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AN EXISTENCE THEOREM FOR LINEAR BOUNDARY VALUE PROBLEMS*

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1. Introduction. Consider a general system of n first-order differential equations

$$y' = F(t, y) \tag{1.1a}$$

subject to the linear two-point boundary conditions

$$B_1y(a) + B_2y(b) = \alpha. \tag{1.1b}$$

The n-vector function F is assumed to be a continuous function of (t, y) for t belonging to $[a, b]$, $b - a > 0$ sufficiently small, and y belonging to a suitable region $R \subset E_n$. The n-vector α is fixed, and B_1 and B_2 are $n \times n$ matrices such that the augmented matrix $[B_1, B_2]$ has rank n. Considerable interest has been shown in the numerical solution of (1.1) (cf. Keller [1], Conti [2], Osborne [3], and Roberts and Shipman [4]). Keller [1], for example, describes shooting methods for the numerical solution of such boundary-value problems. The justification of this approach depends upon the existence and uniqueness of solutions of the linear boundary-value problem

$$y' = A(t)y + f(t), \tag{1.2a}$$

$$B_1y(a) + B_2y(b) = \alpha, \tag{1.2b}$$

where $A(t)$ is a continuous $n \times n$ matrix on $[a, b]$ and $f(t)$ is continuous on $[a, b]$. (In the nonlinear case of (1.1), reduction to (1.2) can be accomplished by Newton's method.) Usually $B_1 + B_2$ is assumed to be nonsingular, although Keller [1, 5] describes a method for obtaining the solution of (1.2) when $B_1 + B_2$ is singular. Keller's method consists of obtaining solution bounds and then applying the Banach lemma.

We shall give a new criterion for the local existence of a unique solution to (1.2) when $B_1 + B_2$ is singular. Our conditions involve only B_1, B_2 and $A(a)$. Our method is then applied to the general two-point boundary-value problem with unmixed boundary conditions.

2. The linear boundary value problem. Let $Y(t)$ denote the fundamental matrix for for (1.2a) satisfying

$$Y'(t) = A(t)Y(t), \qquad Y(a) = I, \qquad I = \text{identity matrix}. \tag{2.1}$$

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From the variation of parameters formula and (1.2b) we have

$$[B_1 + B_2 Y(b)]y(a) = \alpha - B_2 Y(b) \int_a^b Y^{-1}(s)f(s)\,ds, \tag{2.2}$$

and it follows that a necessary and sufficient condition for (1.2) to have a unique solution is that $[B_1 + B_2 Y(b)]$ be nonsingular.

We can write

$$B_1 + B_2 Y(t) = [I + S(t)]UY(t), \tag{2.3}$$

where $U = B_1 P + B_2$ and $S(t) = B_1(Y^{-1}(t) - P)U^{-1}$. Here P is an elementary matrix ($\|P\| = 1$) chosen such that $B_1 P + B_2$ is nonsingular (cf. Keller [1, p. 60]). Consequently, a condition equivalent to $B_1 + B_2 Y(b)$ being nonsingular is that the matrix $S(b)$ not have an eigenvalue $\lambda(b) = -1$. If $B_1 + B_2$ is nonsingular then, from (2.3) with $t = a$, it follows that $S(a)$ does not have an eigenvalue $\lambda(a) = -1$. A standard continuity argument establishes the existence of solutions on $[a, b]$, $b - a > 0$ sufficiently small. In the following a general criterion is established for the case when $B_1 + B_2$ is singular.

3. $B_1 + B_2$ singular. Suppose that $B_1 + B_2$ is singular, rank $[B_1, B_2] = n$ and rank $[B_1 + B_2] = m < n$. Since $B_1 + B_2$ is singular, $I + S(a)$ is singular. Thus there exists an eigenvector x such that

$$S(a)x = -x. \tag{3.1}$$

Now consider the equation

$$S(t)x(t) = \lambda(t)x(t) \tag{3.2}$$

valid for $a \le t \le b$ where $x(t)$ is an eigenvector corresponding to the eigenvalue $\lambda(t)$ of $S(t)$ such that $\lambda(t) \to -1$ as $t \to a$ and $x(t) \to x$ as $t \to a$.

Let y be the right eigenvector of $S(a)$ corresponding to the eigenvalue $\lambda = -1$, i.e.

$$y^T S(a) = -y^T. \tag{3.3}$$

Denoting the right derivative of a function by D_R and taking the right derivative of (3.2) at $t = a$, we obtain

$$D_R S(a)x + S(a)D_R x(a) = D_R\lambda(a)x - D_R x(a). \tag{3.4}$$

Multiplying this equation on the left by y^T yields

$$y^T D_R S(a)x = D_R\lambda(a)y^T x. \tag{3.5}$$

To compute $D_R S(a)$ consider

$$D_R(Y(t)\cdot Y^{-1}(t)) = 0 = D_R Y(t)\cdot Y^{-1}(t) + Y(t)\cdot D_R Y^{-1}(t)$$

and

$$D_R Y(t) = Y'(t) = A(t)Y(t), \qquad Y(a) = I.$$

Then

$$D_R Y^{-1}(a) = -A(a)$$

which yields, from the definition of $S(t)$,

$$D_R S(a) = B_1 D_R Y^{-1}(a)U^{-1} = -B_1 A(a)U^{-1}.$$

If

$$y^T x \neq 0 \quad \text{and} \quad y^T D_R S(a) x \neq 0, \tag{3.6}$$

from (3.5) we have $D_R \lambda(a) \neq 0$. By assumption, the multiplicity of the eigenvalue $\lambda(a) = -1$ is $n - m$. If we could find $n - m$ pairs (y^i, x^i) $i = 1, 2, \cdots, n - m$ satisfying (3.6) having the additional property that both $\{y_1, y_2, \cdots, y_{n-m}\}$ and $\{x_1, \cdots, x_{n-m}\}$ are linearly independent sets spanning R_{n-m}, we could conclude that $D_R \lambda(a) \neq 0$ for every eigenvector pair (y^i, x^i), $i = 1, 2, \cdots, n - m$, corresponding to the eigenvalue $\lambda(a) = -1$. In this case, $I + S(x)$ or, what is the same, $B_1 + B_2 Y(t)$ must be nonsingular if $0 < b - a$ sufficiently small, using a standard continuity of the eigenvalues argument.

We must now find conditions which insure that (3.6) is satisfied. If we can prove the existence of one such pair (y, x), we will then be able to show this yields the existence of the set (y^i, x^i) having the desired properties.

We will now determine conditions which insure the existence of at least one pair (y, x) satisfying (3.6). Consider (3.3): $y^T S(a) = -y^T$. Since

$$S(a) = B_1(I - P)U^{-1},$$

(3.3) can be written

$$y^T B_1 (I - P) U^{-1} = -y^T,$$

or

$$y^T(B_1 + B_2) = 0. \tag{3.7}$$

From (3.1),

$$S(a)x = B_1(I - P)U^{-1}x = -x$$

and letting $x = Uv$, it follows that

$$(B_1 + B_2)v = 0. \tag{3.8}$$

By assumption, rank $[B_1 + B_2] = m < n$. By rearranging equations and identifying the components of y in (1.2), if necessary, $B_1 + B_2$ can always be written in the form

$$B_1 + B_2 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is a nonsingular $m \times m$ matrix. If

$$T_1 = \begin{bmatrix} I_m & 0 \\ -B_{21}B_{11}^{-1} & I_{n-m} \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} I_m & -B_{11}^{-1}B_{12} \\ 0 & I_{n-m} \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix, then

$$T_1(B_1 + B_2)T_2 = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} - B_{22}B_{21}^{-1}B_{12} \end{bmatrix} = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

where $B_{22} - B_{21}B_{11}^{-1}B_{12}$ must vanish in order not to contradict the rank $[B_1 + B_2] = m$. Then

$$y^T(B_1 + B_2) = 0$$

implies

$$y^T T_1^{-1}\begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} B_{11}^T & 0 \\ 0 & 0 \end{bmatrix}(T_1^T)^{-1}y = 0.$$

Thus any vector y satisfying (3.7) can be written as a linear combination of vectors

$$y_j = T_1^T\begin{bmatrix} 0 \\ e_j \end{bmatrix}, \qquad j = 1, 2, \cdots, n - m$$

where $[{}^{0}_{e_j}]$ represents an n-vector with the first m components 0, and e_j is the jth $(n - m)$-dimensional unit vector. In exactly the same manner, it can be shown that any vector v satisfying (3.8) can be written as a linear combination of the vectors

$$v_j = T_2\begin{bmatrix} 0 \\ e_j \end{bmatrix}, \qquad j = 1, 2, \cdots, n - m.$$

Let

$$[0, e_i^T]T_1UT_2\begin{bmatrix} 0 \\ e_j \end{bmatrix} = c_{ij}, \qquad C = (c_{ij}) \tag{3.9}$$

and

$$[0, e_i^T]T_1B_1A(a)T_2\begin{bmatrix} 0 \\ e_j \end{bmatrix} = d_{ij}, \qquad D = (d_{ij}). \tag{3.10}$$

Consider the vectors

$$y = T_1^T\begin{bmatrix} 0 \\ \sum_{i=1}^{n-m} w_ie_i \end{bmatrix} \qquad v = T_2\begin{bmatrix} 0 \\ \sum_{i=1}^{n-m} z_ie_i \end{bmatrix}$$

which span the appropriate subspaces. If $w = (w_1, \cdots, w_{n-m})$, $z = (z_1, \cdots, z_{n-m})$ then

$$y^Tx \neq 0 \quad \text{and} \quad y^TB_1A(a)U^{-1}x \neq 0$$

if there exist constant vectors w, z such that

$$w^TCz \neq 0 \quad \text{and} \quad w^TDz \neq 0. \tag{3.11}$$

The conditions we have been searching for are: *If C and D are both nonzero* then *a w and z can be found satisfying* (3.11). Assume not. Then for every w, z, $(w^TCz)(w^TDz) = 0$. Since w^TCz and w^TDz are polynomials in the polynomial ring $R[w, z]$ which is an integral domain, either $w^T Cz = 0$ for every w, z or $w^T Dz = 0$ for every w, z. This implies that either C or D is the zero matrix, a contradiction.

We have now established the existence of a pair (x, y) satisfying (3.6) which is equivalent to finding a pair (w^1, z^1) satisfying (3.11).

Both $w^T Cz$ and $w^T Dz$ are continuous functions of w and z. Thus there must exist a neighborhood of $(w^1, z^1) \subset R^{n-m} \times R^{n-m}$ where (3.11) is satisfied. It is then possible to find two bases in R^{n-m} one containing w^1 and the other z^1 remaining in the neighborhood of (w^1, z^1). Thus every eigenvalue $\lambda(a) = -1$ of $I + A(a)$ has $D_R\lambda(a) \neq 0$. By continuity, it follows that $I + S(t)$ or $B_1 + B_2 Y(t)$ must be non-singular for $0 < b - a$ sufficiently small.

We have proved the following theorem:

THEOREM. *Assume rank* $[B_1, B_2] = n$, *rank* $[B_1 + B_2] = m < n$. *The boundary value problem* (1.2) *has a unique solution for* $b - a > 0$ *sufficiently small if either of the following equivalent conditions is satisfied:*

(i) there exists right and left eigenvectors y and v corresponding to the eigenvalue $\lambda = 0$ for $B_1 + B_2$ such that $y^T Uv \neq 0$ and $y^T B_1 A(a) v \neq 0$,

(ii) $C \neq 0$ and $D \neq 0$.

Remark. Condition (ii) will usually be easier to verify and we use it in the following corollary.

COROLLARY. *Let* $n = 2k$,

$$A(a) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B_1 = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \end{bmatrix}$$

where $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{21}$ are $k \times k$ matrices, and the rank $B_{11} =$ rank $B_{21} = k$. Then (1.2) has a unique solution for $b - a > 0$ sufficiently small if $B_{21}A_{12}$ has a nonzero element.

Proof. Since

$$U = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{21} \end{bmatrix}, \quad \text{rank} \quad U = 2k,$$

using the definition of T_1, T_2, (ii) of the Theorem yields that both B_{21} and $B_{21}A_{12}$ must have a nonzero element. B_{21} has a nonzero element since rank $B_{21} = k$; hence the corollary is proven.

Remark. If $A(t)$ is in companion form then A_{12} has a one in the $(k, 1)$ position and all other elements are zero. The corollary, in this case, reduces to requiring B_{21} to have a nonzero element in the first row.

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