A TEMPERATURE EQUATION FOR A RIGID HEAT CONDUCTOR WITH MEMORY*

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1. Introduction. The following system arises in a description for the linearized theory of heat flow in a rigid isotropic homogeneous rod consisting of material with thermal memory (cf. [1]–[3]):

$$
\begin{aligned}
e(t, x) &= e_0 + \alpha(0)u(t, x) + \int_{-\infty}^{t} \alpha'(t-s)u(s, x)\,ds, \\
q(t, x) &= -k(0)u_x(t, x) - \int_{-\infty}^{t} k'(t-s)u_x(s, x)\,ds, \\
e_t(t, x) &= -q_x(t, x) + r(t, x),\ t \geq 0,\ 0 \leq x \leq \pi.
\end{aligned}
\tag{1}
$$

Here $u(t, x)$ is the temperature, $e(t, x)$ the internal energy, $q(t, x)$ the heat flux, and $r(t, x)$ represents heat supplied to the rod from external sources. Our aim is to give conditions for boundedness and asymptotic behavior of $u(t, x)$, given $u(t, x)$ for $t \leq 0$, $0 \leq x \leq \pi$, and satisfying the boundary conditions $u(t, 0) = u(t, \pi) = 0$, $t \geq 0$.

Miller in [4] discusses a more general case of (1) where $x = (x_1, \ldots, x_n) \in R^n$ and $\partial/\partial x$ is replaced by the gradient operator ∇. He obtains existence theorems for $u(t, x)$, $t \geq 0$, $x \in B$, an open subset of R^n, where boundary conditions like $u(t, x) = 0$ for all $t \geq 0$ and $x \in \partial B$, the boundary of B, are imposed. He also obtains conditions for the stability and asymptotic stability of the trivial solution in case $r(t, x) \equiv 0$. His results are in terms of three types of solutions: distribution solutions, generalized distribution solutions, and classical solutions. His methods use semigroup theory, and standard results for Volterra integrodifferential equations, using Laplace transform criteria for stability and asymptotic stability.

We propose to study an integrodifferential equation for $u(t, x)$ equivalent to (1) with $\alpha(0) > 0$ and $k(0) \geq 0$, by studying, as Miller does, the equations satisfied by the coefficients of the Fourier series for $u(t, x)$, in our case, a sine series, but instead of using Laplace transform techniques exclusively we use some results obtained in [5]. For simplicity, we also confine our study to classical solutions, in the sense of Miller [4], but similar results for distribution solutions can easily be obtained. In addition to conditions for stability and asymptotic stability for the trivial solution in case $r(t, x) \equiv 0$, we also obtain conditions for asymptotic periodicity and asymptotic almost-periodicity in case $r(t, x)$ is asymptotically periodic or asymptotically almost-periodic, again using results in [5].

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2. An example of instability. The following example shows that for sufficiently strong and large memory effects, the linear model (1) can be unstable in the sense that solutions near the trivial one for $r(t, x) \equiv 0$ can become unbounded as t increases. Take $\alpha(t) \equiv 1$, $k(0) \equiv k_1 \geq 0$, T a positive constant, $k'(t) = T^{-2} t \exp(t/T)$, and $r(t, x) \equiv 0$. Then using (1) we easily obtain

$$u_t(t, x) = k_1 u_{xx}(t, x) + \int_{-\infty}^{t} k'(t - s) u_{xx}(s, x)\, ds. \tag{2}$$

This equation has a solution of the form $u(t)\sin x$, where

$$u'(t) = -k_1 u(t) - \int_{-\infty}^{t} k'(t - s) u(s)\, ds. \tag{3}$$

If we replace t by Tt and $u(Tt)$ by $u(t)$, (3) becomes

$$u'(t) = -T\left(k_1 u(t) + \int_{-\infty}^{t} (t - s)\exp(t - s) u(s)\, ds\right). \tag{4}$$

If we put

$$v(t) = \int_{-\infty}^{t} (t - s)\exp(t - s) u(s)\, ds,$$

then

$$v''(t) + 2v'(t) + v(t) = u(t),$$

and a simple calculation shows that $v(t)$ satisfies

$$v''' + (Tk_1 + 2)v'' + (2Tk_1 + 1)v' + T(k_1 + 1)v = 0. \tag{5}$$

Hence for any complex λ satisfying

$$\lambda^3 + (Tk_1 + 2)\lambda^2 + (2Tk_1 + 1)\lambda + T(k_1 + 1) = 0, \tag{6}$$

$v(t) = v_0 \exp(\lambda t)$ is a solution of (5). Using the Hurwitz criterion, (6) will have a root λ with positive real part if

$$D_2 = \det\begin{pmatrix} Tk_1 + 2 & T(k_1 + 1) \\ 1 & 2Tk_1 + 1 \end{pmatrix} < 0. \tag{7}$$

It follows by straightforward calculations that if $0 < k_1 < 1/8$ and $T_1 < T < T_2$, where

$$T_1 = (1 - 4k_1 - (1 - 8k_1)^{1/2})/4k_1^2,$$
$$T_2 = (1 - 4k_1 + (1 - 8k_1)^{1/2})/4k_1^2,$$

(7) holds. We conclude for such k_1 and T there exists for any $u_0 \neq 0$ a solution $u(t)$ of (4) of the form $u(t) = u_0 p(t)\exp(at)$ where $a > 0$ and $p(t)$ is periodic in t and not identically zero. Hence our assertion is verified.

We note that if $k(0) = k_1 = 0$, a similar analysis shows that if $T > 2$ a similar result for (2) can be obtained; in fact, it is easy to verify that as $k_1 \to 0+$, $T_1 \to 2$ and $T_2 \to +\infty$.

3. The temperature equation. We now assume that $k(0) > 0$. In this case it can be shown that $u(t, x)$ satisfies (1) if and only if it satisfies

$$u_t(t, x) = F(t, x) + cu_{xx}(t, x) + y(0)u(t, x) + \int_0^t y'(t-s)u(s, x)\, ds \tag{8}$$

for $t \geq 0$, $0 \leq x \leq \pi$; here $c = k(0)/\alpha(0)$, $y(t)$ satisfies

$$y(t) = b(t) - a(t) - \int_0^t b(t-s)y(s)\, ds$$

with $b(t) = k'(t)/k(0)$, $a(t) = \alpha'(t)/\alpha(0)$, and $F(t, x)$ depends on $r(t, x)$ and functionally on the given initial values of $u(t, x)$ and $u_{xx}(t, x)$, $t < 0$; we refer to Lemma 3 in [4] for details; we also refer to the proof of Corollary 1 of our paper for an explicit definition of F. Note that $u(0, x)$ is specified.

Following Miller, by a classical solution $u(t, x)$ of (8) we mean a function continuous for $(t, x) \in [0, \infty) \times [0, \pi]$ such that u_t and u_{xx} are continuous on $(0, \infty) \times (0, \pi)$, $u(t, x)$ satisfies (8) on $(0, \infty) \times (0, \pi)$, $u(t, 0) = u(t, \pi) = 0$ for $t \geq 0$, and $u(0, x) = u_0(x)$ is a given function continuous on $[0, \pi]$.

The following conditions are sufficient for the existence of a unique classical solution of (8); cf. Theorem 10 in [4]:

$\alpha(0) > 0, \qquad k(0) > 0;$ (9.1)

$\alpha'''(t)$ and $k''(t)$ are continuous for $t \geq 0$; (9.2)

$\alpha'(t)$, $\alpha''(t)$, and $k'(t)$ are integrable on $[0, \infty)$; (9.3)

the given initial temperature $u(t, x)$ and $u_{xx}(t, x)$ are continuous on $(-\infty, 0) \times (0, \pi)$, $u(0, x)$ is continuous on $[0, \pi]$, and $u(0, \pi) = u(0, 0) = 0$; (9.4)

the functions $r(t, x)$ and

$$\int_{-\infty}^0 [k'(t-s)u_{xx}(s, x) - a''(t-s)u(s, x)]\, ds$$

are locally Hölder-continuous on $[0, \infty) \times (0, \pi)$. (9.5)

We also will need an additional condition which will guarantee the integrability of a certain so-called resolvent kernel which will be needed:

$$k(0) + \int_0^\infty k'(t)\exp(-zt)\, dt \neq 0 \text{ for all complex } z \tag{9.6}$$

with nonnegative real parts.

Clearly a classical solution $u(t, x)$ can be expanded in a convergent Fourier sine series $\sum_{j=1}^\infty u_j(t)\sin jx$ for $t > 0$, $0 \leq x \leq \pi$; we call $u_j(t)$ the mode of the solution corresponding to the frequency j. It follows easily that these modes $u_j(t)$ satisfy

$$u_j'(t) = F_j(t) - (cj^2 + d)u_j(t) + \int_0^t B(t-s)u_j(s)\, ds, \qquad j = 1, 2, \ldots \tag{10j}$$

for $t > 0$; here $d = -y(0)$, $B(t) = y'(t)$ and

$$F_j(t) = (2/\pi) \int_0^\pi F(t, x) \sin jx \, dx.$$

THEOREM 1. Let (9.1)–(9.6) hold; suppose $F(t, x)$ is bounded on $[0, \infty) \times [0, \pi]$, and that there exists a positive integer j_0 such that

(i) $F_j(t) = 0$ for $j < j_0$, and all $t \geq 0$;

(ii) $\int_0^\infty |y'(t)| dt < cj_0^2 + d$; and

(iii) $F_j(t) \to 0$ as $t \to \infty$ for $j \geq j_0$.

Then if $u(t, x)$ is a solution of (8) such that $u_j(0) = 0$ for $j < j_0$, it follows that $u(t, x) \to 0$ as $t \to \infty$. If $j_0 = 1$, the conditions involving $j < j_0$ are here and in the sequel omitted.

Proof. We first use Theorem 4 in [5] on (10*j*) for $j \geq j_0$. Condition (H1) of this paper holds since $B(t) = y'(x)$ is continuous. The fact that $y'(t)$ is integrable on $[0, \infty)$ follows because of (9.6) and our assumption that $b'(t) = k''(t)/k(0)$ is integrable on $[0, \infty)$; cf. the remarks on p. 327 in [4]. Condition (H2) in [5] for $j \geq j_0$ follows from (ii) above. We conclude that for any solution u_j of (10j) with $j \geq j_0$ there exists a constant M_j such that $|u_j(t)| \leq M_j$, $t \geq 0$.

We show next that for $j_1 \geq j_0$ and sufficiently large, there exists $\bar{M}$ such that $M_j \leq \bar{M}$ for $j \geq j_1$. To this end, we write (10j) in integrated form with $\omega_j = cj^2 + d$:

$$u_j(t) = u_j(0)\exp(-\omega_j t) + \int_0^t \exp[-\omega_j(t - \tau)]F_j(\tau) \, d\tau$$
$$+ \int_0^t \exp[-\omega_j(t - \tau)] \int_0^\tau B(\tau - s)u_j(s) \, ds \, d\tau, \ t \geq 0. \tag{10.1j}$$

If $|F(t, x)| \leq B_1$ on $[0, \infty) \times [0, \pi]$ and $\int_0^\infty |B(t)| dt = B_2$, it follows that

$$M_j \leq |u_j(0)| + 2B_1/\omega_j + B_2 M_j/\omega_j \quad \text{for} \quad j \geq j_0, \tag{10.2j}$$

and since $\{|u_j(0)| : j = 1, 2, \ldots\}$ is bounded, it follows that for $j \geq j_1 \geq j_0$, j_1 sufficiently large, $M_j \leq \bar{M}$ for some constant $\bar{M}$, as asserted.

We show next that there exists a constant B such that $|u_j(t)| \leq B/\omega_j$ for $j \geq j_1$ and $t \geq 1$. But this follows easily again from (10.1j) since we now can get an estimate such as (10.2*j*) with M_j replaced by $\bar{M}$ for $j \geq j_1$, and use $|u_j(0)| \exp(-\omega_h t) \leq B_0/\omega_j$ for $t \geq 1$.

We show finally that for each $j = 1, 2, \ldots$, the solution $u_j(t)$ of (10j) satisfies $u_j(t) \to 0$ as $t \to \infty$. For $j < j_0$ the solution $u_j(t)$ is clearly identically zero for $t \geq 0$. For $j \geq j_0$ we use Theorem 5 in [5]; condition (6) in this theorem follows easily from the integrability of $y'(t)$ on $[0, \infty)$. Thus $u_j(t) \to 0$ for $t \to \infty$, $j = 1, 2, \ldots$. But the estimate $|u_j(t) \leq B/\omega_j$ for $j \geq j_1$ and $t \geq 1$ shows that $u(t, x) = \sum_{j=1}^\infty u_j(t) \sin jx \to 0$ since it shows the series to be convergent uniformly on $[1, \infty) \times [0, \pi]$. Q.E.D.

It is interesting to note that if $r(t, x) \equiv 0$ and each solution $u(t, x)$ of (1) with $u(t, x)$ and $u_{xx}(t, x)$ bounded on $(-\infty, 0] \times [0, \pi]$ remains bounded for $t \geq 0$, then (9.6) holds for all z with positive real part; cf. Theorem 5 in [4].

In what follows, we always assume $0 \leq x \leq \pi$.

COROLLARY 1. Let (9.1)–(9.6) hold and $r(t, x) = 0$, $t \geq 0$. If the given $u(t, x)$ and $u_{xx}(t, x)$ are continuous and bounded on $(-\infty, 0] \times [0, \pi]$, $u(t, 0) = u(t, \pi) = 0$, and

$$u_j(t) = (2/\pi) \int_0^\pi u(t, x) \sin jx \, dx = 0$$

for all $t \leq 0$ and $j < j_0$, and j_0 is sufficiently large, then $u(t, x) \to 0$ as $t \to \infty$.

Proof. It can be shown that

$$F(t, x) = f(t, x) - \int_0^t D(t-s) f(s, x) \, ds - D(t) u(0, x) \tag{11}$$

where

$$f(t, x) = \left[r(t, x) + \int_{-\infty}^0 k'(t-s) u_{xx}(s, x) \, ds - \int_{-\infty}^0 \alpha''(t-s) u(s, x) \, ds \right] \Big/ \alpha(0)$$

and $D(t) = b(t) - \int_0^t b(t-s) D(s) \, ds$; cf. [4] for details.

If $j < j_0$, then $u_j(t) = 0$ for $t \leq 0$, and it follows by integration by parts that

$$\int_0^\pi u_{xx}(t, x) \sin jx \, dx = 0 \quad \text{for} \quad t \leq 0.$$

Hence

$$\int_0^\pi f(t, x) \sin jx \, dx = 0 \quad \text{for} \quad t \leq 0,$$

and it follows that

$$F_j(t) = (2/\pi) \int_0^\pi F(t, x) \sin jx \, dx = 0 \quad \text{for} \quad t \leq 0.$$

Thus (i) of Theorem 1 holds, and clearly (ii) also does if j_0 is sufficiently large. To show that (iii) holds we first observe that

$$f(t, x) = \int_t^\infty k'(s) u_{xx}(t-s, s) \, ds - \int_t^\infty \alpha''(s) u(t-s, x) \, ds, \qquad t \geq 0,$$

and from the boundedness of u and u_{xx} for $t \leq 0$ and the integrability of k' and α'' we conclude that $f(t, x) \to 0$ as $t \to \infty$.

Since (9.6) holds, it follows that $D(t)$ is integrable on $[0, \infty)$ (cf. the remark on p. 327 in [4]) and, by a simple argument which we omit, it follows from (11) that $F(t, x) \to 0$ as $t \to \infty$, from which (iii) follows. Q.E.D.

A real-valued function $f(t)$ continuous on $[0, \infty)$ is said to be asymptotically T-periodic if there exists a continuous function $g(t)$ such that $g(t + T) = g(t)$ for all t, and $f(t) - g(t) \to 0$ as $t \to \infty$.

THEOREM 2. Let $r(t, x)$ be asymptotically T-periodic for each x, $0 \leq x \leq \pi$. Let the initial $u(t, x)$ and $u_{xx}(t, x)$ be bounded on $(-\infty, 0] \times [0, \pi]$. Let (9.1)–(9.6) and (ii) of Theorem 1 hold for $j = 1$. Then each solution $u(t, x)$ of (8) is asymptotically T-periodic for each $x \in [0, \pi]$.

Proof. We use Theorem 7 in [5]. Since the additional condition (9) in this theorem holds trivially for our $B(t)$ in each (10j), it follows that each $u_j(t)$ is asymptotically T-periodic, provided each $F_j(t)$ is. To show this, we first observe that $f(t, x) = r_0(t, x) + r_1(t, x)$ where $r_0(t, x) = r(t, x)/\alpha(0)$, and

$$r_1(t, x) = \left[\int_t^\infty k'(s)u_{xx}(t-s, x)\,ds = \int_t^\infty \alpha''(s)u(t-s, x)\,ds\right]\Big/\alpha(0)$$

for $t \geq 0$, $x \in [0, \pi]$; cf. the proof of the corollary. Since k' and α'' are integrable on $[0, \infty)$, it follows easily that $r_1(t, x) \to 0$ as $t \to \infty$.

From the form of F given in (11) it follows that

$$F(t, x) = r_0(t, x) - \int_0^t D(t-s)r_0(s, x)\,ds + G(t, x). \tag{12}$$

But $G(t, x) \to 0$ as $t \to \infty$ which follows easily since $D(t)$ is integrable on $[0, \infty)$; cf. the proof of our corollary. Since $r_0(t, x)$ is asymptotically T-periodic, there exist functions $p(t, x)$ and $q(t, x)$ such that $r_0 = p + q$, p is T-periodic in t, and $q \to 0$ as $t \to \infty$. Using (12) and an argument as above, it follows easily that

$$F(t, x) = p(t, x) - \int_0^t D(t-s)p(s, x)\,ds + G_1(t, x) \tag{13}$$

where $G_1 \to 0$ as $t \to \infty$. But

$$\begin{aligned}\int_0^t D(t-s)p(s, x)\,ds &= \int_{-\infty}^t D(t-s)p(s, x)\,ds - \int_{-\infty}^0 D(t-s)p(s, x)\,ds \\ &= \int_0^\infty D(s)p(t-s, x)\,ds - \int_t^\infty D(s)p(t-s, x)\,ds.\end{aligned}$$

The first integral on the right is clearly T-periodic while the second approaches zero as $t \to \infty$. This shows that F is asymptotically T-periodic in t and, as asserted, it follows that each $F_j(t)$ is also.

It remains to show that if $u_j(t)$ is asymptotically T-periodic then $\sum_{j=1}^\infty u_j(t)\sin jx$ is. This follows in an obvious way, using the condition $|u_j(t)| \leq M/\omega_j$, $j = 1, 2, \ldots, t \geq 0$, which holds as in the proof of Theorem 1. Q.E.D.

We say that a continuous function f on $[0, \infty)$ to R is asymptotically almost-periodic if there exists an almost-periodic function g (in the sense of Bohr) such that $f(t) - g(t) \to 0$ as $t \to \infty$.

THEOREM 3. Let the hypotheses of Theorem 2 hold except that $r(t, x)$ is asymptotically almost-periodic in t. Then each solution $u(t, x)$ of (8) is asymptotically almost periodic in t.

The proof of this theorem uses Theorem 8 in [5]; condition (11) in this theorem again holds trivially in our case. Since the details are almost identical to those in the proof of the previous theorem, we omit the proof.

4. Concluding remarks. Under weaker conditions it is possible to obtain theorems like those of the preceding section for the more general types of solutions mentioned in [4]. For the case $x \in [0, \pi]$ and zero boundary conditions at the end points, hypotheses (H4) and (H5) in [4] are clearly satisfied. For generalizations of our results to cases where $x \in R^n$, these hypotheses would be required.

Extensions to the case $k(0) = 0$, $k'(0) > 0$, apparently cannot be obtained with our methods; recall we used $b(t) = k'(t)/k(0)$. In [4], Miller shows that in this case the temperature equation can be put into the form:

$$u_{tt}(t, x) = y_1(0)u_t(t, x) + y_1'(0)u(t, x) + c_1 u_{xx}(t, x) + F_2(t, x) + \int_0^t y_1''(t - s)u(s, x)\, ds.$$

If $y_1(0) = k''(0)/k'(0) - \alpha'(0)/\alpha(0) < 0$, we would expect to obtain results such as in the previous section; otherwise not. The case $k(0) = 0$ is of interest if the model assumes heat disturbances propagate at a finite speed; cf. [4], [6].

If the temperature-energy relaxation function $\alpha(t)$ and the heat conduction relaxation function $k(t)$ depend on x as well as t, the methods of our paper as those in [4], would lead to a coupled infinite system of integro-differential equations in the modes $u_j(t)$ since we would have to expand $\alpha(t, x)$ and $k(t, x)$ in Fourier series also. Clearly, difficulties arise here even with respect to the existence of solutions in the form of Fourier series expansions.

Finally, from a physical point of view, since there is no mechanism in the model for producing energy, the solutions ought to be stable for any initial values. Both the example in Sec. 2 and the main result, Theorem 1, show that there must therefore be restrictions on the kernels. Clearly under the hypotheses of this theorem, specifically condition (ii), we must have $k(0)/\alpha(0) > k'(0)/k(0) - \alpha'(0)/\alpha(0)$.

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