SOBOLEV-TYPE LOWER BOUNDS ON $||\nabla \psi||^2$ FOR ARBITRARY REGIONS IN TWO-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. This note reports the derivation of lower bounds of the Sobolev type on $||\nabla \psi||^2 = \int_R (\frac{\partial \psi}{\partial x_1})^2 + (\frac{\partial \psi}{\partial x_2})^2) dx_1 dx_2$ for generic real scalar $\psi = \psi(x_1, x_2)$ of function class $C^0$ piecewise $C^2$ which vanish over the boundary of the (bounded or unbounded) region $R$ in Euclidean 2-space.

1. Introduction. It has been shown [1] that for all continuous real scalar functions $\phi = \phi(x_1, x_2, x_3)$ with piecewise continuous second-derivatives we have the Sobolev inequality

$$\int |\nabla \phi|^2 d^3x \geq \frac{3}{2}\left[ \int \phi^6 d^3x \right]^{1/3}$$

satisfied if $\phi$ is such that the integral on the right side of (1) is finite. The proof of (1) was given in [1] for unbounded Euclidean 3-space, but it is obvious that this Sobolev inequality is also valid if the domain of definition for $\phi$ and for the 3-dimensional integrations in (1) is any prescribed (bounded or unbounded) region, provided that $\phi$ is required to vanish over the boundary of the region. It is shown in the present note that useful lower bounds of the Sobolev type can also be established on

$$||\nabla \psi||^2 = \int_R (\frac{\partial \psi}{\partial x_1})^2 + (\frac{\partial \psi}{\partial x_2})^2) dx_1 dx_2$$

for generic real scalar $\psi = \psi(x_1, x_2)$ of function class $C^0$ piecewise $C^2$ which vanish over the boundary of the (bounded or unbounded) region $R$ in Euclidean 2-space.

2. Primary result. Let us consider an unbounded cylindrical region in 3-space that intersects the $x_1 - x_2$ plane in the 2-dimensional region $R$ and has a boundary surface generator parallel to the $x_3$ - axis. Then for $\phi = \psi \exp (-\lambda |x_3|)$ with $\psi = \psi(x_1, x_2)$ and $\lambda$ a disposable positive constant, we have $\phi = 0$ on the boundary of the cylindrical region if $\psi = 0$ on the boundary of $R$. If we introduce the notation

$$N^{(\nu)} = \int_R |\psi|^{\nu} d^2x, \quad \nu = 1, 2, 3, \cdots$$

the Sobolev inequality (1) applies to $\phi = \psi \exp (-\lambda |x_3|)$ through the unbounded cylindrical region and yields

$$\lambda^{-1} ||\nabla \psi||^2 + \lambda N^{(2)} \geq 3\left[ \int \phi^6 d^3x \right]^{1/3}$$

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1 To prove this, one simply makes an extension of the domain of definition of $\phi$ to all 3-space with $\phi = 0$ outside the region and applies the original result for unbounded Euclidean 3-space.
or equivalently

\[ \lambda^{-2/3} ||\nabla \psi||^2 + \lambda^{4/3} N^{(2)} \geq \left( \frac{\sqrt{3} \pi}{2} \right)^{4/3} [N^{(4)}]^{1/3}. \]  

The left side of (5) is minimized by putting \( \lambda = ||\nabla \psi||/[2N^{(2)}]^{1/2} \), and thus we obtain \(^2\)

\[ ||\nabla \psi||^2 \geq \frac{\pi^2}{2 \sqrt{3}} [N^{(4)}/N^{(2)}]^{1/2} \]  

for \( \psi = \psi(x_1, x_2) \) with the specified properties.

It is of interest to compare the primary Sobolev-type lower bound on \( ||\nabla \psi||^2 \) given by (6) with the linear-theoretic result for a bounded region \( R \) of finite area \( A = \int_R d^2x \), namely

\[ ||\nabla \psi||^2 \geq \frac{\pi \alpha_0^2}{A} N^{(2)} \]  

where \( \alpha_0 = .7655\pi \) is the first zero of the zero-order Bessel function, \( J_0(\alpha_0) = 0 \). Because the smallest ground-state eigenvalue is obtained for fixed area \( A \) if \( R \) is a circle of radius \( (A/\pi)^{1/2} \), the numerical coefficient on the right side of (7) follows from the Helmholtz equation eigenvalue problem associated with \( \min_R \{ ||\nabla \psi||^2/N^{(2)} \} \) for \( \psi = \psi(x_1, x_2) \) of function class \( C^0 \) piecewise \( C^2 \) in \( R \) and zero on the boundary of \( R \) (see, for example [2]). Our Sobolev-type result (6) is sharper than (7) for \( \psi \) and \( A \) such that \( [N^{(6)}]^{1/2} > (2\sqrt{3} \alpha_0^2/\pi A)[N^{(2)}]^{3/2} \); moreover, (6) applies for unbounded \( R \) (i.e., \( A = \infty \)) if \( \psi \) is such that the three integrals in (6) exist as finite quantities.

3. Alternative lower bound. Excluding from consideration a trivial \( \psi \) which vanishes identically in \( R \), the functional

\[ \Phi[\psi] = N^{(1)} [N^{(2)}]^{-1} ||\nabla \psi|| \]  

is stationary about solutions to the inhomogeneous Helmholtz equation

\[ \nabla^2 \psi + k^2 \psi = [N^{(1)}]^{-1} ||\nabla \psi||^2 \text{ sgn} (\psi) \]  

where the positive quantity \( k^2 = 2[N^{(2)}]^{-1} ||\nabla \psi||^2 \). In terms of the variable

\[ (\psi - \frac{1}{2} [N^{(1)}]^{-1} N^{(2)} \text{ sgn} (\psi)), \]

Eq. (9) reduces \textit{presque partout} to the homogeneous Helmholtz equation, and thus the established linear theory for proper vibrations of membranes [2] provides the solution to \( \min_R \{ \min_{\Phi[\psi]} \} \) for bounded regions \( R \) of fixed area \( A \). The minimum value of (8) obtains for \( \psi \) of function class \( C^0 \) piecewise \( C^2 \) in \( R \) and zero on the boundary with \( R \) a circle of radius \( r_A = (A/\pi)^{1/2} \) and \( \psi \) proportional to the nonnegative (nodeless) function

\[ \psi = J_0(kr) - J_0(\alpha_1) \approx J_0(kr) + (.4026) \]  

\[ \psi = \psi(x_1, x_2) \]

\(^1\) The somewhat sharper numerical coefficient \( \pi^{3/2}/2^{1/3}3^{1/4} \approx 2.849 \) in place of \( \pi^{3/2}/2^{1/3}3^{1/4} \approx 2.849 \) in (6) if one puts \( \phi = \psi e^{-3/4x^2} \) in place of the form \( \phi = \psi e^{-3/4x^2} \) used here. One is tempted to conjecture that \( \min_{\Phi[\psi]} \{ ||\nabla \psi||^2 \} [N^{(6)}/N^{(2)^{1/2}}] \) equals either 3 or \( \pi \), but the author has not been able to solve the associated nonlinear eigenvalue problem which yields the maximum value for the numerical coefficient in (6).

\(^3\) Along the nodal lines \( \psi = 0 \) the quantity \( \nabla^2 \text{ sgn} (\psi) \) is not defined, and continuity of the solution must be evoked.
in which \( kr_A = \alpha_1 = 1.2197\pi \) is the first positive zero of the first-order Bessel function, \( J_1(\alpha_1) = 0 \). By making use of the definite integrals (for example, \([3]\int_0^1 J_0(\alpha_1 x) x \, dx = 0 \) and \( \int_0^1 J_0(\alpha_1 x)^2 x \, dx = \int_0^1 J_1(\alpha_1 x)^2 x \, dx = \frac{1}{2} J_2(\alpha_1)^2 = \frac{1}{2} J_0(\alpha_1)^2 \)), one obtains the quantities associated with (10)

\[
N^{(1)}(\phi) = \pi r_A^2 |J_0(\alpha_1)|, \quad N^{(2)}(\phi) = 2\pi r_A^2 [J_0(\alpha_1)]^2,
\]

verifies that (10) satisfies (9) with \( k = \alpha_1/r_A \), and evaluates \( \Phi(\phi) = \frac{1}{2} \sqrt{\pi \alpha_1} \). Hence, from (8) and \( \Phi(\phi) \geq \Phi(\phi) \) we get the alternative Sobolev-type lower bound

\[
||\nabla \psi||^2 \geq \frac{\pi}{4} \alpha_1^2 [N^{(2)} / N^{(1)}]^2.
\]

Since the area of the region does not appear on the right side of (12), this result also applies for unbounded \( R \) if \( \psi \) is such that the three integrals in (12) exist as finite quantities. The equality sign in (12) holds only for a circle of finite radius and \( \psi \) proportional to \( \psi \) given by (10), thus for a \( \psi \) which also has its normal derivative equal to zero over the boundary: \( (d\psi/dr)|_{r=r_A} = 0 \). Finally, it should be observed that (12) is sharper than (6) if \([N^{(6)}]^{1/2} < (\sqrt{3} \alpha_1^2 / 2\pi) [N^{(1)}]^{-2}[N^{(2)}]^{5/2} \sim (4.07)[N^{(1)}]^{-2}[N^{(2)}]^{5/2} \), a circumstance not precluded by the general Hölder inequality for all \( \psi \), \([N^{(6)}]^{1/2} > [N^{(1)}]^{-2}[N^{(2)}]^{5/2} \).

References