ON INCIDENCE MATRICES IMPLIED IN MAXWELL'S NETWORK THEORY*

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Introduction. The idea of injecting currents from the outside into the junction points of an electrical network seems to have originated with Maxwell [1, vol. 1, p. 404]. It is a contrivance, like D'Alembert's Principle, to facilitate the perception of the implications of the underlying laws of physics.

Given the branch e.m.f.'s and currents injected at the nodes of a conducting network, Maxwell showed how the resulting branch currents and node voltages can be calculated. In modern symbols, given the problem where v is a vector of node potentials relative to a ground point G,

$$\begin{bmatrix} Z & \Pi_T \\ \Pi & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} e \\ -\iota \end{bmatrix} ; \tag{1}$$

we solve by inverting the matric coefficient, assumed to be possible. Suppose

$$\begin{bmatrix} Z & \Pi_T \\ \Pi & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} ;$$

then

$$A = Y - Y\Pi_T(\Pi Y\Pi_T)^{-1}\Pi Y, \qquad B = Y\Pi_T(\Pi Y\Pi_T)^{-1}, \qquad C = (\Pi Y\Pi_T)^{-1}\Pi Y,$$

and

$$i = Ae - B\iota, \qquad v = Ce + (\Pi Y\Pi_T)^{-1}\iota; \tag{2}$$

Z is the branch-impedance to the current vector i which flows in response to the e.m.f.e when all nodes are grounded; $Y = Z^{-1}$ and Π is the directed-branch-on-node incidence matrix of the network. It is assumed that the network can be represented by a connected system of B line-segments each connecting one junction point with another, the number of junction points being V, the number of vertices of the graph, of which one arbitrarily chosen is the ground point G and the rest are "nodes".

The formulas (2) hold true, it can be shown, when the injected currents, represented by the vector ι, are conducted to the nodes of the network by specific means such as active dipoles of definite impedance, each with one pole grounded.

The idea was broached by the author in 1938 [2, p. 270] that junction-point potentials relative to a ground point G of a connected network, regarded as electromotive forces of constraint, can be thought of as originating in fictitious dipoles bridging, one-to-one, the junction-points and G. A network extended in this way holds, of course, the possibility of an extension of Cauer's cyclomatic matrix Γ to one that is square.

In 1962 Kron [3] took advantage of this fact and found, evidently from cases, that the inverse of the so-extended matrix, assumed to exist, has as a partitioned part a matrix Π which, when its rows and columns are transposed, is Poincaré's "first tableau" [4, p. 280] with one column deleted. This is a sufficiently remarkable result, if generally true, to deserve a proof.

1. The extended cyclomatic matrix and its inverse. Let $\mathfrak{I}$ be any tree in the connected graph $\mathcal{G}$ of V vertices and B branches of a network $\mathfrak{N}$ of dipoles. Let $\mathfrak{N}'$ be an extension of $\mathfrak{N}$ to include $\mathfrak{N}$ and $\tau = V - 1$ active dipoles bridging the ground point G and τ nodes of $\mathfrak{N}$ and so forming a second tree in $\mathcal{G}'$, the graph of $\mathfrak{N}'$. The nullity of $\mathcal{G}'$ thus is B, the number of branches in $\mathcal{G}$, said branches being chords to said second tree. The currents i in the branches of $\mathfrak{N}$ are combinations of the chord-circuit currents i' which circulate in $\mathfrak{N}$ and currents ι' *assumed to be chord-circuit currents* in $\mathfrak{N}'$, relative to $\mathfrak{I}$. Thus

$$i = \Gamma i' + \Gamma' \iota'; \tag{3}$$

Γ, Γ' are directed-circuit-on-directed-branch incidence matrices.

Because all the chords to $\mathfrak{I}$ in $\mathcal{G}'$ are distinct, the chord-circuits in $\mathcal{G}'$ are topologically independent: no null combination of them exists. It follows that if $C = [\Gamma, \Gamma']$ then C^{-1} exists.

Let the branches in $\mathcal{G}'$ corresponding to the bridging dipoles extending $\mathcal{G}$ be directed to the τ nodes and let ι be the vector of co-directed branch currents in and proper to these dipoles as branches. We have assumed [6, p. 140] that in $\mathcal{G}$ the chord-circuit currents are co-directed with the chord currents in the chords. We assume now the opposite for the currents in the branches of $\mathcal{G}'$ which are not in $\mathcal{G}$: as chords to $\mathfrak{I}$ in the graph $\mathcal{G}'$ of $\mathfrak{N}'$ and as branches in $\mathcal{G}'$ we represent these branch currents ι by counter-directed chord-circuit currents, i.e., outside $\mathfrak{N}$, $\iota = -\iota'$.

Let

$$i''_T = (i'_1, i'_2, \cdots, i'_{B-\tau} \quad \iota'_1, \iota'_2, \cdots, \iota'_\tau)$$

then $i = Ci''$. From Kirchhoff's node law we have $\Pi i + \iota = 0$ and thus we have

$$0 = \Pi C i'' + \iota = \Pi(\Gamma i' + \Gamma' \iota') + \iota = (\Pi \Gamma' - I)\iota', \tag{4}$$

an equation in which ι' is perfectly arbitrary since the vector of injected current ι is perfectly arbitrary. It follows that

$$\Pi \Gamma' = I, \qquad \Pi C = (0, I). \tag{5}$$

Thus Π is a partitioned part of C^{-1}, as conjectured by Kron.

Just as the columns of Γ are linearly independent solutions of the diophantine problem $\Pi_\gamma = 0$ as Cauer noted long ago, so the rows of Π are linearly independent solutions of the diophantine problem $\Pi \Gamma = 0$, and so there exists, perhaps, an easier way to find Π, given Γ. Nevertheless, Kron's little theorem has its charm: it implies the rather startling equation

$$i' = \Pi'_\iota \tag{6}$$

in which Π' is the upper partition of C^{-1} and which says that each mesh current is a chord current.

If we write

$$C = [\Gamma, \Gamma'], \qquad C^{-1} = \begin{bmatrix} \Pi' \\ \Pi \end{bmatrix} \tag{7}$$

we get Kron's relations

$$\Pi\Gamma' = I, \qquad \Pi'\Gamma' = 0, \qquad \Pi'\Gamma = I \tag{8}$$

from $C^{-1}C = I$, and the relation

$$\Gamma'\Pi + \Gamma\Pi' = I \tag{9}$$

from $CC^{-1} = I$.

It can be proved that, by proper numbering of branches and junction points, C can be given a form which is its own inverse if, but only if, the graph contains a tree every branch of which has a common terminal.

3. Chord-circuits relative to $\mathfrak{I}$ in $\mathcal{G}'$. If $\mathbf{b}$ is a row-vector of the directed branches in $\mathcal{G}$ then, C being the $B \times B$ matrix of Sec. 1, $\mathbf{b}C$ can be partitioned into a set of τ linearly independent open paths, in $\mathfrak{I}$, and into $B - \tau$ linearly independent chord-circuits in $\mathcal{G}$ relative to $\mathfrak{I}$:

$$\mathbf{b}C = [\mathbf{b}\Gamma, \mathbf{b}\Gamma'] = [\mathbf{c}_{\text{graph}}, \mathbf{c}_{\text{tree}}]; \tag{10}$$

$\mathbf{c}_{\text{tree}}$ is a vector of open chains, $\mathbf{c}_{\text{graph}}$ of closed chains, but $\mathbf{c}_{\text{tree}}$ is closed in $\mathcal{G}'$. Thus Γ' is Cauer's cyclomatic matrix for the graph which remains when, from the extended graph $\mathcal{G}'$, all chords to $\mathfrak{I}$ in $\mathcal{G}$ are removed.

For a column-vector of branch e.m.f.'s co-directed with the branches,

$$C_T e = [\Gamma_T e / \Gamma_T' e], \qquad e_T C = [e_T \Gamma, e_T \Gamma']; \tag{11}$$

$\Gamma_T' e$ are the e.m.f.'s in the τ linearly-independent open paths to ground in $\mathfrak{I}$. There is no restriction, in view of Blakesley's theorem [5] to assume that there is no e.m.f. in any branch of the tree, i.e. to assume the equation $\Gamma_T' e = 0$. We may note now that if we write

$$\Pi_T v = -E, \qquad \Gamma' \iota' = I, \tag{12}$$

we get the system

$$\Gamma_T E = 0, \qquad \Pi i = \iota',$$
$$\Gamma_T' e = 0, \qquad \Pi' I = 0. \tag{13}$$

The equations in E and i represent Kirchhoff's mesh and node laws for the network $\mathfrak{N}$ of $\mathfrak{N}'$. The equation in e is true if and only if there are no e.m.f.'s in $\mathfrak{I}$. The last equation is not a node law: like the mesh law, it is an identity implied by definitions and says nothing about the incidence at the nodes of the components I of the branch currents.

4. On "current-sources". Let $\mathbf{c}$ be the row-vector of closed undirected chains in $\mathcal{G}'$ corresponding to unsigned $[\Gamma, \Gamma']$. All circuits in $\mathcal{G}'$ no one of which includes any branch more than once are obtainable by independent linear combinations, modulo 2, of these circuits relative to $\mathfrak{I}$:

$$\mathbf{c}'_\alpha = \mathbf{c}_\beta \mathfrak{a}^\beta_\alpha, \qquad \det \mathfrak{a} = 1, \qquad \text{modulo 2} \tag{14}$$

[6, p. 147]. From this it follows that not only are there numerous equivalent mesh circuits and currents for the representation of branch currents in $\mathcal{G}$ so, in $\mathcal{G}'$, there are numerous equivalent mesh circuits and currents, some of which circuits in $\mathcal{G}'$ are open paths in $\mathcal{G}$. Among the equivalent circuits are τ circuits in $\mathcal{G}'$ in each of which is a single branch of $\mathcal{G}$, a branch of $\mathfrak{I}$, for conducting the extraneous current in and out of $\mathcal{G}$. Thus we could

have extended our network $\mathfrak{N}$, not by adding grounded dipoles, but by connecting an extraneous dipole in parallel with each branch of the tree. Other extraneous dipoles may be introduced but only to complicate the analysis.

5. The e.m.f. equations. When extraneous active dipoles are connected to $\mathfrak{N}$, extending it to $\mathfrak{N}'$, the dependent variable v, calculable as a function of e, becomes an independent variable. In fact, when the extraneous dipoles are impedanceless and harbor sources of e.m.f. ϵ directed from G to the nodes, then $v = \epsilon$ [7, p. 220]*.

The e.m.f. equation for the chord-circuits in $\mathfrak{N}'$ are obtained from the basic e.m.f. equation

$$e - \Pi_T v = Zi \tag{15}$$

by premultiplication by C_T and the substitutions $i = Ci''$, $v = \epsilon$:

$$\Gamma_T e = \Gamma_T Z \Gamma i' + \Gamma_T Z \Gamma' \iota', \qquad -\epsilon + \Gamma'_T e = \Gamma'_T Z \Gamma i' + \Gamma'_T Z \Gamma' \iota'. \tag{16}$$

In the case of the Wheatstone bridge, for example, we have, corresponding to the graph of Fig. 1 the matrices

$$[\Gamma, \Gamma'] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \Pi' \\ \Pi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}.$$
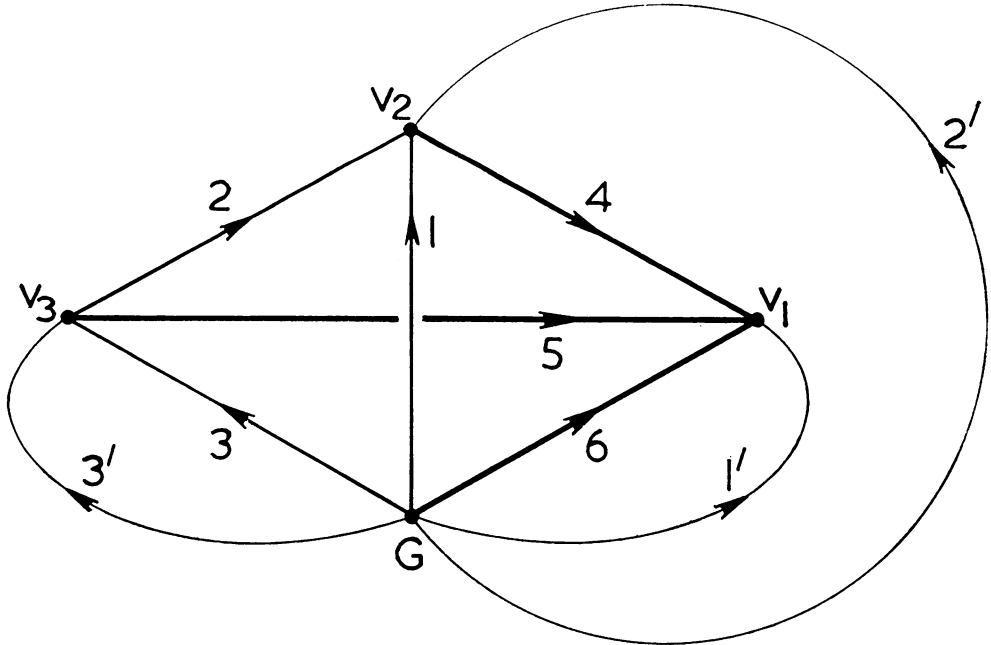

Fig. 1

Having so indexed the branches of the graph that Γ has canonical form [6, p. 147], we see that $\Gamma'_T Z \Gamma'$ involves only the self and mutual impedances of the branches of the tree.

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* The first equation on this page is true if but only if $J = I$.

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Addendum

After the above note had gone to press, it was discovered that the converse of (16) has been sought [1, p. 59]. One interchanges the rows and columns of (9) to get

$$\Pi'_T\Gamma_T + \Pi_T\Gamma'_T = E,$$

say, and so, from (8),

$$\Pi Y\cdot E\cdot Z\Gamma' = I, \qquad \pi'Y\cdot E\cdot Z\Gamma' = 0, \qquad \pi'Y\cdot E\cdot Z\Gamma = I.$$

These relations hold when E_T is substituted for E. One verifies easily that

$$\begin{bmatrix} \Pi'Y\Pi'_T & \Pi'Y\Pi_T \\ \Pi Y\Pi'_T & \Pi Y\Pi_T \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} \Gamma_T Z\Gamma & \Gamma_T Z\Gamma' \\ \Gamma'_T Z\Gamma & \Gamma'_T Z\Gamma' \end{bmatrix}.$$

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