with (2.17) yields

$$\begin{aligned} C(\lambda, \Gamma) = {} & \frac{f_2(0; \lambda, \Gamma)}{\lambda} + (1 - \nu^2) \int_{-\pi/2}^{\pi/2} H(\eta, 0; \lambda)\chi^{\nu} \cos \eta\alpha\, d\eta \\ & - \frac{1}{2\Gamma} \int_{-\pi/2}^{\pi/2} H(\eta, 0; \lambda) \frac{F(\eta, \lambda) - A}{\chi^{2-\nu} \cos \eta\alpha^2}\, d\eta + \int_{-\pi/2}^{\pi/2} G_3(0, \eta)\left(\alpha - \nu\beta - \frac{\Gamma}{2} \psi^2\right) d\eta \\ & - \int_{-\pi/2}^{\pi/2} G_4(0, \eta)\psi\, d\eta. \end{aligned} \tag{4.25}$$

All the terms on the right of (4.25) have a limit as $\lambda \to 0$ except $f_2(0; \lambda, \Gamma)/\lambda$ (cf. (2.19)). In fact, (4.25) implies that

$$\lim_{\lambda\to 0} \left| C(\lambda, \Gamma) - \frac{C_{-1}}{\lambda} - C_0 \right| = 0 \tag{4.26}$$

where

$$C_{-1} = \frac{1 - 2\nu}{2}, \qquad C_0 = 0. \tag{4.27a, b}$$

Combining these results with (4.15) and (4.17), we obtain Eqs. (1.7c) and (1.7d). This completes the proof of Theorem (1.2.)

5. Appendix. In this section the actual details of obtaining Eqs. (2.12) and (2.17) will be indicated. As was mentioned in Sec. 2, these two equations are obtained by a combination of Eqs. (1.3a), (2.4), and (2.11). After an integration (2.11) yields

$$\beta = \frac{B}{\chi^{1+\nu}} + \nu\alpha + \frac{\lambda(1 - \nu^2)}{\chi^{1+\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu} \cos \xi\alpha\, d\xi + \frac{\lambda}{\chi^{1+\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu}\left(\sin \xi\psi - \frac{\Gamma \cos \xi}{2} \psi^2\right) d\xi \tag{5.1}$$

where B is a constant of integration.

Eqs. (2.4) and (5.1) can be combined with Eq. (1.3a) to give a single equation for α. Thus Eq. (1.3a) implies that α must satisfy

$$(\chi\alpha)' = \frac{\lambda B \cos\theta}{\chi^{1+\nu}} + \nu\lambda \cos\theta\alpha + \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1+\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu} \cos\xi\alpha \, d\xi$$
$$+ \frac{\lambda^2 \cos\theta}{\chi^{1+\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu}\left(\sin\xi\psi - \frac{\Gamma\cos\xi}{2}\psi^2\right) d\xi - \tan\theta\chi\alpha + \frac{F(\theta;\lambda) - A}{\cos\theta}. \tag{5.2}$$

This rather complicated expression can be rewritten

$$\frac{\cos\theta}{\chi^{1-\nu}}\left(\frac{\chi^{1-\nu}}{\cos\theta}\alpha\right)' = \frac{\lambda B\cos\theta}{\chi^{2+\nu}} + \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{2+\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu}\cos\xi\alpha \, d\xi$$
$$+ \frac{\lambda^2\cos\theta}{\chi^{2+\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu}\left(\sin\xi\psi - \frac{\Gamma\cos\xi}{2}\psi^2\right) d\xi + \frac{F(\theta;\lambda) - A}{\chi\cos\theta}. \tag{5.3}$$

There is an apparent difficulty in integrating (5.3), i.e. neither $\chi^{1-\nu}\alpha/\cos\theta$ nor $(F - A)/(\chi\cos\theta)$ is defined at $\theta = \pm\pi/2$. However, this difficulty is removed by rewriting (5.3) in the form

$$\left[\frac{\chi^{1-\nu}}{\cos\theta}\left(\alpha - \frac{(F(\theta;\lambda) - A)}{\chi}\sin\theta\right)\right]' = \frac{\lambda B}{\chi^{1+2\nu}} + \frac{\lambda^2(1-\nu^2)}{\chi^{1+2\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu}\cos\xi\alpha \, d\xi$$
$$+ \frac{\lambda^2}{\chi^{1+2\nu}} \int_{-\pi/2}^{\theta} \chi^{\nu}\left(\sin\xi\psi - \frac{\Gamma\cos\xi}{2}\psi^2\right) d\xi - \sin\theta\left(\chi^{1-\nu} - \nu\lambda\frac{F(\theta;\lambda) - A}{\chi^{1+\nu}}\right). \tag{5.4}$$

The conditions (2.6) imply that

$$\frac{\chi^{1-\nu}}{\cos\theta}\left(\alpha - \frac{F(\theta;\lambda) - A}{\chi}\sin\theta\right)$$

is well defined at $\theta = \pm\pi/2$. Thus after integration of (5.4) we find

$$\alpha(\theta;\lambda,\Gamma) = \frac{C\cos\theta}{\chi^{1-\nu}} + \frac{(F(\theta;\lambda) - A)\sin\theta}{\chi} + \frac{\lambda B\cos\theta}{\chi^{1-\nu}} h(\theta;\lambda)$$
$$+ \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\theta} h'(\eta;\lambda) \int_{-\pi/2}^{\eta} \chi^{\nu}\cos\xi\alpha \, d\xi \, d\eta$$
$$+ \frac{\lambda^2\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\theta} h'(\eta;\lambda) \int_{-\pi/2}^{\eta} \chi^{\nu}\left(\sin\xi\psi - \frac{\Gamma\cos\xi}{2}\psi^2\right) d\xi \, d\eta$$
$$- \frac{\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\theta} \sin\eta\left(\chi^{1-\nu} - \lambda\nu\frac{F(\eta;\lambda) - A}{\chi^{1+\nu}}\right) d\eta, \tag{5.5}$$

where C is a constant of integration and $h(\theta;\lambda)$ is given by (2.14). It is easily verified that the function $\alpha(\theta;\lambda,\Gamma)$ defined by (5.5) satisfies the conditions (1.6) regardless of the values of B and C. The constants B and C are determined from the boundary conditions (1.5). The boundary condition $\alpha'(-\pi/2) = 0$ implies that

$$C = 0 \tag{5.6}$$

and the boundary condition $\alpha'(\pi/2) = 0$ yields

$$\lambda B = -\frac{\lambda^2(1-\nu^2)}{h(\pi/2;\lambda)} \int_{-\pi/2}^{\pi/2} h'(\eta;\lambda) \int_{-\pi/2}^{\eta} \chi^\nu \cos\xi \, d\xi \, d\eta$$
$$- \frac{\lambda^2}{h(\pi/2;\lambda)} \int_{-\pi/2}^{\pi/2} h'(\eta;\lambda) \int_{-\pi/2}^{\eta} \chi^\nu \left(\sin\xi\psi - \frac{\Gamma\cos\xi}{2}\psi^2\right) d\xi \, d\eta$$
$$+ \frac{1}{h(\pi/2;\lambda)} \int_{-\pi/2}^{\pi/2} \sin\eta \left(\chi^{1-\nu} - \lambda\nu \frac{(F(\eta;\lambda)-A)}{\chi^{1+\nu}}\right) d\eta. \tag{5.7}$$

Eqs. (5.6) and (5.7) reduce (5.5) to a nonlinear Fredholm integral equation for α. In fact, noting that

$$\frac{h(\theta;\lambda)}{h(\pi/2;\lambda)} \int_{-\pi/2}^{\pi/2} h'(\eta;\lambda) \int_{-\pi/2}^{\eta} f(\xi)\, d\xi \, d\eta - \int_{-\pi/2}^{\theta} h'(\eta;\lambda) \int_{-\pi/2}^{\eta} f(\xi)\, d\xi \, d\eta$$
$$= \int_{-\pi/2}^{\pi/2} G(\theta,\eta;\lambda) f(\eta)\, d\eta, \tag{5.8}$$

where $G(\theta, \eta; \lambda)$ is given by (2.13), Eq. (5.5) becomes

$$\alpha = \frac{(F(\theta;\lambda)-A)\sin\theta}{\chi} + \frac{\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} H(\theta,\eta;\lambda) \sin\eta \left(\chi^{1-\nu} - \lambda\nu \frac{F(\eta;\lambda)-A}{\chi^{1+\nu}}\right) d\eta$$
$$- \frac{\lambda^2(1-\nu^2)\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta,\eta;\lambda)\chi^\nu \cos\eta\alpha \, d\eta$$
$$- \frac{\lambda^2\cos\theta}{\chi^{1-\nu}} \int_{-\pi/2}^{\pi/2} G(\theta,\eta;\lambda)\chi^\nu \left(\sin\eta\psi - \frac{\Gamma\cos\eta}{2}\psi^2\right) d\eta \tag{5.9}$$

where $H(\theta, \eta; \lambda)$ is given by (2.16). Eq. (5.9) can be further simplified since (2.4) implies that

$$\sin\eta\psi - \frac{\Gamma\cos\eta}{2}\psi^2 = \frac{\sin^2\eta}{2\Gamma\cos\eta} - \frac{(F(\eta;\lambda)-A)^2}{2\Gamma\cos\eta\chi^2\alpha^2}. \tag{5.10}$$

We conclude that α is a solution of (2.12). If we assume that $\alpha \neq 0$ for $-\pi/2 \leq \theta \leq \pi/2$, each of the integrals in (2.12) and (2.15) is convergent.

It remains to find an expression for β. However, Eq. (2.17) is an immediate consequence of Eqs. (5.1) and (5.7).

References

[1] E. Bromberg and J. J. Stoker, *Non-linear theory of curved elastic sheets*, Quart. Appl. Math. **3** (1945)

[2] P. F. Jordan, *Stresses and deformations of the thin-walled pressurized torus*, J. Aerospace Sci. 29 (1962)

[3] J. L. Sanders, Jr. and A. Liepins, *The toroidal membrane under internal pressure*, AIAA Launch and Space Vehicle Shell Structures Conf., 1963.

[4] H. B. Keller and E. L. Reiss, *Iterative solutions for the nonlinear bending of circular plates*, Comm. Pure Appl. Math. **11** (1958)

[5] R. W. Dickey, *The plane circular elastic surface under normal pressure, Arch. Rat. Mech. Anal.* **26** (1967)

[6] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 1, Interscience, New York, 1953