ON THE MATERIAL TIME DERIVATIVE OF ARBITRARY ORDER*

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Abstract. The material time derivative of arbitrary order of a function of space and time is established. In the case of a function of a single space variable, it is reduced to Faa de Bruno’s formula for derivatives of a function of a function.

This note is concerned with the material time derivative of arbitrary order of a function of space and time. These derivatives, in addition to their own merits, are needed in problems with non-stationary boundaries [1, 2]. The result is quite different from the Rivlin-Erickson tensors for material time derivatives of the square of an infinitesimal arc length.

Let \( f = f(x_1, x_2, x_3, t) \) and \( x_i = x_i(t) \). Using a multinomial expansion, we have the Taylor’s series

\[
f = \sum \sum \sum \partial_t^{m_0} \partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} f \bigg|_0 h^{m_0} k_1^{m_1} k_2^{m_2} k_3^{m_3} (m_0! m_1! m_2! m_3!)
\]

where \( \partial_t = \partial / \partial t, \partial x_i = \partial / \partial x_i, h = t - t_0 \) and \( k_i = x_i - x_i(t_0) \). The Taylor’s series of \( k_i^{m_i} \) is

\[
k_i^{m_i} = \left( \sum_{r_i} D^p x_i(t_0) h^p / p! \right)^{m_i} = \sum_{r_i = m_i} m_i! A_{m_i} r_i(x_i) |_0 h^{r_i}
\]

where \( D = d/dt \). The coefficient of \( h^{r_i} \) in this series is

\[
m_i! A_{m_i} r_i(x_i) |_0 = \sum_{\alpha_k} \frac{m_i!}{\alpha_1! \alpha_2! \cdots \alpha_{r_i}!} (D x_1 |_0)^{\alpha_1} (D^2 x_1 |_0 / 2!)^{\alpha_2} \cdots (D^{r_i} x_1 |_0 / r_i!)^{\alpha_{r_i}}.
\]

The sum is extended to all values of \( \alpha_k = 0, 1, 2, \cdots \) which satisfy

\[
\alpha_1 + \alpha_2 + \cdots + \alpha_{r_i} = m_i,
\]

\[
\alpha_1 + 2\alpha_2 + \cdots + r_i \alpha_{r_i} = r_i.
\]

Similarly, we obtain

\[
k_2^{m_2} = \sum_{r_2 = m_2} m_2! A_{m_2} r_2(x_2) |_0 h^{r_2}, \quad k_3^{m_3} = \sum_{r_3 = m_3} m_3! A_{m_3} r_3(x_3) |_0 h^{r_3}
\]

where

\[
A_{m_k} r_k(x_k) |_0 = \sum_{\beta_k} \prod_{i=1}^{r_k} \left[ (D^k x_k |_0 / k! \beta_k) / \beta_k! \right],
\]

\[
A_{m_3} r_3(x_3) |_0 = \sum_{\gamma_k} \prod_{i=1}^{r_3} \left[ (D^k x_3 |_0 / k! \gamma_k) / \gamma_k! \right]
\]

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subject to
\[ \sum \beta_k = m_2, \quad \sum k\beta_k = r_2, \]  
\[ \sum \gamma_k = m_3, \quad \sum k\gamma_k = r_3. \]
Substitutions of (2) and (5) into (1) yield
\[ f = \sum_{m_0=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} A_{m_1}r_1(x_1)_0 A_{m_2}r_2(x_2)_0 A_{m_3}r_3(x_3)_0 \]  
\[ \partial_t^{m_0} \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \partial_{x_3}^{m_3} f |_{A^N/m_0^N}. \]  
where \( N = m_0 + r_1 + r_2 + r_3 \). A change of the order of summations yields
\[ f = \sum_{m_0=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{m_1=0}^{r_1} \sum_{m_2=0}^{r_2} \sum_{m_3=0}^{r_3} \cdots \]
The function \( f \) can also be written as
\[ f = \sum_{N=0}^{\infty} \frac{D^N f|_0}{h^N/N!}. \]
Matching the coefficients of equal powers of \( h \) and using the fact that they are valid for arbitrary \( t_0 \), we obtain
\[ D^N f = \sum_{m_0=0}^{N} \sum_{r_1=0}^{N-m_0} \sum_{r_2=0}^{n-m_1} \sum_{m_1=0}^{r_1} \sum_{m_2=0}^{r_2} \sum_{m_3=0}^{r_3} \frac{N!}{m_0!} \]  
\[ A_{m_1}r_1(x_1) A_{m_2}r_2(x_2) A_{m_3}r_3(x_3) \partial_t^{m_0} \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \partial_{x_3}^{m_3} f \]  
where \( r_3 = N - m_0 - r_1 - r_2 \). This is the required result.
In the one-dimensional case as needed in [1, 2], this result is reduced to
\[ D^N f = \sum_{n=0}^{N} \sum_{m=0}^{N-n} (N!/n!) A_m^{N-n}(x) \partial_t^n \partial_x^m f. \]  
Also, if \( f \) does not depend on \( t \) explicitly, then \( n = 0 \). It is further reduced to Faa de Bruno's formula of derivatives of a function of a function [3],
\[ D^N f = \sum_{m=0}^{N} N! A_m^{N}(x) d^m f / dx^m. \]

References