—NOTES—

ON THE MATERIAL TIME DERIVATIVE OF ARBITRARY ORDER*

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Abstract. The material time derivative of arbitrary order of a function of space and time is established. In the case of a function of a single space variable, it is reduced to Faa de Bruno's formula for derivatives of a function of a function.

This note is concerned with the material time derivative of arbitrary order of a function of space and time. These derivatives, in addition to their own merits, are needed in problems with non-stationary boundaries [1, 2]. The result is quite different from the Rivlin-Ericksen tensors for material time derivatives of the square of an infinitesimal arc length.

Let $f = f(x_1, x_2, x_3, t)$ and $x_i = x_i(t)$. Using a multinomial expansion, we have the Taylor's series

$$f = \sum\sum_{m_i=0}^{\infty}\sum\sum \partial_t^{m_0}\partial_{x_1}^{m_1}\partial_{x_2}^{m_2}\partial_{x_3}^{m_3} f|_0 h^{m_0}k_1^{m_1}k_2^{m_2}k_3^{m_3}/(m_0!m_1!m_2!m_3!) \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_{x_i} \equiv \partial/\partial x_i$, $h = t - t_0$ and $k_i = x_i - x_i(t_0)$. The Taylor's series of $k_1^{m_1}$ is

$$k_1^{m_1} = \left(\sum_1^{\infty} D^p x_1(t_0)h^p/p!\right)^{m_1} = \sum_{r_1=m_1}^{\infty} m_1!\, A_{m_1}^{r_1}(x_1)|_0 h^{r_1} \tag{2}$$

where $D \equiv d/dt$. The coefficient of h^{r_1} in this series is

$$m_1!\, A_{m_1}^{r_1}(x_i)|_0 = \sum_{\alpha_k} \frac{m_1!}{\alpha_1!\,\alpha_2!\cdots\alpha_{r_1}!}(Dx_1|_0)^{\alpha_1}(D^2x_1|_0/2!)^{\alpha_2}\cdots(D^{r_1}x_1|_0/r_1!)\alpha_{r_1}. \tag{3}$$

The sum is extended to all values of $\alpha_k = 0, 1, 2, \cdots$ which satisfy

$$\begin{aligned} \alpha_1 + \alpha_2 + \cdots + \alpha_{r_1} &= m_1, \\ \alpha_1 + 2\alpha_2 + \cdots + r_1\alpha_{r_1} &= r_1. \end{aligned} \tag{4}$$

Similarly, we obtain

$$k_2^{m_2} = \sum_{r_2=m_2}^{\infty} m_2!\, A_{m_2}^{r_2}(x_2)|_0 h^{r_2}, \quad k_3^{m_3} = \sum_{r_3=m_3}^{\infty} m_3!\, A_{m_3}^{r_3}(x_3)|_0 h^{r_3} \tag{5}$$

where

$$\begin{aligned} A_{m_2}^{r_2}(x_2)|_0 &= \sum_{\beta_k}\prod_1^{r_2}[(D^k x_2|_0/k!)^{\beta_k}/\beta_k!], \\ A_{m_3}^{r_3}(x_3)|_0 &= \sum_{\gamma_k}\prod_1^{r_3}[(D^k x_3|_0/k!)^{\gamma_k}/\gamma_k!] \end{aligned} \tag{6}$$

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subject to

$$\sum \beta_k = m_2, \qquad \sum k\beta_k = r_2 , \qquad \sum \gamma_k = m_3 , \qquad \sum k\gamma_k = r_3 . \tag{7}$$

Substitutions of (2) and (5) into (1) yield

$$f = \sum_{m_0=0}^{\infty} \sum \sum_{m_i=0}^{\infty} \sum \sum \sum_{r_i=m_i}^{\infty} \sum A_{m_1}{}^{r_1}(x_1)|_0 \, A_{m_2}{}^{r_2}(x_2)|_0 A_{m_3}{}^{r_3}(x_3)|_0 \, \partial_t{}^{m_0}\partial_{x_1}{}^{m_1}\partial_{x_2}{}^{m_2}\partial_{x_3}{}^{m_3}f|_0 h^N/m_0! \tag{8}$$

where $N = m_0 + r_1 + r_2 + r_3$. A change of the order of summations yields

$$f = \sum_{m_0=0}^{\infty} \sum \sum_{r_i=0}^{\infty} \sum \sum \sum_{m_i=0}^{r_i} \sum \cdots \tag{9}$$

The function f can also be written as

$$f = \sum_{N=0}^{\infty} D^N f|_0 \, h^N/N!. \tag{10}$$

Matching the coefficients of equal powers of h and using the fact that they are valid for arbitrary t_0 , we obtain

$$D^N f = \sum_{m_0=0}^{N} \sum_{r_1=0}^{N-m_0} \sum_{r_2=0}^{N-m_0-r_1} \sum \sum_{m_i=0}^{r_i} \sum \frac{N!}{m_0!} A_{m_1}{}^{r_1}(x_1) A_{m_2}{}^{r_2}(x_2) A_{m_3}{}^{r_3}(x_3) \;\; \partial_t{}^{m_0}\partial_{x_1}{}^{m_1}\partial_{x_2}{}^{m_2}\partial_{x_3}{}^{m_3}f \tag{11}$$

where $r_3 = N - m_0 - r_1 - r_2$. This is the required result.

In the one-dimensional case as needed in [1, 2], this result is reduced to

$$D^N f = \sum_{n=0}^{N} \sum_{m=0}^{N-n} (N!/n!) \, A_m{}^{N-n}(x) \; \partial_t{}^n \partial_x{}^m f. \tag{12}$$

Also, if f does not depend on t explicitly, then $n = 0$. It is further reduced to Faa de Bruno's formula of derivatives of a function of a function [3],

$$D^N f = \sum_{0}^{N} N! \, A_m{}^N(x) \, d^m f/dx^m. \tag{13}$$

References

[1] D. D. Joseph and R. L. Fosdick, *The free surface on a liquid between cylinders rotating at different speeds—I*, Arch. Rat. Mech. Anal. **49**, 321–380 (1973)

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[3] C. Jordan, *Calculus of finite differences*, Chelsea, 1965