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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

—BOOK REVIEW SECTION—

Methods of intermediate problems for eigenvalues. By Alexander Weinstein and William Stenger. *Mathematics in science and engineering*, v. 89, Academic Press, New York and London, 1972, xi + 236 pp. $14.00.

This book provides a detailed introduction to the best-known methods of eigenvalue approximation using variational techniques. The first five chapters develop the basic variational inequalities, the famous Rayleigh-Ritz method, the methods of intermediate problems of the first kind (due to Weinstein) and of the second kind (due to Aronszajn), and various other related methods (due to Bayley and Fox, Weinberger, Fichera among others). Remaining chapters treat Weinstein's new maximum-minimum theory, numerous eigenvalue inequalities, and a few related results in perturbation theory. There are two appendices—one giving the basic facts about Hilbert space used in the text, the other giving tables of eigenvalue approximations, obtained using the methods of the text, for fifteen problems from mechanics.

All the results obtained apply to operators in Hilbert space—specifically, to selfadjoint operators bounded below whose spectrum has a lower part consisting of isolated eigenvalues of finite multiplicity. Only such initial eigenvalues are considered. Results and theorems are stated formally and precisely, and proofs are complete. Readers interested primarily in applications may be assured, however, that the emphasis is always on applicability of the techniques to concrete problems, and that numerous examples from physics, mechanics, and quantum theory are discussed in detail.

Interspersed among the formal results are facts and opinions about the historical development of the methods, the relative merits of various methods, and errors in the works of others (e.g., Ritz, Trefftz, Krein, Dolberg). Most of the comments are interesting and valuable, and they form an important part of the book. Occasionally, however, the authors seem to miss target. For instance, they maintain that intermediate problems are not perturbations, arguing that they are not perturbations of the *given* problem. In fact, intermediate problems are finite dimensional perturbations of the *base* problem (see Aronszajn and the reviewer, *Finite-dimensional perturbations of spectral problems*, Stud. Math. **36**, 1–74 (1970)), and many of the results common to the various approximation methods (e.g., Aronszajn's rule) can be deduced from this fact.

One important subject which is not discussed in this book is the application of the approximation methods to general differential problems. Thus the book under review cannot completely replace all other books on the subject (e.g., S. H. Gould's book on *Variational methods for eigenvalue problems*, University of Toronto Press). However, it gives an exposition of the various variational methods which is up to date, well organized, well referenced, and easily understood, and it includes material not previously published in book form. It should be useful to applied mathematicians, numerical analysts, and physicists.

ROBERT D. BROWN (*Lawrence, Kansas*)

Branching processes. By K. B. Athreya and P. E. Ney. Springer-Verlag, Berlin, Heidelberg, New York, 1972. xi + 287 pp. (Vol. 196 of *Grundlehren.*) $24.10.

Branching processes describe the growth of populations influenced by chance. The simplest branching process, named after Galton and Watson (who were to some extent anticipated by Bienaymé), is a Markov process $Z_0, Z_1, Z_2, \cdots$ with integer values ≥ 0. Taking $Z_0 = 1$ and Prob $(Z_1 = i) = p_i$, $i = 0, 1, 2, \cdots$, it is assumed that if $Z_n = k$ (k objects in the nth generation), then Z_{n+1} is distributed as the sum of k independent random variables, each distributed like Z_1. This represents independent procreation of the k objects in the nth generation. In more complex processes the objects may be of

several different types or a continuum of types (e.g. age or energy, immigration may occur, the law of procreation may vary with time, even in a random manner. However, we cannot get far from the assumption of independence if we want the full benefit of the method of generating functions. The basic fact for the Galton-Watson process (with counterparts for the related processes mentioned above) is that if $f(s) = p_0 + p_1 s + p_2 s^2 + \cdots$ is the generating function of Z_1, then the generating function of Z_n is the nth functional iterate $f_n(s) = f(f(\cdots (s) \cdots))$.

Since the reviewer's book surveyed the subject in 1963, research has been very lively and a new book is appropriate.

The present book concentrates on advances since 1962. The most noticeable features are the sharp limit theorems, including local limit theorems and individual ratio theorems, and the use of potential theory and the theory of the Martin boundary. Some recently invented processes are discussed, but for the most part the authors have concentrated on getting the best possible results about familiar processes.

Chapters 1 and 2 are on the Galton-Watson process. Letting $m = EZ_1$, the expected value of Z_1, the process is called critical, subcritical, or supercritical according as $m = 1$, $m < 1$, or $m > 1$. It has long been known that Z_n/m_n converges with probability 1 to a random variable W which is 0 if $m < 1$ (and also if $m = 1$ in non-trivial cases.) If $m > 1$ and $EZ_1^2 < \infty$, it is easy to see that $W \not\equiv 0$. As an example of the newer results, it is shown that if $m > 1$, then $W \not\equiv 0$ if and only if $E(Z_1 \log Z_1) < \infty$. However, even if $E(Z_1 \log Z_1) = \infty$, there is a norming factor C_n such that Z_n/C_n converges to a non-trivial limit.

Chapter 2 develops potential theory and the theory of the Martin boundary for branching processes. Much light is thrown on stationary measures, of which R. A. Fisher found the first example in a genetical application. Chapter 3 treats a continuous-time version of the Galton-Watson process, giving sharp forms of limit theorems. There are recent results on times of splitting and on random modification of the time scale. New results are given on the problem of embedding discrete-time processes in continuous ones. (On p. 132 it was not clear to the reviewer how we know that $F(s, t)$ is a generating function if $\hat{F}(s, t)$ is.)

Chapter 4 treats age-dependent processes, where objects have life lengths with arbitrary distributions, rather than the exponential distributions of Chapter 3. There are nonlinear integral equations for the generating function and renewal equations for the moments. Earlier results are sharpened, and new results are found through the "sub-exponential" class of life-length distributions.

Chapter 5 treats multitype processes, where the results get more complicated. There is again an appropriate definition of criticality. In the supercritical case, if the size of a family goes to infinity, the ratios of different types approach non-random limits. Again older results are refined and there are new kinds of results. For example, if $Z_i(t)$ is the number of the ith type at time t, $i = 1, \cdots, p$, there are limit theorems about certain combinations $\sum c_i Z_i(t)$ where the c_i do not all have the same sign.

The sixth and last chapter treats a number of special processes, including random walks and diffusion with branching, splitting that involves sharing of mass or energy, processes where the population size is continuous, and processes with random environments where the generating function varies from one generation to the next in a random manner.

The book does not have a systematic discussion of processes with several non-communicating types, of some of the more general age-dependent processes, and of general spaces of types. Some of these topics are treated in a book by C. J. Mode (1971), some in a book by B. A. Sevast'yanov (1971, in Russian), and some in a series of papers by N. Ikeda, M. Nagasawa, and S. Watanabe, beginning in 1965.

Within the limits they have chosen, the authors have given a well-written exposition of the many newer results, including much work of their own. I believe they make good the statement in their preface that "the subject has developed and matured significantly" in the past 10 years.

T. E. Harris (*Los Angeles*)

The genetic structure of populations. By A. Jacquard. Vol. 5 of *Biomathematics.* Springer-Verlag, New York-Heidelberg-Berlin, 1974. $39.40.

Parts 1–3 of this book are translations, by D. and B. Charlesworth of the Department of Genetics, University of Liverpool, of *Structures génétiques des Populations*, published in Paris (1970). Part 4 consists of three new chapters concerned with the study of human population structure, employing multivariate analysis and the concept of genetic distance.

In this lucid and elegant presentation of population genetics, Professor Jacquard fuses demographic and genetic concepts, illustrating it with many examples from human populations. Part 1 gives the basic facts and concepts of the Mendelian theory of inheritance and of the probabilistic tools necessary for the field. Part 2 deals with the model in which evolutionary factors other than those imposed by the genetic mechanism itself are absent: the Hardy-Weinberg principle, the equilibrium for two loci, the inheritance of quantitative characters, the genetic relationships between relatives and the structure of populations with overlapping generations. Part 3 investigates the causes of evolutionary changes in populations and thus treats models closer to the real world than does Part 2. In the five chapters of this part, each of the five assumptions of Part 2 are in turn dropped: no migration, no mutation, no selection, random mating, and no stochastic changes in the genetic composition of the population.

As Professor Lewontin points out in his introduction, the appearance of this superb treatment of the theory of evolutionary genetics in an excellent English translation is greatly to be welcomed.

WALTER FREIBERGER (*Providence, R. I.*)