ON SYMMETRIC-TENSOR-VALUED ISOTROPIC FUNCTIONS OF TWO SYMMETRIC TENSORS

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1. Introduction. We show that any symmetric-tensor-valued isotropic polynomial function $T(A, B)$ of two symmetric tensors $A$ and $B$ is expressible as

$$T(A, B) = (h_0 + h_1I_{10} + h_2I_{12})I + (h_3 + h_4I_{10})A + (h_5 + h_6I_{10})B$$

$$+ (h_7 + h_8I_{10})A^2 + (h_9 + h_{10}I_{10})(AB + BA) + (h_{11} + h_{12}I_{10})B^2$$

$$+ (h_13 + h_{14}I_{10})(A^2B + BA^2) + (h_{15} + h_{16}I_{10})(AB^2 + B^2A) + (h_{17} + h_{18}I_{10})(AB^2 + B^2A)$$

where $h_0, \cdots, h_{17}$ are polynomials in the isotropic invariants $I_1, \cdots, I_9$ defined by

$$I_1, \cdots, I_9 = \text{tr } A, \text{tr } B, \text{tr } A^2, \text{tr } AB, \text{tr } B^2, \text{tr } A^3, \text{tr } A^2B, \text{tr } AB^2, \text{tr } B^3$$

and where

$$I_{10} = \text{tr } A^2B^2.$$  

It has been shown by Rivlin [1] that any symmetric-tensor-valued isotropic polynomial function of the symmetric tensors $A$ and $B$ is expressible as

$$T(A, B) = \gamma_0I + \gamma_1A + \gamma_2B + \gamma_3A^2 + \gamma_4(AB + BA)$$

$$+ \gamma_5B^2 + \gamma_6(A^2B + BA^2) + \gamma_7(AB^2 + B^2A) + \gamma_8(A^2B^2 + B^2A^2)$$

where the $\gamma_k$ are polynomials in the isotropic invariants $I_1, \cdots, I_{10}$ defined by (1.2) and (1.3). There are a number of redundant terms in the expression (1.4). In Sec. 2 we outline the procedures employed to generate the matrix identities which enable us to eliminate these redundant terms and thus to proceed from the expression (1.4) for $T(A, B)$ to that defined by (1.1), (1.2), (1.3). In Sec. 3 we show that there are no redundant terms in the expression for $T(A, B)$ given by (1.1) and hence no further simplification of the expression for $T(A, B)$ is possible.

2. Reduction procedure. We may also write the expression (1.1) in the form

$$T(A, B) = \sum_{i, j, k} a_{ijk}H_{ijk}$$

where the $a_{ijk}$ are constants and where $H_{ijk}$ ($k = 1, 2, \cdots$) denote the matrices of degree $i, j$ in $A, B$ which appear in the expansion of (1.1); for example,

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1 "Tensor" means three-dimensional second-order tensor.
We now outline the procedures used to generate the matrix identities which enable us to reduce (1.4) to (1.1).

(i) Let $\alpha, \beta, \gamma$ denote $3 \times 3$ skew-symmetric matrices. Then we have [2] the identity
\[
\alpha \beta \gamma - \gamma \beta \alpha - \alpha \gamma \beta - \beta \gamma \alpha = 2 I \text{ tr } \alpha \beta \gamma.
\] (2.3)

Substitution of
\[
\alpha = A^2B - BA^2, \quad \beta = B^2A - AB^2, \quad \gamma = AB - BA
\] (2.4)
into (2.3) will yield, upon application of various of the matrix identities given in [1], a matrix identity of the form
\[
(tr A^2B^2)(A^2B^2 + B^2A^2) = \sum_p \alpha_p H_{44p}.
\] (2.5)

(ii) Let $\alpha$ and $\beta$ denote skew-symmetric $3 \times 3$ matrices and let $c$ denote a symmetric $3 \times 3$ matrix. Then we have [2] the identity
\[
\alpha \beta c + c \beta \alpha + \alpha \beta c + \beta \alpha c = (\alpha \beta - \beta \alpha) \text{ tr } c + c \text{ tr } \alpha \beta + I (2 \text{ tr } \alpha \beta c - \text{ tr } c \text{ tr } \alpha \beta).
\] (2.6)

We substitute
\[
\alpha = \beta = (A^2B - BA^2), \quad c = AB^2 + B^2A
\] (2.7)
into (2.6). The resulting identity may be reduced, upon application of (2.5) and identities found in [1], to a matrix identity of the form
\[
(tr A^2B^2)^2A = \sum_p \delta_p H_{64p}.
\] (2.8)

Interchanging $A$ and $B$ in (2.8) yields
\[
(tr A^2B^2)^2B = \sum_p \gamma_p H_{45p}.
\] (2.9)

(iii) We add the two identities obtained by multiplying the identity resulting from substitution of (2.7) into (2.6) on the right by $A$ and on the left by $A$. Upon application of identities (2.5), (2.8), and identities appearing in [1], we obtain
\[
(tr A^2B^2)^2A^2 = \sum_p \delta_p H_{44p}.
\] (2.10)

Interchanging $A$ and $B$ in (2.10) yields
\[
(tr A^2B^2)^2B^2 = \sum_p \epsilon_p H_{16p}.
\] (2.11)

(iv) We add the two identities obtained by multiplying the identity resulting from substitution of (2.7) into (2.6) on the right by $B$ and on the left by $B$. The resulting

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*It has been noted by the referee that the identities (2.3) and (2.6) are special cases of an identity given by Spencer and Rivlin [3 (see equation (4.13) on page 55)].*
identity may be reduced, upon application of (2.5), (2.8), (2.9) and identities appearing in [1], to a matrix identity of the form

\[(\text{tr } A^2B^2)^2(\text{AB} + \text{BA}) = \sum \lambda_p H_{55p} . \] (2.12)

(v) We add the two identities obtained by multiplying the identity resulting from substitution of (2.7) into (2.6) on the right by \(B^2\) and on the left by \(B^2\). The resulting identity may be reduced, upon application of (2.5), (2.8), \cdots and identities appearing in [1], to a matrix identity of the form

\[(\text{tr } A^2B^2)^2(\text{AB}^2 + B^2\text{A}) = \sum \mu_p H_{55p} . \] (2.13)

Interchanging \(A\) and \(B\) in (2.13) yields

\[(\text{tr } A^2B^2)^2(A^2\text{B} + B^2\text{A}) = \sum \nu_p H_{65p} . \] (2.14)

(vi) We multiply the identity obtained by substituting (2.4) into (2.3) on the left by \(A^2B^2\) and then take the trace of the resulting identity. This yields, upon application of identities given in [1],

\[(\text{tr } A^2B^2)^3 = \beta_0 + \beta_1 \text{tr } A^2B^2 + \beta_2 (\text{tr } A^2B^2)^2 \] (2.15)

where \(\beta_0, \cdots, \beta_2\) are polynomials in the invariants \(I_1, \cdots, I_9\) defined by (1.2).

With the aid of the identities (2.5), (2.8), \cdots, (2.15), we readily see that the expression (1.4) for \(T(A, B)\) reduces to the expression for \(T(A, B)\) defined by (1.1), (1.2) and (1.3). We note that in (2.5), (2.8), \cdots, (2.14), the \(\alpha_p, \cdots, \nu_p\) are constants and the \(H_{i,j,p}\) are matrices defined as in (2.1) and (2.2).

3. Irreducibility of (1.1). Let \(g_{mn}\) denote the number of linearly independent symmetric-tensor-valued isotropic polynomial functions of degree \(m, n\) in \(A, B\). Let \(p_{mn}\) denote the number of monomial terms of degree \(m, n\) in \(A, B\) appearing in the expression (1.1). We shall see below that \(p_{mn} = g_{mn}\) for all \(m, n\). Suppose that there are still redundant terms of degree \(m, n\) in \(A, B\) present in the expression (1.1). If we were to eliminate these, the resulting expression \(\tilde{T}(A, B)\) would contain \(\tilde{p}_{mn} < g_{mn}\) monomial terms of degree \(m, n\) in \(A, B\). This would mean that not every symmetric-tensor-valued isotropic polynomial function of degree \(m, n\) in \(A, B\) would be expressible in the form \(T(A, B)\) and hence would also not be expressible in the form (1.1). However, we have shown above that every symmetric-tensor-valued isotropic polynomial function of \(A, B\) is expressible in the form (1.1). We conclude that by showing \(p_{mn} = g_{mn}\) for all \(m, n\) we have verified that there are no redundant terms in the expression (1.1). Hence no further simplification of the expression for \(T(A, B)\) is possible. We now proceed to show that \(p_{mn} = g_{mn}\).

It may be shown from group-theoretic considerations that the number \(g_{mn}\) of linearly independent symmetric-tensor-valued isotropic polynomial functions of degree \(m\) and \(n\) respectively in the symmetric tensors \(A\) and \(B\) is given by the coefficient of \(a^m b^n\) in the expansion of the function

\[G(a, b) = (2\pi)^{-1} \int_0^{2\pi} (e^{2i\theta} + e^{i\theta} + 2 + e^{-i\theta} + e^{-2i\theta})F(a, \theta)F(b, \theta)(1 - \cos \theta) d\theta \] (3.1)

where
\[ F(a, \theta) = [(1 - ae^{2i\theta})(1 - ae^{i\theta})(1 - a)^2(1 - ae^{-i\theta})(1 - ae^{-2i\theta})]^{-1}. \] (3.2)

The integral (3.1) may be converted into a contour integral by setting \( e^{i\theta} = z \) and evaluated by the method of residues. We obtain, after a lengthy computation,

\[ G(a, b) = H(a, b)/K(a, b) \] (3.3)

where

\[ H(a, b) = 1 + a + b + a^2 + ab + b^2 + a^2b + ab^2 + 2a^2b^2 + a^3b^2 + a^2b^3 + a^4b^2 + a^3b^3 + a^2b^4 + a^5b^2 + a^4b^3 + a^6b^2, \]

\[ K(a, b) = (1 - a)(1 - a^2)(1 - a^3)(1 - ab)(1 - b)(1 - b^3)(1 - ab^2)(1 - b^3). \] (3.4)

The coefficients \( h_i \) appearing in (1.1) are polynomials in \( I_1, \cdots, I_9 \) and are expressible as

\[ h_i = h_{i_1}^{(1)}I_1^{i_1}I_2^{i_2} \cdots I_9^{i_9} \] (3.5)

where the \( h_{i_1}^{(1)}, \cdots, h_{i_9}^{(1)} \) are constants. We note that

\[ (1 + a + a^2 + \cdots)(1 + b + b^2 + \cdots) \cdots (1 + b^3 + b^6 + \cdots) \]

is equal to the sum of all of the monomial terms appearing in the expression (3.5) for \( h_i \). The number of monomial terms of degree \( m, n \) in \( A, B \) in the expression (3.6) and hence also in (3.5) is given by the coefficient of \( a^m b^n \) in the expression obtained from (3.6) by replacing \( I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9 \) by \( a, b, a^2, ab, b^2, a^3, ab, ab^2, b^3 \) respectively. This yields

\[ (1 + a + a^2 + \cdots)(1 + b + b^2 + \cdots) \cdots (1 + b^3 + b^6 + \cdots) = [K(a, b)]^{-1} \] (3.7)

where we have employed formal expansions such as \( (1 - a)^{-1} = 1 + a + a^2 + a^3 + \cdots \). We then see that the number of monomial terms of degree \( m, n \) in \( A, B \) appearing in the terms

\[ h_0 I, h_1 (\text{tr } A^2 B^2) I, h_2 (\text{tr } A^2 B^2) I, h_3 A, h_4 (\text{tr } A^2 B^2) A, \cdots \] (3.8)

is given by the coefficient of \( a^m b^n \) in the expansions of

\[ \frac{1}{K(a, b)} \frac{a^2 b^2}{K(a, b)} \frac{a^2 b^4}{K(a, b)} \frac{a}{K(a, b)} \frac{a^3 b^2}{K(a, b)} \cdots \] (3.9)

respectively. From (1.1) and (3.9), we see that the number \( p_{mn} \) of monomial terms of degree \( m, n \) in \( A, B \) contained in the expression (1.1) for \( T(A, B) \) is given by the coefficient of \( a^m b^n \) in the expansion of \( G(a, b) \) defined by (3.3) and (3.4). Since this also gives the number of linearly independent symmetric-tensor-valued isotropic polynomial functions of degree \( m, n \) in \( A, B \), we have verified that \( p_{mn} = g_{mn} \).

References