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A UNIQUENESS THEOREM FOR RIGID HEAT CONDUCTORS WITH MEMORY*

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1. Introduction. Recently there have appeared [1, 2] two theories for the behavior of rigid conductors of heat composed of materials with memory. The theory developed in [1] has associated with it finite wave speeds, a feature of considerable interest in light of the well-known shortcomings of the classical theory.

In [1], Gurtin and Pipkin deduced the linearized form assumed by their constitutive equations for isotropic and homogeneous conductors. Nunziato [3] established the corresponding linearized version of the results of [2], finding for the internal energy e and heat flux $\mathbf{q}$,

$$e = c\theta + \int_0^\infty \alpha(s)\theta(t - s)\, ds, \qquad \mathbf{q} = -\kappa\nabla\theta - \int_0^\infty \beta(s)\nabla\theta(t - s)\, ds. \tag{1.1}$$

Here θ denotes the departure of the temperature from its reference value, c and κ stand for the respective instantaneous heat capacity and thermal conductivity, and α and β designate the energy and heat-flux relaxation functions. For $\kappa = 0$, the constitutive relations (1.1) reduce to their counterparts in [1].

Nunziato's investigation [3] also contains two uniqueness theorems for history-value problems appropriate to the linearized theories. One theorem entails assuming

$$\kappa > 0, \qquad c > 0, \qquad \alpha(0) > 0, \tag{1.2}$$

and the other

$$\kappa = 0, \quad c > 0, \quad \beta(0) > 0, \quad \alpha(0) \geq 0, \quad \alpha'(0) > 0. \tag{1.3}$$

Further results on uniqueness were found by Finn and Wheeler [4] in an investigation aimed mainly at wave propagation aspects. Their hypotheses include

$$\kappa = 0, \quad c > 0, \quad \beta(0) > 0, \quad \beta'(0) < 0, \quad \alpha(0) > 0. \tag{1.4}$$

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Finally, Nunziato reconsidered uniqueness for the case $\kappa = 0$ in [5], and concluded that the conditions

$$c > 0, \qquad \beta(0) > 0 \tag{1.5}$$

are adequate in this case.

In the present paper we are able to establish that uniqueness holds if either

$$\kappa > 0, \qquad c > 0, \tag{1.6}$$

or, as Nunziato asserts in [5],

$$\kappa = 0, \quad c > 0, \quad \beta(0) > 0. \tag{1.7}$$

The conditions (1.6) are weaker than those hitherto invoked for the case $\kappa \neq 0$. Perhaps equally interesting is the approach of the present paper, which differs significantly from those employed in [3], [4], [5].

2. Uniqueness. For a rigid stationary heat conductor, the energy-balance law assumes the form

$$\dot{e} = -\nabla\cdot\mathbf{q} + r \quad \text{on} \quad R \times (-\infty, \infty), \tag{2.1}$$

in which r designates the heat supply and R denotes the region of space occupied by the conductor. Since the divergence theorem is essential to the developments that follow, we assume that R henceforth stands for the interior of a *bounded regular region* as defined in Kellogg's book [6]. For such regions, the divergence theorem is applicable to vector fields in class $\mathcal{C}^1$ on the closure, $\bar{R} = R \cup \partial R$, of R, where ∂R refers to the boundary of R.

THEOREM 1. Let e_γ, $\mathbf{q}_\gamma$ ($\gamma = 1, 2$) be in class $\mathcal{C}^1$ on $\bar{R} \times (-\infty, \infty)$ and assume that

$$\dot{e}_\gamma = -\nabla\cdot\mathbf{q}_\gamma + r_\gamma \quad \text{on} \quad R \times (-\infty, \infty). \tag{2.2}$$

Then

$$\int_R \left[\frac{\partial}{\partial t}(e_1 - e_2)^2 - 2(\nabla e_1 - \nabla e_2)\cdot(\mathbf{q}_1 - \mathbf{q}_2)\right] dV$$
$$= 2\int_R (e_1 - e_2)(r_1 - r_2)\, dV - 2\int_{\partial R} (e_1 - e_2)(\mathbf{q}_1 - \mathbf{q}_2)\cdot\mathbf{n}\, dA, \tag{2.3}$$

where $\mathbf{n}$ denotes the unit outward normal to ∂R.

Proof. By (2.2),

$$(e_1 - e_2)(\dot{e}_1 - \dot{e}_2) = -(e_1 - e_2)(\nabla\cdot\mathbf{q}_1 - \nabla\cdot\mathbf{q}_2) + (e_1 - e_2)(r_1 - r_2).$$

Thus,

$$\frac{1}{2}\frac{\partial}{\partial t}(e_1 - e_2)^2 = -\nabla\cdot[(e_1 - e_2)(\mathbf{q}_1 - \mathbf{q}_2)] + (\nabla e_1 - \nabla e_2)\cdot(\mathbf{q}_1 - \mathbf{q}_2) + (e_1 - e_2)(r_1 - r_2).$$

The desired identity (2.3) now follows with the aid of the divergence theorem, which completes the proof.

It is worth mentioning that the foregoing theorem is free of assumptions concerning the material of the conductor. Conceivably, (2.3) may have interesting implications for conductors whose constitutive behavior is not accounted for by (1.1).

THEOREM 2. Let e_γ, $\mathbf{q}_\gamma$, r_γ ($\gamma = 1, 2$) obey the hypothesis of Theorem 1, let $T > 0$, and let $t_0 \in (0, T)$ be such that

(a) for every $\mathbf{x} \in R$,

$$e_1(\mathbf{x}, t_0) = e_2(\mathbf{x}, t_0); \tag{2.4}$$

for every $t \in [t_0, T]$,

$$\text{(b)} \quad \int_{t_0}^{t} \int_{\partial R} [e_1(\mathbf{x}, \tau) - e_2(\mathbf{x}, \tau)][\mathbf{q}_1(\mathbf{x}, \tau) - \mathbf{q}_2(\mathbf{x}, \tau)] \cdot \mathbf{n}(\mathbf{x})\, dA\, d\tau = 0; \tag{2.5}$$

$$\text{(c)} \quad -\int_{t_0}^{t} \int_{R} [\nabla e_1(\mathbf{x}, \tau) - \nabla e_2(\mathbf{x}, \tau)] \cdot [\mathbf{q}_1(\mathbf{x}, \tau) - \mathbf{q}_2(\mathbf{x}, \tau)]\, dV\, d\tau \geq 0; \tag{2.6}$$

(d) $r_1(\cdot, t) = r_2(\cdot, t)$ on R.

Then $e_1 = e_2$ on $R \times [t_0, T]$.

Proof. By (a), (b), (d) and Theorem 1,

$$\frac{1}{2} \int_R [e_1(\mathbf{x}, t) - e_2(\mathbf{x}, t)]^2\, dV = \int_{t_0}^{t} \int_R [\nabla e_1(\mathbf{x}, \tau) - \nabla e_2(\mathbf{x}, \tau)] \cdot [\mathbf{q}_1(\mathbf{x}, \tau) - \mathbf{q}_2(\mathbf{x}, \tau)]\, dV\, d\tau.$$

for every $t \in [t_0, T]$. Thus, (c) furnishes

$$\int_R (e_1 - e_2)^2\, dV \leq 0 \quad \text{on} \quad [t_0, T], \tag{2.7}$$

which, since the integrand is continuous and non-negative, implies $e_1 = e_2$ on $R \times [t_0, T]$. The proof is now complete.

The conjecture that Theorem 1 has significant implications beyond those of the present paper pertains as well to Theorem 2, whose *validity* is likewise not contingent upon the existence of constitutive relations. On the other hand, the *usefulness* of Theorem 2 may rest on the connection between (2.6) and restrictions on the response functionals, as suggested by

THEOREM 3. Let θ_γ ($\gamma = 1, 2$) be in class $\mathcal{C}^2$ on $\bar{R} \times (-\infty, \infty)$, and suppose that α, β are in $\mathcal{C}^2$ on $[0, \infty)$. Suppose that the functions e_γ, $\mathbf{q}_\gamma$ defined for every $(\mathbf{x}, t) \in \bar{R} \times (-\infty, \infty)$ through

$$e_\gamma(\mathbf{x}, t) = c\theta_\gamma(\mathbf{x}, t) + \int_0^\infty \alpha(s)\theta_\gamma(\mathbf{x}, t - s)\, ds,$$

$$\mathbf{q}_\gamma(\mathbf{x}, t) = -\kappa \nabla \theta_\gamma(\mathbf{x}, t) - \int_0^\infty \beta(s) \nabla \theta_\gamma(\mathbf{x}, t - s)\, ds \tag{2.8}$$

are in class $\mathcal{C}^1$ on $\bar{R} \times (-\infty, \infty)$ and obey (2.2).

Let $T > 0$ and assume

(a) $\theta_1 = \theta_2$ on $R \times (-\infty, 0]$, $\quad r_1 = r_2$ on $R \times (-\infty, T]$;

$$\text{(b)} \quad \int_0^t \int_{\partial R} [e_1(\mathbf{x}, \tau) - e_2(\mathbf{x}, \tau)][\mathbf{q}_1(\mathbf{x}, \tau) - \mathbf{q}_2(\mathbf{x}, \tau)] \cdot \mathbf{n}(\mathbf{x})\, dA\, d\tau = 0$$

for every $t \in [0, T]$;

(c) either

$$c > 0, \qquad \kappa > 0$$

or

$$\kappa = 0, \qquad c > 0, \qquad \beta(0) > 0.$$

Then

$$\theta_1 = \theta_2 \quad \text{on} \quad \bar{R} \times (-\infty, T) \tag{2.9}$$

Proof. For convenience, let

$$\bar{\theta} = \theta_1 - \theta_2, \qquad \bar{e} = e_1 - e_2, \qquad \bar{\mathbf{q}} = \mathbf{q}_1 - \mathbf{q}_2. \tag{2.10}$$

Define

$$t_0 = \sup \left\{ s \in [0, T] \;\middle|\; \int_R \bar{e}^2 \, dV = 0 \quad \text{on} \quad [0, s] \right\}. \tag{2.11}$$

If $t_0 = T$, then

$$\bar{e} = 0 \quad \text{on} \quad \bar{R} \times [0, T],$$

and it follows at once from $(2.8)_1$, and the assumption that $\theta_1 = \theta_2$ on $\bar{R} \times (-\infty, 0]$, that

$$c\bar{\theta}(\mathbf{x}, t) + \int_0^t \alpha(s)\bar{\theta}(\mathbf{x}, t - s) \, ds = 0, \tag{2.12}$$

for every $(\mathbf{x}, t) \in \bar{R} \times [0, T]$. Since $c > 0$, (2.12) implies

$$\bar{\theta} = \theta_1 - \theta_2 = 0 \quad \text{on} \quad \bar{R} \times [0, T]. \tag{2.13}$$

It therefore suffices to show that $t_0 = T$ in order to reach the desired conclusion (2.9).

Assume to the contrary that $t_0 < T$. By (2.11), there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset (t_0, T]$ such that

$$\lim_{n \to \infty} t_n = t_0, \tag{2.14}$$

$$\int_R \bar{e}^2(\mathbf{x}, t_n) \, dV \neq 0 \qquad (n = 1, 2, \cdots). \tag{2.15}$$

Moreover, (2.15), (a), (b), and Theorem 2 imply that the function D defined for all $t \in [t_0, T]$ by

$$D(t) = \int_{t_0}^t \int_R \nabla\bar{e}(\mathbf{x}, \xi) \cdot \bar{\mathbf{q}}(\mathbf{x}, \xi) \, dV \, d\xi \tag{2.16}$$

obeys

$$D(t_n) < 0 \; (n = 1, 2, \cdots). \tag{2.17}$$

From (2.16), (2.10), (2.8) there follows

$$D(t) = \int_{t_0}^t \int_R \left\{ c\kappa \, |\bar{\mathbf{g}}(\tau)|^2 + \int_0^{\tau - t_0} \delta(s)\bar{\mathbf{g}}(\tau) \cdot \bar{\mathbf{g}}(\tau - s) \, ds \right.$$
$$\left. + \left[\int_0^{\tau - t_0} \alpha(s)\bar{\mathbf{g}}(\tau - s) \, ds \right] \cdot \left[\int_0^{\tau - t_0} \beta(s)\bar{\mathbf{g}}(\tau - s) \, ds \right] \right\} dV \, d\tau, \tag{2.18*}$$

* Here and in what follows, matters are somewhat simplified by dropping the explicit representation of position dependence.

where $\delta(s) = c\beta(s) + \kappa\alpha(s)$ and $\bar{\mathbf{g}} = \nabla\bar{\theta}$. Introduce

$$\mathbf{h}(\xi) = \int_{t_0}^{t} \bar{\mathbf{g}}(s)\, ds, \tag{2.19}$$

and integrate by parts to get

$$\int_0^{\tau-t_0} \delta(s)\bar{\mathbf{g}}(\tau)\cdot\bar{\mathbf{g}}(\tau - s)\, ds = \dot{\mathbf{h}}(\tau)\cdot\mathbf{h}(\tau)\ \delta(0) + \dot{\mathbf{h}}(\tau)\cdot\int_0^{\tau-t_0} \delta'(s)\mathbf{h}(\tau - s)\, ds.$$

A second integration by parts furnishes

$$\int_0^t \int_0^{\tau-t_0} \delta(s)\bar{\mathbf{g}}(\tau)\cdot\bar{\mathbf{g}}(\tau - s)\, ds = \frac{\delta(0)}{2}\, |\mathbf{h}(t)|^2 + \mathbf{h}(t)\cdot\int_0^{t-t_0} \delta'(s)\mathbf{h}(t - s)\, ds$$
$$- \int_{t_0}^{t} \mathbf{h}(\tau)\cdot\int_0^{\tau-t_0} \delta'(s)\dot{\mathbf{h}}(\tau - s)\, ds\, d\tau.$$

Finally, integrate by parts in the last term to arrive at

$$\int_{t_0}^t \int_0^{\tau-t_0} \delta(s)\bar{\mathbf{g}}(\tau)\cdot\bar{\mathbf{g}}(\tau - s)\, ds\, d\tau = \frac{\delta(0)}{2}\, |\mathbf{h}(t)|^2 + \mathbf{h}(t)\cdot\int_0^{t-t_0} \delta'(s)\mathbf{h}(t - s)\, ds$$
$$- \delta'(0)\int_{t_0}^{t} |\mathbf{h}(\tau)|^2\, d\tau - \int_{t_0}^{t}\int_0^{\tau-t_0} \delta''(s)\mathbf{h}(\tau)\cdot\mathbf{h}(\tau - s)\, ds\, d\tau. \tag{2.20}$$

Similarly,

$$\left[\int_0^{\tau-t_0} \alpha(s)\mathbf{g}(\tau - s)\, ds\right]\cdot\left[\int_0^{\tau-t_0} \beta(\xi)\mathbf{g}(\tau - \xi)\, d\xi\right]$$
$$= \left[\alpha(0)\mathbf{h}(\tau) + \int_0^{\tau-t_0} \alpha'(s)\mathbf{h}(\tau - s)\, ds\right]\cdot\left[\beta(0)\mathbf{h}(\tau) + \int_0^{\tau-t_0} \beta'(\xi)\mathbf{g}(\tau - \xi)\, d\xi\right]$$
$$= \alpha(0)\beta(0)\ |\mathbf{h}(\tau)|^2 + \int_0^{\tau-t_0}\int_0^{\tau-t_0} \alpha'(s)\beta'(\xi)\mathbf{h}(\tau - s)\cdot\mathbf{h}(\tau - \xi)\, ds\, d\xi$$
$$+ \mathbf{h}(\tau)\cdot\int_0^{\tau-t_0} [\alpha(0)\beta'(s) + \beta(0)\alpha'(s)]\mathbf{h}(\tau - s)\, ds.$$

Substitute from this equation and (2.20) into (2.18) to get

$$D(t) = c\kappa\int_{t_0}^t\int_R |\dot{\mathbf{h}}(\tau)|^2\, d\tau\, dV + \frac{\delta(0)}{2}\int_R |\mathbf{h}(t)|^2\, dV + \int_R\int_0^{t-t_0} \delta'(s)\mathbf{h}(t)\cdot\mathbf{h}(t - s)\, ds\, dV$$
$$+ [\alpha(0)\beta(0) - \delta'(0)]\int_{t_0}^t\int_R |\mathbf{h}(\tau)|^2\, d\tau\, dV$$
$$+ \int_{t_0}^t\int_0^{\tau-t_0}\int_R [\alpha(0)\beta'(s) + \beta(0)\alpha'(s) - \delta''(s)]\mathbf{h}(\tau)\cdot\mathbf{h}(\tau - s)\, dV\, ds\, d\tau$$
$$+ \int_{t_0}^t\int_0^{\tau-t_0}\int_0^{\tau-t_0}\int_R \alpha'(s)\beta'(\xi)\mathbf{h}(\tau - s)\cdot\mathbf{h}(\tau - \xi)\, dV\, ds\, d\xi\, d\tau. \tag{2.21}$$

Therefore,

$$D(t) \geq c\kappa \int_{t_0}^{t} \int_R |\dot{\mathbf{h}}(\tau)|^2 \, dV \, d\tau + \frac{\delta(0)}{2} \int_R |\mathbf{h}(t)|^2 \, dV$$
$$- M\left\{\int_0^{t-t_0} \int_R |\mathbf{h}(t)| \, |\mathbf{h}(t-s)| \, dV \, ds + \int_{t_0}^{t} \int_R |\mathbf{h}(\tau)|^2 \, dV \, d\tau \right.$$
$$\left. + \int_{t_0}^{t} \int_0^{\tau-t_0} \int_R |\mathbf{h}(\tau)| \, |\mathbf{h}(\tau-s)| \, dV \, ds \, d\tau + \int_{t_0}^{t} \int_R \left[\int_0^{\tau-t_0} |\mathbf{h}(\tau-s)| \, ds\right]^2 dV \, d\tau\right\}, \tag{2.22}$$

provided

$$M = \sup_{[0,T]} (|\delta'| + |\alpha\beta| + |\delta''| + |\alpha\beta'| + |\alpha'\beta'| + |\alpha'\beta|). \tag{2.23}$$

The inequality $2ab \leq a^2 + b^2$ is valid for a and b real numbers. Consequently, (2.22) and the Schwartz inequality for integrals give rise to

$$D(t) \geq c\kappa \int_{t_0}^{t} \int_R |\dot{\mathbf{h}}(\tau)|^2 \, dV \, d\tau + \frac{\delta(0)}{2} \int_R |\mathbf{h}(t)|^2 \, dV$$
$$- M\left\{\frac{(t-t_0)}{2} \int_R |\mathbf{h}(t)|^2 \, dV + \frac{3}{2} \int_0^{t-t_0} \int_R |\mathbf{h}(t-s)|^2 \, dV \, ds \right.$$
$$\left. + \frac{1}{2} \int_{t_0}^{t} \int_R (\tau - t_0) \, |\mathbf{h}(\tau)|^2 \, dV \, d\tau + \int_{t_0}^{t} \int_0^{\tau-t_0} \int_R (\tfrac{1}{2} + \tau - t_0) \, |\mathbf{h}(\tau-s)|^2 \, dV \, ds \, d\tau\right\}.$$

Put $k(\xi) = \int_R |\mathbf{h}(\xi)|^2 \, dV$ to get

$$D(t) \geq c\kappa \int_{t_0}^{t} \int_R |\dot{\mathbf{h}}(\tau)|^2 \, dV \, d\tau - M\left\{\frac{1}{2}\left[(t-t_0) - \frac{\delta(0)}{M}\right]k(t) \right.$$
$$\left. + \int_{t_0}^{t} \left(\frac{3}{2} + \frac{s-t_0}{2}\right)k(s) \, ds + \int_{t_0}^{t} \int_{t_0}^{\tau} \left[\frac{1 + 2(\tau - t_0)}{2}\right]k(s) \, ds \, d\tau\right\}. \tag{2.24}$$

But

$$k(t) = \int_R \left|\int_{t_0}^{t} \tilde{\mathbf{g}}(\tau) \, d\tau\right|^2 dV \leq (t - t_0) \int_R \int_{t_0}^{t} |\tilde{\mathbf{g}}(\tau)|^2 \, d\tau \, dV,$$

and the right-hand member in this inequality is nondecreasing in t. Accordingly, (2.24) yields

$$D(t) \geq \left\{c\kappa - 5M(t-t_0)\left[(t-t_0) + (t-t_0)^2 + (t-t_0)^3 - \frac{\delta(0)}{10M}\right]\right\}$$
$$\cdot \int_R \int_{t_0}^{t} |\tilde{\mathbf{g}}(\tau)^2| \, d\tau \, dV.$$

Therefore, and because of (2.17),

$$\left\{c\kappa - 5M(t_n - t_0)\left[(t_n - t_0) + (t_n - t_0)^2 + (t_n - t_0)^3 - \frac{\delta(0)}{10M}\right]\right\}$$
$$\cdot \int_R \int_0^{t_n} |\tilde{\mathbf{g}}(\tau)|^2 \, d\tau \, dV < 0,$$

for $n = 1, 2, \cdots$, which implies

$$c\kappa + \frac{\delta(0)}{2}(t_n - t_0) < 5M(t_n - t_0)[(t_n - t_0) + (t_n - t_0)^2 + (t_n - t_0)^3].$$

Since $\delta(0) = c\beta(0) + \kappa\alpha(0)$ and $(t_n - t_0) \to 0+$ as $n \to \infty$, this contradicts (c). The proof is now complete.

This theorem pertains to problems of the history-value type, as evidenced by hypothesis (a). The class of boundary conditions to which the theorem applies is determined by hypothesis (b). It is clearly sufficient to prescribe on ∂R the temperature or the heat flux, or each of these on complementary subsets of ∂R. The standard boundary conditions are therefore included, although they by no means exhaust the possibilities.

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