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ON A FUNCTIONAL EQUATION ARISING IN THE STABILITY THEORY OF DIFFERENCE-DIFFERENTIAL EQUATIONS*

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Abstract. The functional differential equation

$$Q'(t) = AQ(t) + BQ^T(\tau - t), \qquad -\infty < t < \infty,$$

where A, B are $n \times n$ constant matrices, $\tau \geq 0$, $Q(t)$ is a differentiable $n \times n$ matrix and $Q^T(t)$ is its transpose, is studied. Existence, uniqueness and an algebraic representation of its solutions are given.

This equation, of considerable interest in its own right, arises naturally in the construction of Liapunov functionals of difference-differential equations of the type $\dot{x}(t) = Cx(t) + Dx(t - \tau)$, where C, D are constant n $\times$ n matrices. The role played by the matrix $Q(t)$ is analogous to the one played by a positive definite matrix in the construction of Liapunov functions for ordinary differential equations.

In this paper, we show that, in spite of the functional nature of this equation, the linear vector space of its solutions is n^2; moreover, we give a complete algebraic characterization of its solutions and indicate computationally simple methods for obtaining these solutions, which we illustrate through an example. Finally, we briefly indicate how to obtain solutions for the nonhomogeneous problem, through the usual variation of constants method.

1. Introduction.

The study of difference-differential equations has received considerable attention in recent times [2, 6, 7], the overwhelming interest being devoted to equations with positive delays.

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In this brief paper we wish to study the matrix functional equation

$$Q'(t) = AQ(t) + BQ^T(\tau - t), \qquad -\infty < t < \infty \tag{1.1}$$

where A, B are constant $n \times n$ matrices and $\tau \geq 0$. This equation is neither of the retarded nor advanced type. We show that, unlike the infinite dimensionality of the vector space of solutions of functional differential equations, the linear vector space of solutions of this equation is n^2. Moreover, we give a simple algebraic characterization of these n^2 linearly independent solutions which parallels the one for ordinary differential equations, indicate some methods of computation of these solutions and allude to the variation of constants formula for the nonhomogeneous problem.

This equation, of interest in its own right, is particularly important since it arises naturally in the process of constructing Liapunov functionals for retarded differential equations of the form $x'(t) = Cx(t) + Dx(t - \tau)$. Datko [4] has encountered this equation in a somewhat different form, but has not studied it. Repin [11] in his construction of Liapunov functionals uses this equation, but does so erroneously in replacing, in (1.1), the term $BQ^T(\tau - t)$ by $BQ(\tau - t)$, making the analysis trivial.

This equation has been used by Infante and Walker [9] in the construction of the Liapunov functional for a scalar difference-differential equation. The study presented here arose in the use of the solutions of this equation in a forthcoming paper [8] which treats the construction of Liapunov functionals for matrix difference-differential equations.

2. Existence, uniqueness and algebraic structure of the solutions. Consider the equation

$$Q'(t) = AQ(t) + BQ^T(\tau - t), \qquad -\infty < t < \infty, \tag{2.1}$$

with the condition

$$Q\left(\frac{\tau}{2}\right) = K, \tag{2.2}$$

where K is an arbitrary $n \times n$ matrix; this equation is intimately related to the differential equation

$$\begin{aligned} Q'(t) &= AQ(t) + BR(t), \\ R'(t) &= -Q(t)B^T - R(t)A^T, \end{aligned} \qquad -\infty < t < \infty, \tag{2.3}$$

with the initial conditions

$$Q\left(\frac{\tau}{2}\right) = K, \quad R\left(\frac{\tau}{2}\right) = K^T. \tag{2.4}$$

Moreover, for any two $n \times n$ matrices P, S, let the $n^2 \times n^2$ matrix $P \otimes S$ denote their Kronecker (or direct) product [1, 10] and introduce the notation for the $n \times n$ matrix

$$S = (s_{ij}) = \left(\frac{s_{1*}}{s_{n*}}\right) = \begin{bmatrix} s_{1*} \\ \vdots \\ s_{n*} \end{bmatrix}$$

where s_{i*} and s_{*j} are, respectively, the ith row and the jth column of S; further, let there correspond to the $n \times n$ matrix S the n^2-vector $s = (s_{1*}, \cdots, s_{n*})^T$.

With this notation [1, 10], Eqs. (2.3) and (2.4) can be rewritten as

$$\begin{bmatrix} q(t) \\ r(t) \end{bmatrix}' = \begin{bmatrix} A \otimes I & B \otimes I \\ -I \otimes B & -I \otimes A \end{bmatrix} \begin{bmatrix} q(t) \\ r(t) \end{bmatrix}, \tag{2.5}$$

and

$$q\left(\frac{\tau}{2}\right) = [k_{1*}, \cdots, k_{n*}]^T, \quad r\left(\frac{\tau}{2}\right) = [k_{*1}{}^T, \cdots, k_{*n}{}^T]^T, \tag{2.6}$$

which, with the obvious correspondence and for simplicity of notation, we denote as

$$p'(t) = Cp(t), \tag{2.7}$$

$$p(\tau/2) = p_{\tau/2}. \tag{2.8}$$

Here $p(t)$ is an $2n^2$-vector and C is a $2n^2 \times 2n^2$ constant matrix.

The use of the Kronecker product, which has allowed us to reduce (2.3)–(2.4) to (2.7)–(2.8), permits us to prove our first result.

THEOREM 1: Eq. (2.1) with condition (2.2) has a unique solution $Q(t)$ for $-\infty < t < \infty$.

Proof: If Eq. (2.1) with condition (2.2) has a differentiable solution $Q(t)$ then, defining $R(t) = Q^T(t - \tau)$, the pair of matrices $Q(t)$, $R(t)$ will satisfy Eqs. (2.3) and (2.4); hence, with the notation introduced above, the pair of vectors $q(t)$ and $r(t)$ will satisfy Eqs. (2.5), (2.6). These remarks, the linearity of all the equations involved, and the uniqueness of the solutions of (2.5)–(2.6) immediately imply that if a solution $Q(t)$ exists it is unique.

On the other hand, (2.5)–(2.6) has a unique solution defined for $-\infty < t < \infty$, and this implies the existence of a unique pair of differentiable matrices $Q(t)$ and $R(t)$ defined for $-\infty < t < \infty$ and satisfying (2.3)–(2.4). But these last equations can be rewritten as

$$\frac{d}{dt} Q(t) = AQ(t) + BR(t),$$

$$\frac{d}{dt} R(\tau - t) = AR^T(\tau - t) + BQ^T(\tau - t), \tag{2.9}$$

with initial condition

$$Q\left(\frac{\tau}{2}\right) = K = R^T\left(\frac{\tau}{2}\right), \tag{2.10}$$

from which it follows, from uniqueness, that $R(t) = Q^T(\tau - t)$, completing the proof.

Examination of the above proof makes it clear that knowledge of the solution of (2.5)–(2.6) immediately yields the solution of (2.1)–(2.2). But (2.5)–(2.6) is a standard initial-value problem in ordinary differential equations; the structure of the solutions of such problems is well known [3, 5]. Moreover, since the $2n^2 \times 2n^2$ matrix C has a very special structure, it should be possible to recover the structure of the solutions of Eq. (2.1).

Let us consider, for the moment, the solutions of Eq. (2.5). Recall [3, 5] that it has $2n^2$ linearly independent solutions which can be obtained in the following fashion. Let $\lambda_1, \cdots, \lambda_p$, $p = 2n^2$, be the distinct eigenvalues of the matrix C, that is solutions of the determinental equation

$$\det [\lambda I - C] = \det \begin{bmatrix} (\lambda I - A) \otimes I & -B \otimes I \\ T \otimes B & I \otimes (\lambda I + A) \end{bmatrix} = 0, \tag{2.11}$$

each λ_j, $j = 1, \cdots, p$ with algebraic multiplicity m_j and geometric multiplicities $n_j{}^r$,

$\sum_{r=1}^{s} n_j^r = m_j$, $\sum_j m_j = 2n^2$. Then $2n^2$ linearly independent solutions of (2.5) (or (2.7)) are given by

$$\phi_{j,r}^q(t) = \exp\left(\lambda_j\left(t - \frac{T}{2}\right)\right) \sum_{i=1}^{q} \frac{\left(t - \frac{T}{2}\right)^{q-i}}{(q-i)!} e_{j,r}^i, \tag{2.12}$$

where $q = 1, \cdots, n_j^r$, $r = 1, \cdots, s$, $\sum_{r=1}^{s} n_j^r = m_j$, $\sum_j m_j = 2n^2$, and the $2n^2$ linearly independent eigenvectors and generalized eigenvectors are given by

$$[\lambda_j I - C]e_{j,r}^i = -e_{j,r}^{i-1}, \quad e_{j,s}^0 = 0. \tag{2.13}$$

A change of notation, and a return from the vector to the matrix form, shows that $2n^2$ linearly independent solutions of (2.3) are given by

$$\begin{bmatrix} \Phi_{j,r}^q(t) \\ \Upsilon_{j,r}^q(t) \end{bmatrix} = \exp\left(\lambda_j\left(t - \frac{T}{2}\right)\right) \sum_{i=1}^{q} \left[\frac{\left(t - \frac{T}{2}\right)^{q-i}}{(q-i)!} \begin{bmatrix} L_{j,r}^i \\ M_{j,r}^i \end{bmatrix}\right], \tag{2.14}$$

for $q = 1, \cdots, n_j^r$, $r = 1, \cdots, s, j = 1, \cdots, p$, $\sum_{n=1}^{s} n_j^r = m_j$, $\sum_j m_j = 2n^2$, and where the generalized eigenmatrix pair $(L_{j,r}^i, M_{j,r}^i)$ associated with the eigenvalue λ_j satisfy the equations

$$\begin{aligned} (\lambda_j I - A)L_{j,r}^i - BM_{j,r}^i &= -L_{j,r}^{i-1}, \\ L_{j,r}^i B^T + M_{j,r}^i(\lambda_j I + A^T) &= -M_{j,r}^{i-1} \end{aligned} \tag{2.15}$$

for $i = 1, \cdots, n_j^r$, $r = 1, \cdots, s$, $L_{j,s}^0 = M_{j,s}^0 = 0$.

The structure of these equations is a most particular one; indeed, if they are multiplied by -1, transposed, and written in reverse order, they yield

$$\begin{aligned} (-\lambda_j I - A)M_{j,r}^{iT} - BL_{j,r}^{iT} &= M_{j,r}^{i-1T}, \\ M_{j,r}^{iT}B_{j,r}^{iT} + L_{j,r}^{iT}(-\lambda_j I + A^T) &= L_{j,r}^{i-lT}, \end{aligned} \tag{2.16}$$

for $i = 1, \cdots, n_j^r$, $r = 1, \cdots, s$, $L_{j,r}^{0T} = M_{j,r}^{0T} = 0$. But this result demonstrates that if λ_j is a solution of (2.11), $-\lambda_j$ will also be a solution; moreover, λ_j and $-\lambda_j$ have the same geometric multiplicities and the same algebraic multiplicity. Hence, the distinct eigenvalues always appear in pairs $(\lambda_j, -\lambda_j)$, and an examination of Eqs. (2.15) and (2.16) shows that if the generalized eigenmatrix pairs corresponding to λ_j are $(L_{j,r}^i, M_{j,r}^j)$, the generalized eigenmatrix pairs corresponding to $-\lambda_j$ will be $((-1)^{i+1}M_{j,r}^{iT}, (-1)^{i+1}L_{j,r}^{iT})$.

But these remarks imply that if the solution (2.12) corresponding to λ_j is added to the solution (2.12) corresponding to $-\lambda_j$ multiplied by $(-1)^{q+1}$, the n^2 linearly independent solutions of (2.3) given by

$$\begin{aligned} \begin{bmatrix} \Xi_{j,s}^q(t) \\ \Pi_{j,s}^q(t) \end{bmatrix} &= \exp\left(\lambda_j\left(t - \frac{T}{2}\right)\right) \sum_{i=1}^{q} \frac{\left(t - \frac{T}{2}\right)^{q-i}}{(q-i)!} \begin{bmatrix} L_{j,r}^i \\ M_{j,r}^i \end{bmatrix} \\ &+ \exp\left(-\lambda_j\left(t - \frac{T}{2}\right)\right) \sum_{i=1}^{q} \frac{\left(t - \frac{T}{2}\right)^{q-i}}{(q-i)!} (-1)^{q+i} \begin{bmatrix} M_{j,r}^{iT} \\ L_{j,r}^{iT} \end{bmatrix} \end{aligned}$$

satisfy the condition

$$\Xi_{j,r}{}^{q}\left(\frac{\tau}{2}\right) = \Pi_{j,r}{}^{q\,T}\left(\frac{\tau}{2}\right).$$

But this is precisely condition (2.4); it therefore follows that

$$\Xi_{j,r}{}^{q}(t) = \sum_{i=1}^{q} \frac{\left(t - \frac{\tau}{2}\right)^{q-i}}{(q-i)!}$$

$$\cdot\left[\exp\left(\lambda_j\left(t - \frac{\tau}{2}\right)\right) L_{j,r}{}^{i} + (-1)^{q+i} \exp\left(-\lambda_j\left(t - \frac{\tau}{2}\right)\right) M_{j,r}{}^{i\,T}\right], \quad (2.17)$$

for $q = 1, \cdots, n_j{}^r$, $\sum_{r=1}^{s} n_j{}^r = m_j$, $\sum_{2j} m_j = 2n^2$, are n^2 linearly independent solutions of (2.1). We have, therefore proven

THEOREM 2: Eq. (2.1) has n^2 linearly independent solutions given by Eq. (2.17), where the generalized eigenmatrix pairs $(L_{j,s}{}^{i}, M_{j,s}{}^{i})$ satisfy Eq. (2.17) for one of the elements of the pair $(\lambda_j, -\lambda_j)$, each of which is a solution of Eq. (2.11).

It is interesting to remark that the determinental equation (2.11), involving a $2n^2 \times 2n^2$ determinant, given the commutativity of its elements, can always be rewritten as

$$\det\left[(\lambda I - A)\otimes(\lambda I + A) + B\otimes B\right] = 0. \quad (2.18)$$

Theorem 2 gives the desired algebraic representation of the solutions, a representation which is completely analogous to that for ordinary differential equations. It is surprising that the vector space of solutions of (2.1) has dimension n^2.

3. Some further characterizations. Theorem 2 of the previous section gives a complete characterization of the solutions of our original functional equation. It is possible, however, to give some further properties of the eigenmatrix pairs in certain particular cases; these further characterizations of the eigenmatrix pairs are very useful from a computational viewpoint, as we demonstrate in the next section.

LEMMA 1: If, for a λ_j satisfying Eq. (2.18), there exists an $\alpha_j \neq 0$ and corresponding vectors x_j, y_j such that

$$[\lambda_j I - A - \alpha_j B]x_j = 0,$$

$$\left[\lambda_j I + A + \frac{1}{\alpha_j} B\right] y_j = 0, \quad (3.1)$$

then Eq. (2.15) is satisfied for $i = 1$ with $L_{j,r}{}^{1} = x_j y_j{}^T$, $M_{m,r}{}^{1} = \alpha_j L_{j,r}{}^{1}$.

Proof: A simple substitution of the indicated result into Eq. (2.15) shows that this equation is satisfied.

The proofs of the following two lemmas are equally obvious.

LEMMA 2: If, for a λ_j satisfying Eq. (2.18), there exist vectors x_j, y_j such that

$$Bx_j = 0, \quad (\lambda_j I + A)y_j = 0 \quad (3.2)$$

then Eq. (2.15) is satisfied for $i = 1$ with $L_{j,r}{}^{1} = 0$, $M_{j,r}{}^{1} = x_j y_j{}^T$.

LEMMA 3: If, for a λ_j satisfying Eq. (2.18) there exist vectors x_j, y_j such that

$$(\lambda_j I - A)x_j = 0, \; By_j = 0 \quad (3.3)$$

then Eq. (2.15) is satisfied for $i = 1$ with $L_{j,r}{}^{1} = x_j y_j{}^T$, $M_{j,r}{}^{1} = 0$.

It is noted that, if the assumptions of any of these three lemmas hold, then the form of our eigenmatrix pairs is dyadic. Moreover, determination of the x_j depends on simultaneous solutions of the determinantal equations, for a given λ_j satisfying (2.18),

$$\det[\lambda_j I - A - \alpha_j B] = 0,$$
$$\det\left[\lambda_j I + A + \frac{1}{\alpha_j} B\right] = 0 \tag{3.4}$$

where for Lemma 2 we let $1/\alpha_j = 0$ and for Lemma 3 $\alpha_j = 0$. The computations involved for the determination of α_j, x_j and y_j are much simpler than those implied by Eq. (2.15).

If B and either $(\lambda_j I - A)$ or $(\lambda_j I + A)$ are not invertible, then Lemmas 2 and 3 apply. It is not difficult to show that, if such is not the case, there always exists at least one $\alpha_j \neq 0$ such that (3.4) is satisfied, which implies the applicability of Lemma 1.

LEMMA 4: If either B is invertible or $(\lambda_j I - A)$ and $(\lambda_j I + A)$ are invertible, then there exist an $\alpha_j \neq 0$ which satisfies (2.22).

Proof: If B is invertible, for a given j, r and $i = 1$, Eq. (2.15) are equivalent to

$$B^{-1}(\lambda_j I - A)L_{j,r}{}^1(\lambda_j I + A)^T B^{T^{-1}} = -L_{j,r}{}^1$$
$$B^{-1}(\lambda_j I - A)M_{j,r}{}^1(\lambda_j I + A)^T B^{T^{-1}} = -M_{j,r}{}^1. \tag{3.5}$$

But this implies that the matrix $[B^{-1}(\lambda_j I - A)] \otimes [B^{-1}(\lambda_j I + A)]$ has at least one eigenvalue that equals -1. Since [10] the eigenvalues of a Kronecker product are the products of the eigenvalues of the two matrices appearing in the product, then there exist an $\alpha_j \neq 0$ and vectors x_j and y_j such that

$$B^{-1}(\lambda_j I - A)x_j = \alpha_j x_j,$$
$$B^{-1}(\lambda_j I + A)y_j = -\frac{1}{\alpha_j} y_j, \tag{3.6}$$

which is equivalent to the assumptions of Lemma 1.

If B is not invertible but $(\lambda_j I - A)$ and $(\lambda_j I + A)$ are both invertible, then (2.15), for $i = 1$, are equivalent to

$$(\lambda_j I - A)^{-1}B\, L_{j,r}B^T(\lambda_j I + A^T)^{-1} = -L_{j,r}{}^1$$
$$(\lambda_j I - A)^{-1}B\, M_{j,r}{}^1B^T(\lambda_j I + A^T)^{-1} = -M_{j,r}{}^1$$

and a repetition of the above argument leads to the same conclusion.

These last four Lemmas imply that, associated with each distinct eigenvalue pair $(\lambda_j, -\lambda_j)$, a solution (2.17) to our equation exists in which $L_{j,r}{}^1$ and $M_{j,r}{}^1$ are for some r dyadic and linearly dependent. It is also worth remarking that Lemmas 1-3 are not exclusive, and that more than one set (α_j, x_j, y_j) could, in some cases, be obtained from the appropriate equations, yielding a similar result for other values of r.

4. An example. Here we present, as an illustration of the computations involved, the construction of the linearly independent solutions for 2×2 systems of the form

$$Q'(t) = AQ(t) + BQ^T(\tau - t), \qquad -\infty < t < \infty \tag{4.1}$$

where, for illustrative purposes, we have chosen

$$A = \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}. \tag{4.2}$$

For the computation of the eigenvalues, the determinental equation (2.18) gives

$$\det \begin{bmatrix} (\lambda I - A)\otimes I & -B\otimes I \\ I\otimes B & I\otimes(\lambda I + A) \end{bmatrix} = \det\,[(\lambda I - A)\otimes(\lambda I + A) + B\otimes B]$$

$$= \det \begin{bmatrix} (\lambda + 5)(\lambda - 5) + 16 & 0 & 0 & 0 \\ 8 & (\lambda + 5)(\lambda - 2) & 0 & 0 \\ 8 & 0 & (\lambda + 2)(\lambda - 5) & 0 \\ 4 & 0 & 0 & (\lambda + 2)(\lambda - 2) \end{bmatrix} = 0,$$

from which we obtain the four pairs of roots $(\lambda_1, -\lambda_1) = (5, -5)$, $(\lambda_2, -\lambda_2) = (3, -3)$, $(\lambda_3, -\lambda_3) = (2, -2)$, $(\lambda_4, -\lambda_4) = (2, -2) = (\lambda_3, -\lambda_3)$.

For the first pair, note that $\lambda_1 = 5$ is an eigenvalue of $-A$ and that B is not invertible. Application of Lemma 2 (Eq. (3.2)) yields that

$$Bx_1 = 0, \quad \text{or} \quad \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} x_1 = 0,$$

$$(\lambda_1 I + A)y_1 = 0, \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} y_1 = 0;$$

hence we have

$$L_{1,1}{}^1 = 0 \quad \text{and} \quad M_{1,1}{}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1, 0],$$

yielding the solution

$$\Xi_{1,1}{}^1(t) = \exp(-5(t - 3)) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4.3}$$

The second pair, $(\lambda_2, -\lambda_2) = (3, -3)$, does not consist of eigenvalues of either A or $-A$. Hence, we search for an α such that Eqs. (3.4) are satisfied, that is

$$\det \begin{bmatrix} 3 + 5 - \alpha_2 4 & 0 \\ -2\alpha_2 & 3 + 2 \end{bmatrix} = 0$$

and

$$\det \begin{bmatrix} 3 - 5 + \dfrac{1}{\alpha_2} 4 & 0 \\ + \dfrac{1}{\alpha_2} 2 & 3 - 2 \end{bmatrix} = 0,$$

which yields the value $\alpha_2 = 2$. Eqs. (3.1) with $\lambda_j = 3$ and $\alpha_j = 2$ now yield

$$x_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \qquad y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and from our first lemma we obtain the solution

$$\Xi_{2,1}{}^{1}(t) = \exp(3(t-3))\begin{bmatrix} 5 & 5 \\ 4 & -4 \end{bmatrix} + 2\exp(-3(t-3))\begin{bmatrix} 5 & 4 \\ -5 & -4 \end{bmatrix}. \tag{4.4}$$

The last pair of eigenvalues is of algebraic multiplicity two, but it is easy to check that they are of geometric multiplicity one; hence we can attempt to treat them once again through our lemmas.

In the first case, $\lambda_3 = 2$ is an eigenvalue of $-A$. Then eqs. (3.2) become

$$B_{x3,1} = 0, \quad \text{or} \quad \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} x_{3,1} = 0,$$

$$(\lambda_3 I + A)y_{3,1} = 0, \quad \text{or} \quad \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} y_{3,1} = 0,$$

yielding

$$L_{3,1}{}^{1} = 0 \quad \text{and} \quad M_{3,1}{}^{1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0, 1]$$

and, therefore, the solution

$$\Xi_{3,1}{}^{1}(t) = \exp(-2(t-s))\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.5}$$

In the second case, $\lambda_3 = +2$ is an eigenvalue of A; still, Eq. (3.4) are satisfied for an $\alpha_3 \neq 0$; indeed, we require

$$\det[\lambda_3 I - A - \alpha_3 B] = \det\begin{bmatrix} 2 + 5 - \alpha_3 4 & 0 \\ -\alpha_3 2 & 2 + 2 \end{bmatrix} = 0,$$

$$\det\left[\lambda_3 I + A + \frac{1}{\alpha_3}B\right] = \det\begin{bmatrix} 2 - 5 + \dfrac{4}{\alpha_3} & 0 \\ \dfrac{2}{\alpha_3} & 2 - 2 \end{bmatrix} = 0,$$

which are clearly satisfied for $\alpha_3 = 7/4$. The corresponding vectors are immediately computed as

$$x_{3,2} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}, \quad y_{3,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

yielding, through Lemma 1, the solution

$$\Xi_{3,2}{}^{1}(t) = \exp(2(t-3))\begin{bmatrix} 0 & 8 \\ 0 & 7 \end{bmatrix} + \frac{7}{4}\begin{bmatrix} 0 & 0 \\ 8 & 7 \end{bmatrix}\exp(-2(t-3)). \tag{4.6}$$

It is easily seen that these four solutions are linearly independent.

5. The nonhomogeneous problem. We briefly consider the nonhomogeneous problem

$$Q'(t) = AQ(t) + BQ^{T}(\tau - t) + F(t), \qquad -\infty < t < \infty \tag{5.1}$$

where A, B are constant $n \times n$ matrices and $F(t)$ is a continuous $n \times n$ matrix. We seek a particular solution of this problem.

For simplicity of notation, let the n^2 linearly independent solutions given by (2.17) be relabeled as $Z_k(t) = (z_{i,j}{}^k(t))$, $k = 1, \cdots, n^2$. Then, in a manner completely analogous to that for ordinary differential equations, we obtain

THEOREM 3: Eq. (5.1) has a particular solution given by

$$Q(t) = \sum_{k=1}^{n^2} \left(\int_{T/2}^{t} r_k(s)\, ds \right) Z^k(t), \tag{5.2}$$

where the $r_k(s)$ are given below.

Proof: Substitution of (5.2) into (5.1), given that the $Z^k(t)$ are solutions of the homogeneous equation, yields that the $r_k(s)$ must satisfy the equation

$$\sum_{k=1}^{n^2} r_k(t) Z^k(t) = F(t). \tag{5.3}$$

In the notation of Sec. 2, define the n^2-vectors $\zeta_k(t) = (z(t)_{1*}{}^k, \cdots, z_{n*}{}^k(t))^T$, $r(t) = (r_1(t), \cdots, r_2(t)^T, f(t) = (f_{1*}, \cdots, f_{n*})^T$, and the $n^2 \times n^2$ matrix $\Theta(t) = (\zeta_1, \cdots, \zeta_{n^2})$. Then (5.3) is equivalent to

$$\Theta(t) r(t) = f(t).$$

$\Theta(t)$ is clearly nonvanishing for all t, given the linear independence of the solutions $Z^k(t)$; hence

$$r(t) = \Theta^{-1}(t) f(t)$$

and this concludes the proof.

REFERENCES

[1] R. Bellman, *Introduction to matrix theory,* McGraw Hill, 1960

[2] R. Bellman, and K. Cooke, *Differential difference equations,* Academic Press, 1963

[3] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations,* McGraw-Hill, New York, 1955

[4] R. Datko, *An algorithm for computing Liapunov functionals for some differential difference equations,* in *Ordinary differential equations,* 1971 NRL-MRC Conference, Academic Press, 1972, p. 387–398

[5] J. K. Hale, *Ordinary differential equations,* Wiley-Interscience, 1969

[6] J. K. Hale, *Functional differential equations,* Appl. Math. Science Series, Springer-Verlag, 1971

[7] J. K. Hale, *Theory of functional differential equations,* Appl. Math. Science Series, Springer-Verlag, 1977

[8] E. F. Infante and W. B. Castelan, *A Liapunov functional for a matrix differential difference equation,* to appear

[9] E. F. Infante and J. A. Walker, *A Liapunov functional for a scalar differential difference equation,* Proc. Roy. Soc. Edinburgh (to appear)

[10] P. Lancaster, *Theory of matrices,* Academic Press, 1969

[11] I. M. Repin, *Quadratic Liapunov functionals for systems with delays,* Prikl. Math. Mech. **29**, 564–566 (1965)