

A MATHEMATICAL MODEL FOR LINEAR ELASTIC SYSTEMS WITH STRUCTURAL DAMPING*

BY

G. CHEN (*Pennsylvania State University*)

AND

D. L. RUSSELL (*University of Wisconsin, Madison*)

Abstract. We present a mathematical model exhibiting the empirically observed damping rates in elastic systems. The models studied are of the form (A the relevant elasticity operator)

$$\ddot{x} + B\dot{x} + Ax = 0$$

with B related in various ways to the positive square root, $A^{1/2}$, of A . Comparison with existing "ad hoc" models is made.

1. Semigroup background. A wide variety of conservative linear elastic systems may be represented by a second-order differential equation

$$\ddot{x} + Ax = 0 \tag{1.1}$$

where \cdot means d/dt , $x \in X$, a Hilbert space with inner product (\cdot, \cdot) and associated norm $\|\cdot\|$, and A is a positive self-adjoint operator on X , ordinarily bounded with domain $\mathcal{D}(A)$ dense in X . Under these circumstances A has a non-negative self-adjoint square root $A^{1/2}$ defined on a domain $\mathcal{D}(A^{1/2}) \subset X$. Throughout the paper we will assume that A is bounded below, i.e.,

$$(x, Ax) \geq \alpha \|x\|^2, \quad x \in X,$$

for some fixed $\alpha > 0$. Then the spectrum of A is bounded away from zero and, as a consequence $\mathcal{D}(A^{1/2}) \supset \mathcal{D}(A)$; indeed $\mathcal{D}(A^r) \supset \mathcal{D}(A^p)$ if r and p are positive numbers with $r < p$. Associated with (1.1) is the energy form

$$E(x, \dot{x}) = \frac{1}{2}(\|\dot{x}\|^2 + \|A^{1/2}x\|^2) \tag{1.2}$$

which is conserved when $x(t)$ is a solution of (1.1). More on this shortly.

Perhaps the most notable disadvantage associated with conservative systems is the fact that they do not occur in nature. Always there are dissipative mechanisms acting within the system causing the energy to decrease during any positive time interval. The most widely accepted mathematical model exhibiting such dissipative behavior takes the form

$$\ddot{x} + B\dot{x} + Ax = 0 \tag{1.3}$$

* Received August 14, 1980. Sponsored by the United States Army under Contract Nos. DAAG29-75-C-0024 and DAAG29-80-C-0041, and the Air Force Office of Scientific Research under Grant AFOSR79-0018.

where B is again a positive self-adjoint operator on X with domain $\mathcal{D}(B)$ dense in X . If $x(t)$ is a solution of (1.2), twice strongly continuously differentiable with $x(t) \in \mathcal{D}(A)$, $\dot{x}(t) \in \mathcal{D}(B)$, then

$$\begin{aligned} \frac{d}{dt} E(x(t), \dot{x}(t)) &= \frac{d}{dt} \frac{1}{2} [\dot{x}(t), \dot{x}(t)] + (A^{1/2}x(t), A^{1/2}x(t)) = (\dot{x}(t), \ddot{x}(t) + Ax(t)) \\ &= -(\dot{x}(t), B\dot{x}(t)) \leq 0. \end{aligned}$$

For the moment this is all formal since we have not discussed the existence of, or the nature of, solutions of (1.3).

Letting

$$z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \quad z^1, z^2 \in X,$$

the system (1.3) is formally equivalent, under the transformation $z^1 = x, z^2 = \dot{x}$, to

$$\dot{z} = \mathcal{A}_0 z, \tag{1.4}$$

where

$$\mathcal{A}_0 z \equiv \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} z^2 \\ -Az^1 \end{pmatrix}. \tag{1.5}$$

We digress, briefly, to consider the case where A is only non-negative. In this case we may write

$$X = X^+ \oplus X^0, \quad x \in X \Rightarrow x = \begin{pmatrix} x^+ \\ x^0 \end{pmatrix},$$

and, for $x \in \mathcal{D}(A)$,

$$Ax = \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^+ \\ x^0 \end{pmatrix} = \begin{pmatrix} A^+ x^+ \\ 0 \end{pmatrix}.$$

Here X^0 is the null space of A . We will assume that A^+ is bounded below. Then (1.5) is the same as

$$\begin{pmatrix} \dot{z}_1^0 \\ \dot{z}_2^0 \end{pmatrix} = \begin{pmatrix} 0 & I^0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix}, \tag{1.6}$$

$$\begin{pmatrix} \dot{z}_1^+ \\ \dot{z}_2^+ \end{pmatrix} = \begin{pmatrix} 0 & I^+ \\ -A^+ & 0 \end{pmatrix} \begin{pmatrix} z_1^+ \\ z_2^+ \end{pmatrix}. \tag{1.7}$$

Since (1.6) is almost trivial, only (1.7) need be studied. As a consequence, we may as well assume that A is bounded away from zero in (1.1), (1.5) when proving theorems about these systems. The space X^0 is usually finite-dimensional, encompassing free rigid body motion.

With A bounded away from zero, $A^{-1}, A^{-1/2}$ are non-negative bounded self-adjoint operators on X . The transformation

$$z^1 = A^{-1/2}w^1, \quad z^2 = w^2 \tag{1.8}$$

carries (1.5) into

$$\begin{pmatrix} \dot{w}^1 \\ \dot{w}^2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & 0 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \equiv \mathcal{L}_0 w. \tag{1.9}$$

It should be noted that the transformation (1.8) maps $\mathcal{D}(A^{1/2}) \oplus X$ onto $W \equiv X \oplus X$. The energy form (1.2) maps into

$$\frac{1}{2}\|w\|^2 = \frac{1}{2}(\|w^1\|^2 + \|w^2\|^2). \tag{1.10}$$

Applying the inequality

$$\left\| \begin{pmatrix} \lambda I & -A^{1/2} \\ A^{1/2} & \lambda I \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \right\|^2 \geq \lambda^2(\|w^1\|^2 + \|w^2\|^2), \quad \lambda \text{ real,}$$

which is easily verified for $w^1, w^2 \in \mathcal{D}(A^{1/2})$, we see that

$$\|R(\lambda, \mathcal{L}_0)\| \leq 1/|\lambda|, \quad \lambda \neq 0 \quad \text{and real.}$$

Then the Hille-Yoshida theorem ([2]) applies to give

THEOREM 1.1. The operator

$$\mathcal{L}_0 = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & 0 \end{pmatrix}$$

generates a strongly continuous group of bounded operators $\mathcal{S}_0(t)$ on $W = X \oplus X$, the solutions

$$w(t) = \mathcal{S}_0(t)w_0, \quad w_0 = \begin{pmatrix} w_0^1 \\ w_0^2 \end{pmatrix}$$

being strongly continuously differentiable and satisfying (1.9) in X for all t just in case

$$w_0^1, w_0^2 \in \mathcal{D}(A^{1/2}).$$

For B positive and self-adjoint in (1.3) the first-order system comparable to (1.9) is

$$\dot{z} = \mathcal{A}z, \tag{1.11}$$

$$\mathcal{A}_B z = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}. \tag{1.12}$$

Treating first the case where A is strictly positive, $A^{-1/2}$ is bounded and the transformation (1.8) yields

$$\begin{pmatrix} \dot{w}^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \equiv \mathcal{L}_B w. \tag{1.13}$$

Unless B is bounded it cannot be expected that \mathcal{L}_B will generate a group on W ; a semigroup is all we get. For $\lambda > 0$, our assumptions imply that $\lambda I + B$ is self-adjoint and non-negative. We compute

$$\begin{aligned} \left\| \begin{pmatrix} \lambda I & -A^{1/2} \\ A^{1/2} & \lambda I + B \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \right\|_W^2 &= \lambda^2 \|w^1\|_X^2 + \|A^{1/2}w\|_X^2 \\ &+ ((\lambda I + B)w^2, (\lambda I + B)w^2)_X + \|A^{1/2}w^2\|_X^2 + (A^{1/2}w^1, Bw^2)_X + (Bw^2, A^{1/2}w^1)_X. \end{aligned}$$

Since

$$\begin{aligned} ((\lambda I + B)w^2, (\lambda I + B)w^2)_X &= \lambda^2 \|w^2\|_X^2 + 2\lambda(w^2, Bw^2)_X + (Bw^2, Bw^2)_X \\ &\geq \lambda^2 \|w^2\|_X^2 + \|Bw^2\|_X^2, \quad \lambda > 0, \end{aligned}$$

and since

$$\|A^{1/2}w^1\|_X^2 + \|Bw^2\|_X^2 + (A^{1/2}w^1, Bw^2)_X + (Bw^2, A^{1/2}w^1)_X = \|A^{1/2}w^1 + Bw^2\|_X^2 \geq 0$$

we are able to conclude that

$$\left\| \begin{pmatrix} \lambda I & -A^{1/2} \\ A^{1/2} & \lambda I + B \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \right\|^2 \geq \lambda^2 (\|w^1\|^2 + \|w^2\|^2), \quad \lambda \geq 0 \tag{1.14}$$

as defined on the domain

$$w^1 \in \mathcal{D}(A^{1/2}), \quad w^2 \in \mathcal{D}(A^{1/2}) \cap \mathcal{D}(B). \tag{1.15}$$

For such $w = (w^1, w^2)$, then,

$$\|R(\lambda, \mathcal{L}_B)\| \leq 1/\lambda, \quad \lambda > 0.$$

The Hille-Yoshida theorem applies if the domain of \mathcal{L}_B is dense in W and \mathcal{L}_B is closed. For the first of these requirements we assume

$$\mathcal{D}(A^{1/2}) \cap \mathcal{D}(B) \text{ is dense in } X.$$

For the second we observe that if $\{(w_k^1, w_k^2)\}$ converges in W and $\mathcal{L}_B(w_k^1, w_k^2)$ converges in W , the latter implies that

$$\{A^{1/2}w_k^2\} \text{ converges in } X, \tag{1.16}$$

$$\{A^{1/2}w_k^1 + Bw_k^2\} \text{ converges in } X. \tag{1.17}$$

From (1.16) together with the fact that $A^{1/2}$, being self-adjoint, is closed, we conclude that $w^2 \equiv \lim_{k \rightarrow \infty} w_k^2 \in \mathcal{D}(A^{1/2})$. It is not, however, easy to conclude from (1.17) that either $\{A^{1/2}w_k^1\}$ or $\{Bw_k^2\}$ is convergent. Indeed, take the case where $B = A$. Using a coordinate system based on an orthonormal system for A , we may represent

$$A = B = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots), \quad A^{1/2} = (\lambda_1^{1/2}, \lambda_2^{1/2}, \lambda_3^{1/2}, \dots)$$

and vectors $x \in X$ may be represented by their expansion coefficients:

$$x = (x_1, x_2, x_3, \dots), \quad \sum_{k=1}^{\infty} |x_k|^2 < \infty.$$

Assuming that $\lim_{k \rightarrow \infty} \lambda_k = +\infty$, a sequence $\{x_k\}$ with positive elements may be found such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty, \quad \sum_{k=1}^{\infty} |\lambda_k^{1/2} x_k|^2 = \sum_{k=1}^{\infty} \lambda_k |x_k|^2 = \infty.$$

Assuming no $\lambda_k = 0$, let

$$w_k^2 = (x_1/\lambda_1^{1/2}, x_2/\lambda_2^{1/2}, \dots, x_k/\lambda_k^{1/2}, 0, 0, \dots), \quad w_k^1 = (-x_1, -x_2, \dots, -x_k, 0, 0, \dots).$$

Then $\{w_k^1\}, \{w_k^2\}$ are convergent, $\{A^{1/2}w_k^2\}$ is convergent, $\{A^{1/2}w_k^1 + Aw_k^2\} \equiv \{0\}$ is convergent but neither

$$\{A^{1/2}w_k^1\} = \{(-\lambda_1^{1/2}x_1, -\lambda_2^{1/2}x_2, \dots, -\lambda_k^{1/2}x_k, 0, 0, \dots)\}$$

nor

$$\{Aw_k^2\} = \{(\lambda_1^{1/2}x_1, \lambda_2^{1/2}x_2, \dots, \lambda_k^{1/2}x_k, 0, 0, \dots)\}$$

is convergent. Thus the operator \mathcal{L}_A is not closed on the domain (1.15), which in this case is

$$w^1 \in \mathcal{D}(A^{1/2}), \quad w^2 \in \mathcal{D}(A^{1/2}) \cap \mathcal{D}(A) = \mathcal{D}(A). \tag{1.18}$$

The operator \mathcal{L}_A is closable because it can be shown to have a complete set of eigenvectors in $W = X \oplus X$ and from that, via the Galerkin method [6], it can be shown that there is a strongly continuous semigroup, $\mathcal{S}_A(t)$, of bounded operators which satisfy

$$\frac{d}{dt} \mathcal{S}_A(t)w = \mathcal{L}_A \mathcal{S}_A(t)w \quad \text{whenever } w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \text{ satisfies (1.18).}$$

Then $\mathcal{S}_A(t)$ has a closed generator, which we will still call \mathcal{L}_A , defined by

$$\mathcal{L}_A w = \lim_{t \downarrow 0} \frac{1}{t} (\mathcal{S}_A(t) - I)w$$

for all w such that this limit exists. Denoting this set of w by $\mathcal{D}(\mathcal{L}_A)$, \mathcal{L}_A will be closed. In general this domain is larger than the one described in (1.18).

Despite this eventually positive outcome for the case $B = A$, the example shows that it will not be easy to characterize all instances wherein \mathcal{L}_B is a closed operator or, at least, has a closed extension.

At least two procedures come to mind. For some applications it is reasonable to assume that B is $A^{1/2}$ -bounded, i.e., the domain of B includes the domain of $A^{1/2}$ and there is a positive number M such that

$$\|Bx\| \leq M(\|x\| + \|A^{1/2}x\|), \quad x \in \mathcal{D}(A^{1/2}). \tag{1.19}$$

In such an event the domain of \mathcal{L}_B is precisely $\{(w^1, w^2) \mid w^1 \in \mathcal{D}(A^{1/2}), w^2 \in \mathcal{D}(A^{1/2})\}$, (1.16) implies that $\{Bw_k^2\}$ converges in X and that, with (1.17) implies that $\{A^{1/2}w_k^1\}$ converges in X . Then \mathcal{L}_B is closed as defined on (1.15).

A second possibility is to show that \mathcal{L}_B is maximal dissipative or that it has a maximal dissipative extension. The theory of Phillips [7] then applies to show that the maximally extended dissipative operator generates a semigroup $\mathcal{S}_B(t)$ and is a closed operator. This is essentially what we have already carried out for the case $B = A$.

2. Structural damping and holomorphic semigroups: implications. The basic property of structural damping, which is said to be consistent with empirical studies [3, 9], is that the amplitudes of the normal modes of vibration are attenuated at rates which are proportional to the oscillation frequencies. We will see that this is an important property, implying as it does that many distributed systems act in a manner more like finite-dimensional systems than would otherwise be the case.

The subject of energy dissipation in elastic systems has been extensively studied in the literature (see, e.g. [3]). Nevertheless, mathematical modelling appears to be rather primitive and *ad hoc*. In [3] and [4] the representation

$$\ddot{x} + (1 \pm \delta i)Ax = 0 \tag{2.1}$$

is used. Assuming that A has discrete spectrum $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$, with corresponding orthonormal eigenvectors $\phi_1, \phi_2, \phi_3, \dots$, (2.1) has solutions

$$x_k(t) = e^{\sigma_k t} \phi_k$$

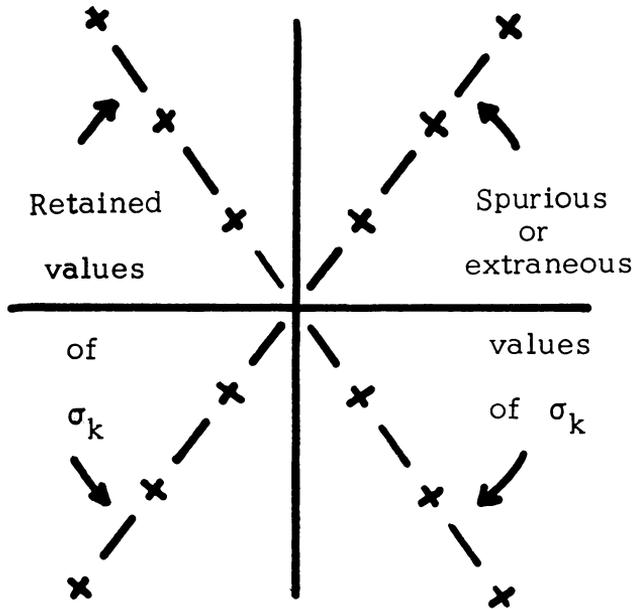


FIG. 1. Spurious spectrum, $\ddot{x} = (1 \pm \delta i)Ax$.

if and only if

$$\sigma_k^2 + (1 \pm \delta i)\lambda_k = 0,$$

giving

$$\sigma_k = -(1 \pm \delta i)\lambda_k^{1/2}.$$

Since

$$-(1 \pm \delta i)\lambda_k = \omega_k^2 \sqrt{1 + \delta^2} e^{\pm i(\tan^{-1} \delta + \pi)},$$

where $\omega_k = \sqrt{\lambda_k}$, we have

$$\sigma_k = \pm \omega_k \sqrt{1 + \delta^2} e^{\pm i\psi}, \quad \psi = \frac{1}{2}(\tan^{-1} \delta + \pi),$$

four values for each integer k . These lie in an "X" pattern in the complex plane, symmetric with respect to both the real and imaginary axes (Fig. 1). In engineering use, those lying in the right half-plane are rejected as extraneous, those in the left half-plane are retained. This does not correspond to choosing one to the signs $+$ or $-$ in (2.1), however.

Eq. (2.1) has numerous disadvantages. First of all, it is not properly an equation and there is no associated strongly continuous semigroup. Secondly, if \hat{A} denotes a positive symmetric matrix representing a discretization of the elastic operator A , the equation

$$\ddot{\hat{x}} + (1 \pm \delta i)\hat{A}\hat{x} = 0, \quad \hat{x} \in E^n,$$

is still not computationally useful for generating approximate solutions. There is the obvious problem of introducing complex numbers into an equation which is supposed to represent a real system. Further, the "extraneous" solutions will grow and eclipse the

decaying solutions which are actually desired. This remains true for the real fourth-order equation

$$\hat{x}^{(iv)} + 2\hat{A}\hat{x}'' + (1 + \delta^2)\hat{A}^2\hat{x} = 0 \tag{2.2}$$

which has the same solutions as (2.1).

The form of (2.2) is, nevertheless, suggestive of the prototype model for this paper which is

$$\ddot{x} + 2\rho A^{1/2}\dot{x} + Ax = 0, \tag{2.3}$$

where $A^{1/2}$ denotes the positive self-adjoint square root of A . For (2.3), trial solutions

$$x_k(t) = e^{\sigma_k t} \phi_k$$

lead to the equation

$$\sigma_k^2 + 2\rho\sigma_k\lambda_k^{1/2} + \lambda_k = 0$$

or

$$\sigma_k = \frac{-2\rho\omega_k \pm \sqrt{4\rho^2\lambda_k - 4\lambda_k}}{2} = \omega_k(-\rho \pm i\sqrt{1-\rho^2}) = \omega_k e^{\pm i\psi}, \quad \psi = \tan^{-1}\left(\frac{\rho}{\sqrt{1-\rho^2}}\right), \tag{2.4}$$

if $\rho^2 < 1$, which we would normally anticipate for lightly damped structures. In (2.4) we have exponents forming a pattern “>” in the complex plane, a pattern quite similar to the retained σ_k of the earlier model (2.1) (see Fig. 2).

The system (2.3) has many advantages in addition to not producing extraneous spectral values. It is a bona fide dynamical system; we will see in the next two sections

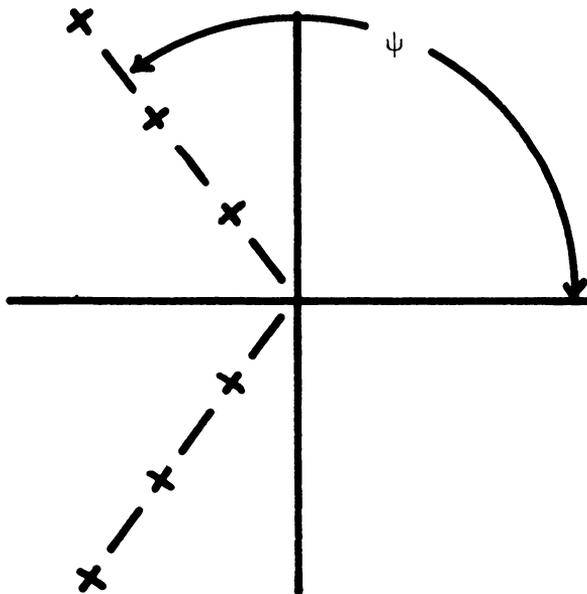


FIG. 2. Spectrum for the system (2.3).

that this system corresponds to a system (1.13) with

$$\mathcal{L}_B = \mathcal{L}_{2\rho A^{1/2}} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -2\rho A^{1/2} \end{pmatrix}$$

such that $\mathcal{L}_{2\rho A^{1/2}}$ generates a strongly continuous (in fact, holomorphic) semigroup in $W = X \oplus X$, and that this remains true for the operator \mathcal{L}_B if B “resembles” $2\rho A^{1/2}$ appropriately.

The eigenvectors of $\mathcal{L}_{2\rho A^{1/2}}$ also have properties which are desirable from the analytical point of view. If we denote the normalized eigenvectors of A (equivalently $A^{1/2}$) in X by ϕ_k , $k = 1, 2, 3, \dots$, then the normalized eigenvectors of the antihermitian operator

$$\mathcal{L}_0 = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & 0 \end{pmatrix} \quad (2.5)$$

corresponding to $i\omega_k$, $-i\omega_k$ are seen to be

$$\theta_k^+ = \begin{pmatrix} \phi_k/\sqrt{2} \\ i\phi_k/\sqrt{2} \end{pmatrix}, \quad \theta_k^- = \begin{pmatrix} \phi_k/\sqrt{2} \\ -i\phi_k/\sqrt{2} \end{pmatrix}, \quad k = 1, 2, \dots, \quad (2.6)$$

and it is easy to verify from the corresponding properties of the ϕ_k in X that the vectors (2.6) form an orthonormal basis for $W = X \oplus X$ —which would also follow from the fact that the operator (2.5) is antihermitian. The normalized eigenvectors of $\mathcal{L}_{2\rho A^{1/2}}$ corresponding to the eigenvalues $\omega_k e^{\pm i\psi}$ are

$$\Psi_k^+ = \begin{pmatrix} \phi_k/\sqrt{2} \\ e^{i\psi}\phi_k/\sqrt{2} \end{pmatrix}, \quad \Psi_k^- = \begin{pmatrix} \phi_k/\sqrt{2} \\ e^{-i\psi}\phi_k/\sqrt{2} \end{pmatrix}, \quad k = 1, 2, \dots,$$

which may be seen to be related to the vectors (2.6) by the transformation

$$\Psi_k^{(\pm)} = \begin{pmatrix} I & 0 \\ \cos \psi I & \sin \psi I \end{pmatrix} \theta_k^{(\pm)}$$

in $W = X \oplus X$. Since this transformation is bounded and boundedly invertible for $\pi/2 \leq \psi < \pi$, which corresponds to $0 \leq \rho < 1$, we conclude that the $\Psi_k^{(\pm)}$ form a uniform, or Riesz, basis [1, 8] for $W = X \oplus X$; that is, given $w \in W$, w may be expanded in the convergent series

$$w = \sum_{k=1}^{\infty} (w_k^+ + \Psi_k^+ + w_k^- - \Psi_k^-)$$

and there are positive constants c, C , such that

$$c^{-2} \|w\|_W^2 \leq \sum_{k=1}^{\infty} (|w_k^+|^2 + |w_k^-|^2) \leq C^2 \|w\|_W^2.$$

Among other things, this implies that $\mathcal{L}_{2\rho A^{1/2}}$ is similar to a normal operator—which is also easily seen from the operator identity

$$\begin{pmatrix} \frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \\ e^{i\psi} I & e^{-i\psi} I \end{pmatrix}^{-1} \begin{pmatrix} 0 & A^{1/2} \\ A^{1/2} & -2\rho A^{1/2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \\ e^{i\psi} I & -\frac{e^{-i\psi}}{\sqrt{2}} I \end{pmatrix} = \begin{pmatrix} e^{i\psi} A^{1/2} & 0 \\ 0 & e^{-i\psi} A^{1/2} \end{pmatrix}. \quad (2.7)$$

These observations lead to a number of significant consequences. Let

$$b = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}, \quad c = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}$$

be vectors in X . The transfer function for the forced system

$$\dot{w} = \begin{pmatrix} \dot{w}^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -2\rho A^{1/2} \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} f = \mathcal{L}_{2\rho A^{1/2}} w + bf \quad (2.8)$$

with output

$$\omega(t) = \left[\begin{pmatrix} w^1(t) \\ w^2(t) \end{pmatrix}, \quad \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} \right]_w = (w(t), c)_w$$

is the function

$$R(\lambda) = [(\lambda I - \mathcal{L}_{2\rho A^{1/2}})^{-1} b, c]_w.$$

This function is invariant under any transformation

$$w = Ty,$$

in particular for (cf. (2.7))

$$T = T_\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \\ e^{i\psi} & e^{-i\psi} \\ \frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \end{pmatrix}, \quad (2.9)$$

in the sense that

$$R(\lambda) = ((\lambda I - T^{-1} \mathcal{L}_{2\rho A^{1/2}} T)^{-1} T^{-1} b, T^* C)_w,$$

and in the case (2.9) this gives

$$R(\lambda) = \left[\begin{pmatrix} (\lambda I - e^{i\psi} A^{1/2})^{-1} & 0 \\ 0 & (\lambda I - e^{-i\psi} A^{1/2})^{-1} \end{pmatrix} T_\psi^{-1} b, T_\psi^* C \right]_w.$$

From the fact that A , and hence $A^{1/2}$, is positive, we see that if $|\arg(\lambda) - \Psi|$, $|\arg(\lambda) + \psi|$ are bounded away from zero, $|R(\lambda)| \leq K/|\lambda|$ for some positive K . In particular, as long as $0 < \rho < 1$, we see that the frequency response function $R(i\omega)$ has the property

$$|R(i\omega)| \leq K/|\omega|. \quad (2.10)$$

This is emphatically not true for the undamped operator \mathcal{L}_0 , or for the operator $\mathcal{L}_{\gamma I}$, $\gamma > 0$, corresponding to viscous damping. In those cases all that can be asserted is that $|R(i\omega)|$ is bounded. The inequality

$$\|(\lambda I - \mathcal{L}_B)^{-1}\| \leq M/|\lambda|, \quad |\arg \lambda| < \psi - \varepsilon, \quad (2.11)$$

which follows from the above considerations for $B = 2\rho A^{1/2}$, is the fundamental hypothesis required in order that \mathcal{L}_B should generate a holomorphic semigroup in W . An inequality of the form (2.10) easily follows from (2.11). Since systems whose frequency response functions satisfy (2.10) may be modeled effectively by finite-dimensional

systems—as will be shown elsewhere—structural damping and the closely allied property of holomorphic semigroup generation have significant practical modeling implications.

To conclude this section we remark that, although A being a differential operator does not at all ensure that $A^{1/2}$ is such, this does not impede the usefulness of (2.3) in computations. If one approximates A by a positive, self-adjoint finite-dimensional matrix \hat{A} , the positive self-adjoint square root matrix $\hat{A}^{1/2}$ may be calculated readily and the finite dimensional system

$$\ddot{\hat{w}} + 2\rho\hat{A}^{1/2}\dot{\hat{w}} + \hat{A}\hat{w} = 0$$

is then available for computational use.

3. A sufficient condition for general damping operators B. We wish now to consider the general second order equation

$$\ddot{x} + B\dot{x} + Ax = 0 \tag{3.1}$$

with, as we have seen, the equivalent first-order representation

$$\dot{w} = \begin{pmatrix} \dot{w}^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \equiv \mathcal{L}_B w. \tag{3.2}$$

Following the discussion of the previous section, our objective is to determine conditions on B sufficient in order that \mathcal{L}_B should generate a holomorphic semigroup in W —and we want to do this without making overly restrictive assumptions on B , such as $B = \gamma A^{1/2}$, $B = \gamma A$, or, for that matter, $B = \gamma f(A)$ where $f(\lambda)$ is an analytic function of λ in a region containing the positive real λ axis with $f(\lambda) > 0$ for $\lambda > 0$. Nevertheless, in the present paper, we are concerned with operators B which are closely related to $A^{1/2}$. Specifically, we will assume at the outset that B is $A^{1/2}$ -bounded so that, as noted in Sec. 1, \mathcal{L}_B is closed.

From [5] we know that a sufficient condition for \mathcal{L}_B to generate a holomorphic semigroup is the following: that \mathcal{L}_B should be closed and that there should exist a positive number θ_1 , $0 < \theta_1 \leq \pi/2$, such that

$$\|R(\lambda, \mathcal{L}_B)\| = \|(\lambda I - \mathcal{L}_B)^{-1}\| \leq M_1/|\lambda|, \quad M_1 = M_1(\theta), \tag{3.3}$$

for $|\lambda|$ sufficiently large in any sector

$$|\arg(\lambda)| \leq \frac{\pi}{2} + \theta, \quad 0 < \theta < \theta_1 \tag{3.4}$$

of the complex λ plane. When this is true the semigroup $\mathcal{S}_B(t)$ generated by \mathcal{L}_B is defined for t in the dual sector

$$\{0\} \cup \{t \mid |\arg t| < \theta_1\}$$

and for each $w \in W$, $\mathcal{S}_B(t)w$ is a holomorphic (i.e., differentiable with respect to $\| \cdot \|_W$) vector-valued function in $\{t \mid t > 0, |\arg t| < \theta_1\}$.

We have already presented, in Sec. 2, a discussion of the significance of the holomorphicity of $\mathcal{S}_B^l(t)$. From that work we know that solutions $w(t) = \mathcal{S}_B^l(t)w$ corresponding to holomorphic semigroups $\mathcal{S}_B^l(t)$ exhibit a number of properties characteristic of structural damping.

THEOREM 3.1. Let B , positive and self-adjoint, be $A^{1/2}$ -bounded and, in addition, let B satisfy

$$B = 2\rho A^{1/2} + CA^{1/2} \tag{3.5}$$

where C is a bounded operator. Then there exists a positive number $m(\rho)$ (depending only on ρ and A) such that whenever

$$\|C\| \leq m(\rho),$$

\mathcal{L}_B generates a holomorphic semigroup on $W = X \oplus X$.

Proof. We begin by noting that $R(\lambda, \mathcal{L}_B)$ can be computed explicitly. Representing $\lambda I - \mathcal{L}_B: W = X \oplus X \rightarrow X \oplus X$ in operator matrix form as

$$\lambda I - \mathcal{L}_B = \begin{pmatrix} \lambda I & -A^{1/2} \\ A^{1/2} & \lambda I + B \end{pmatrix},$$

it must be true that

$$(\lambda I - \mathcal{L}_B)(\lambda I - \mathcal{L}_B)^{-1} = I. \tag{3.6}$$

Representing the resolvent in matrix form also:

$$(\lambda I - \mathcal{L}_B)^{-1} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

(3.6) is seen to be equivalent to

$$\lambda W - A^{1/2}Y = I, \quad \lambda X - A^{1/2}Z = 0, \tag{3.7}$$

$$A^{1/2}W + (\lambda I + B)Y = 0, \quad A^{1/2}X + (\lambda I + B)Z = I. \tag{3.8}$$

From (3.7), we have, immediately,

$$W = \lambda^{-1}(I + A^{1/2}Y), \quad X = \lambda^{-1}A^{1/2}Z. \tag{3.9}$$

Substituting (3.9) into (3.8) and multiplying the result by λ , we have

$$A^{1/2} + AY + (\lambda^2 I + \lambda B)Y = 0,$$

$$AZ + (\lambda^2 I + \lambda B)Z = \lambda I.$$

These give

$$Y = -(\lambda^2 I + \lambda B + A)^{-1}A^{1/2}, \quad Z = \lambda(\lambda^2 I + \lambda B + A)^{-1}. \tag{3.10}$$

Then, returning to (3.9), we have

$$W = \lambda^{-1}(I + A^{1/2}Y) = \lambda^{-1}(I - A^{1/2}(\lambda^2 I + \lambda B + A)^{-1}A^{1/2}), \tag{3.11}$$

$$X = \lambda^{-1}A^{1/2}Z = A^{1/2}(\lambda^2 I + \lambda B + A)^{-1}.$$

Combining these results, and letting

$$P(\lambda, A, B) = \lambda^2 I + \lambda B + A,$$

we have

$$R(\lambda, \mathcal{L}_B) = \begin{pmatrix} \lambda^{-1}(I - A^{1/2}P(\lambda, A, B)^{-1}A^{1/2}) & A^{1/2}P(\lambda, A, B)^{-1} \\ -P(\lambda, A, B)^{-1}A^{1/2} & \lambda P(\lambda, A, B)^{-1} \end{pmatrix}. \tag{3.12}$$

Since $\|\lambda R(\lambda, \mathcal{L}_B)\| = |\lambda| \|R(\lambda, \mathcal{L}_B)\|$, all we need to show is that there is an $M_1 = M_1(\theta)$ such that

$$\|\lambda R(\lambda, \mathcal{L}_B)\| \leq M \tag{3.13}$$

for $|\lambda|$ sufficiently large, $|\arg \lambda| \leq (\pi/2) + \theta$. Since A, B are self-adjoint,

$$P(\lambda, A, B)^* = P(\bar{\lambda}, A, B), \quad (P(\lambda, A, B)^{-1}A^{1/2})^* = A^{1/2}P(\bar{\lambda}, A, B)^{-1},$$

and it is enough to show that the three operators

$$A^{1/2}P(\lambda, A, B)^{-1}A^{1/2}, \lambda A^{1/2}P(\lambda, A, B)^{-1}, \quad \lambda^2P(\lambda, A, B)^{-1} \tag{3.14}$$

are all bounded in the indicated region.

Let ρ be the positive number referred to in the theorem. We have

$$P(\lambda, A, B) = \lambda^2I + \lambda B + A = \lambda^2 + 2\rho\lambda A^{1/2} + A + \lambda(B - 2\rho A^{1/2}).$$

By analogy with the quadratic formula, we set

$$\begin{aligned} \lambda^2I + 2\rho\lambda A^{1/2} + A &= (\lambda I + [\rho + (\rho^2 - 1)^{1/2}]A^{1/2})(\lambda I + [\rho - (\rho^2 - 1)^{1/2}]A^{1/2}) \\ &(\equiv Q^+(\lambda, \rho, A^{1/2})Q^-(\lambda, \rho, A^{1/2})). \end{aligned} \tag{3.15}$$

Then

$$\begin{aligned} P(\lambda, A, B)^{-1} &= (\lambda I + [\rho - (\rho^2 - 1)^{1/2}]A^{1/2})^{-1}[I + (\lambda I + [\rho + (\rho^2 - 1)^{1/2}]A^{1/2})^{-1} \\ &\times \lambda(B - 2\rho A^{1/2})(\lambda I + [\rho - (\rho^2 - 1)^{1/2}]A^{1/2})^{-1}]^{-1}(\lambda I + [\rho + (\rho^2 - 1)^{1/2}]A^{1/2})^{-1} \\ &= Q^-(\lambda, \rho, A^{1/2})^{-1}[I + Q^+(\lambda, \rho, A^{1/2})^{-1}\lambda(BA^{-1/2} - 2\rho I)A^{1/2} \\ &\times Q^-(\lambda, \rho, A^{1/2})^{-1}]^{-1}Q^+(\lambda, \rho, A^{1/2})^{-1}. \end{aligned} \tag{3.16}$$

Now suppose we can establish that the operators

$$\lambda Q^\pm(\lambda, \rho, A^{1/2})^{-1}, \quad A^{1/2}Q^\pm(\lambda, \rho, A^{1/2})^{-1} \tag{3.17}$$

are uniformly bounded for $|\arg \lambda| \leq (\pi/2) + \theta$. Then, using this in (3.16) we see that the operators (3.14) are all bounded just in case

$$[I + Q^+(\lambda, \rho, A^{1/2})^{-1}\lambda(BA^{-1/2} - 2\rho I)A^{1/2}Q^-(\lambda, \rho, A^{1/2})^{-1}]^{-1}$$

is bounded. Again using the boundedness of the operators (3.17), this will be true if

$$\|BA^{-1/2} - 2\rho I\| \leq m(\rho),$$

where $m(\rho)$ is such that

$$m(\rho)\|Q^+(\lambda, \rho, A^{1/2})^{-1}\| \|A^{1/2}Q^-(\lambda, \rho, A^{1/2})^{-1}\| \leq \gamma < 1 \tag{3.18}$$

uniformly for sufficiently large $|\lambda|$ in the sector $|\arg \lambda| \leq (\pi/2) + \theta$.

Letting $E(\mu)$ be the spectral measure associated with the positive self-adjoint operator $A^{1/2}$, the operators (3.18) have the representations

$$\begin{aligned} \lambda Q^\pm(\lambda, \rho, A^{1/2})^{-1} &= \int_{\sigma(A^{1/2})} \lambda/(\lambda + [\rho \pm (\rho^2 - 1)^{1/2}]\mu) dE(\mu), \\ A^{1/2}Q^\pm(\lambda, \rho, A^{1/2})^{-1} &= \int_{\sigma(A^{1/2})} \mu/(\lambda + [\rho \pm (\rho^2 - 1)^{1/2}]\mu) dE(\mu), \end{aligned}$$

and we have the desired result if the functions

$$p^\pm(\lambda, \rho, \mu) = \frac{\lambda}{\lambda + [\rho \pm (\rho^2 - 1)^{1/2}]\mu}, \quad q^\pm(\lambda, \rho, \mu) = \frac{\mu}{\lambda + [\rho \pm (\rho^2 - 1)^{1/2}]\mu}$$

are uniformly bounded for $\mu \in \sigma(A^{1/2})$, $|\lambda|$ sufficiently large in the sector $|\arg \lambda| \leq (\pi/2) + \theta$.

We consider first the case where $\rho > 1$. Then $(\rho^2 - 1)^{1/2}$ is positive and

$$\min\{\rho + (\rho^2 - 1)^{1/2}, \rho - (\rho^2 - 1)^{1/2}\} = \rho - (\rho^2 - 1)^{1/2} \equiv r^- > 0.$$

$$\max\{\rho + (\rho^2 - 1)^{1/2}, \rho - (\rho^2 - 1)^{1/2}\} = \rho + (\rho^2 - 1)^{1/2} \equiv r^+ > 0.$$

In this case we can take θ to be any positive number $< \pi/2$. The modulus of $p^\pm(\lambda, \rho, \mu)$ is then the ratio of the distance from λ to the origin to the distance from λ to one of the points $-\mu r^+$ or $-\mu r^-$. Let ψ be the angle between λ and the negative real axis (see Fig. 3). Clearly

$$\psi > \frac{\pi}{2} - \theta > 0.$$

Then, by the law of cosines,

$$|\lambda + \mu r^\pm|^2 = |\lambda|^2 + |\mu r^\pm|^2 - 2 \cos \psi |\lambda| |\mu r^\pm|$$

so that

$$\left(\frac{|\lambda|}{|\lambda + \mu r^\pm|}\right)^2 + \left(\frac{|\mu r^\pm|}{|\lambda + \mu r^\pm|}\right)^2 - 2 \cos \psi \left(\frac{|\lambda|}{|\lambda + \mu r^\pm|} \frac{|\mu r^\pm|}{|\lambda + \mu r^\pm|}\right) = 1. \quad (3.19)$$

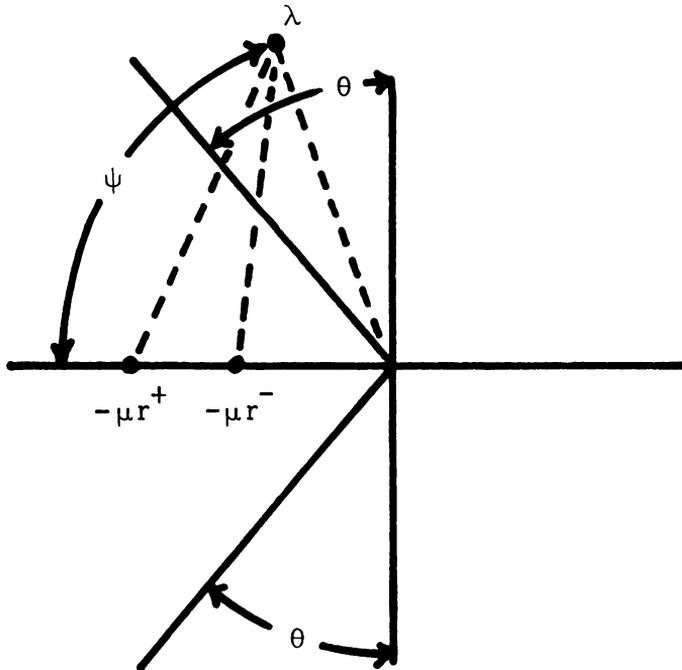


FIG. 3.

Now it is clear geometrically that

$$\frac{|\mu r^\pm|}{|\lambda + \mu r^\pm|} \leq \csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta \tag{3.20}$$

so that

$$|q^\pm(\lambda, \rho, \mu)| \leq \frac{1}{r^-} \left| \frac{r^\pm \mu}{|\lambda + \mu r^\pm|} \right| \leq \frac{\sec \theta}{r^-}. \tag{3.21}$$

Then, using (3.20) in (3.19), we have

$$\left(\frac{|\lambda|}{|\lambda + \mu r^\pm|} \right)^2 - 2 \cos \Psi \sec \theta \frac{|\lambda|}{|\lambda + \mu r^\pm|} - 1 \leq 0$$

which implies that

$$\frac{|\lambda|}{|\lambda + \mu r^\pm|} \leq \cos \psi \sec \theta + \sqrt{\cos^2 \Psi \sec^2 \theta + 1}. \tag{3.22}$$

If $\rho < 1$ the argument is not very different. In this case the numbers $-\rho \pm (\rho^2 - 1)^{1/2} = \rho \pm i(1 - \rho^2)^{1/2} \equiv r^\pm$ are conjugate complex numbers and $-\mu r^\pm$ lies on one of the rays

$$|\arg \lambda| = \frac{\pi}{2} + \tan^{-1} \frac{\rho}{(1 - \rho^2)^{1/2}} = \frac{\pi}{2} + \sin^{-1} \rho$$

in the left half-plane as shown in Fig. 4, where we have set

$$\theta_1 = \sin^{-1} \rho.$$

The symmetry of the situation allows us to consider λ in the upper half of this sector only. We let ψ be the angle between λ and the ray $\arg \lambda = (\pi/2) + \theta_1$. Then

$$\psi \geq \theta_1 - \theta > 0.$$

Using the law of cosines in precisely the same way as before, we have (3.20) modified to

$$|\mu r^\pm| / |\lambda + \mu r^\pm| \leq \csc(\theta_1 - \theta) \tag{3.23}$$

and (3.22) is replaced by

$$\frac{|\lambda|}{|\lambda + \mu r^\pm|} \leq \cos \psi \csc(\theta_1 - \theta) + \sqrt{\cos^2 \psi \csc^2(\theta_1 - \theta) + 1}. \tag{3.24}$$

This completes the proof.

The result shows, in effect, that the set of operators B for which \mathcal{L}_B generates a holomorphic semigroup includes a neighborhood, relative to the operator topology $\|B\|_{A^{1/2}} = \|BA^{-1/2}\|$ about $B = 2\rho A^{1/2}$ for any positive number ρ . Inspection of the case $C = cI$ shows very quickly that this is the best result one can obtain in terms of $\| \cdot \|_{A^{1/2}}$. However, the result is not satisfactory for many purposes. What we would eventually like to be able to prove is that if B is a positive self-adjoint operator such that

$$\rho_1 A^{1/2} \leq B \leq \rho_2 A^{1/2}, \quad x \in \mathcal{L}(A^{1/2}), \tag{3.25}$$

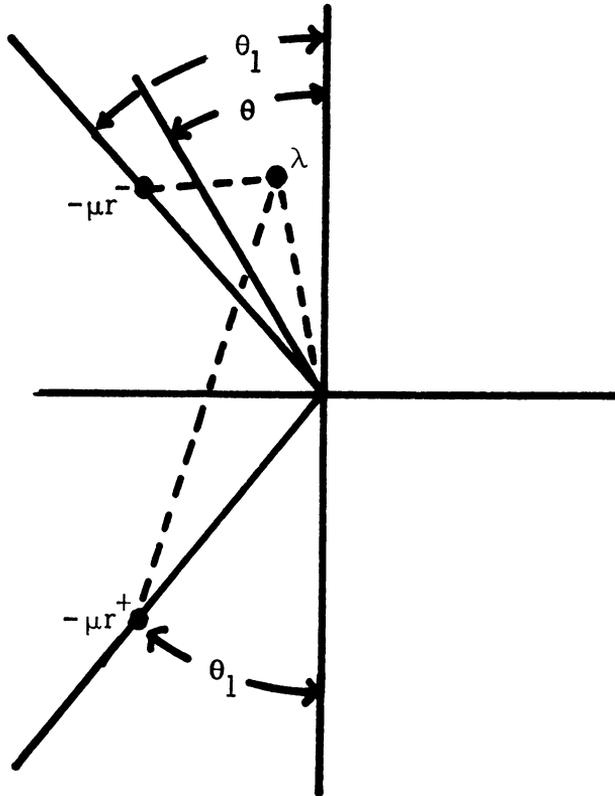


FIG. 4.

or else (not, in general, equivalent)

$$\rho_1^2 A \leq B^2 \leq \rho_2^2 A, \quad x \in \mathcal{D}(A); \tag{3.26}$$

then \mathcal{L}_B generates a holomorphic semigroup on $W = X \oplus X$.

We can offer a partial result in this direction in the form of the following corollary to Theorem 3.1.

COROLLARY 3.2. For each $\rho > 0$ there exists $\varepsilon(\rho)$ such that if B is a positive self-adjoint operator satisfying

$$[2\rho - \varepsilon(\rho)]A^{1/2} \leq B \leq [2\rho + \varepsilon(\rho)]A^{1/2}, \tag{3.27}$$

then \mathcal{L}_B generates a holomorphic semigroup on $W = X \oplus X$.

Proof. Multiplying (3.27) on the right and on the left by $A^{-1/4}$ we have

$$[2\rho - \varepsilon(\rho)]I \leq A^{-1/4}BA^{-1/4} \leq [2\rho + \varepsilon(\rho)]I.$$

Then

$$-\varepsilon(\rho)I \leq A^{-1/4}BA^{-1/4} - 2\rho I \leq \varepsilon(\rho)I$$

and since $A^{-1/4}BA^{-1/4}$ is self-adjoint, we conclude that

$$\|A^{-1/4}BA^{-1/4} - 2\rho I\| \leq \varepsilon(\rho). \tag{3.28}$$

Now the term $Q^+(\lambda, \rho, A^{1/2})^{-1}\lambda(B - 2\rho A^{1/2})Q^-(\lambda, \rho, A^{1/2})^{-1}$ which occurs in (3.16) can be rewritten as

$$Q^+(\lambda, \rho, A^{1/2})^{-1}\lambda^{1/2}A^{1/4}(A^{-1/4}BA^{-1/4} - 2\rho I)\lambda^{1/2}A^{1/4}Q^-(\lambda, \rho, A^{1/2})^{-1}.$$

The proof then proceeds as before except that it is now necessary to establish the boundedness of the operators

$$Q^+(\lambda, \rho, A^{1/2})^{-1}\lambda^{1/2}A^{1/4}, \quad \lambda^{1/2}A^{1/4}Q^-(\lambda, \rho, A^{1/2})^{-1}.$$

Going over to the spectral analysis again, it is sufficient to establish the uniform boundedness of the functions

$$S^\pm(\lambda, \rho, \mu) = \frac{\lambda^{1/2}\mu^{1/2}}{\lambda + [\rho \pm (\rho^2 - 1)^{1/2}]\mu}$$

for $\mu \in \sigma(A^{1/2})$, $|\lambda|$ sufficiently large in the sector $|\arg \lambda| \leq (\pi/2) + \theta$. But since

$$|S^\pm(\lambda, \rho, \mu)|^2 = |p^\pm(\lambda, \rho, \mu)| |q^\pm(\lambda, \rho, \mu)|.$$

This follows immediately from the work already done in the proof of Theorem 3.1 and Corollary 3.2 is proved.

It is possible to prove Theorem 3.1 in a slightly different way by first carrying out the proof for the case

$$B = 2\rho A^{1/2}$$

and then applying a result in [5] on perturbation of holomorphic semigroups. One needs only to observe that, with

$$\mathcal{A} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -2\rho A^{1/2} \end{pmatrix} \tag{3.29}$$

the operator

$$\mathcal{C} = \begin{pmatrix} 0 & 0 \\ 0 & CA^{1/2} \end{pmatrix}$$

is \mathcal{A} -bounded with \mathcal{A} norm tending to 0 as $\|C\|$ tends to zero. The cited result in [5] then shows that $\mathcal{A} + \mathcal{C}$ generates a holomorphic semigroup. The proof that (3.29) generates a holomorphic semigroup is almost immediate when one notes that, as pointed out in Sec. 2, A is similar to a normal operator.

4. A result for strong structural damping. We have noted our conjecture that \mathcal{L}_B in (1.13) should generate a holomorphic semigroup if B is self-adjoint and positive and (3.25) or (3.26) is valid. Though unable to obtain a result of this strength at the moment, we can present a theorem valid under hypotheses significantly different from those made in Theorem 3.1. Assuming that

$$\rho_1^2 \|x\|^2 \leq \|BA^{-1/2}x\|^2 \leq \rho_2^2 \|x\|^2, \tag{4.1}$$

equivalently,

$$\rho_1^2 A \leq B^2 \leq \rho_2^2 A, \quad \rho_2 > \rho_1 > 0. \tag{4.2}$$

The theorem applies, basically, when ρ_1 is large and ρ_2 is not “too large” in relation to ρ_1 . We will call this the case of “strong” structural damping.

Let us note, first of all, that (4.1), (4.2) imply that $\|Bx\| \leq k_1 \|A^{1/2}x\|$ for some k_1 not greater than ρ_2 . We will see later that (4.1) implies

$$\rho_1^2 \|x\|^2 \leq \|A^{-1/2}Bx\|^2 \leq \rho_2^2 \|x\|^2 \tag{4.3}$$

so that $\|A^{-1/2}BA^{-1/2}A^{1/2}x\| \leq k_2 \|x\|$ with k_2 not greater than ρ_2 . We let $\tilde{k} = \sup(k_1, k_2)$. Then

$$\|Bx\| \leq k \|A^{1/2}x\|, \|A^{-1/2}BA^{-1/2}y\| \leq \tilde{k} \|A^{-1/2}y\|. \tag{4.4}$$

THEOREM 4.1. Let B be positive and self-adjoint; let A be as in Sec. 1. Let $BA^{-1/2}$ satisfy (4.1) and let the range of $BA^{-1/2}$ be X —so that $BA^{-1/2}$ is bounded and boundedly invertible. Then the operator \mathcal{L}_B (cf. (1.13)) generates a holomorphic semigroup $\mathcal{S}_B(t)$ for t in the interior of the sector

$$\sum_{\vartheta} = \{t \in \mathbb{C} \mid |\arg t| \leq \tan^{-1}\vartheta\},$$

for some $\vartheta > 0$, provided that for some $\varepsilon, 0 < k\varepsilon < 1$,

$$(i) \quad 1 - \left(\frac{k}{4\varepsilon} + 2\right) \frac{1}{\rho_1^2} > 0;$$

$$(ii) \quad 1 + \frac{2}{\rho_2^2} - \frac{1}{1 - k\varepsilon} > 0;$$

$$(iii) \quad \left|1 - \left(\frac{k}{4\varepsilon} + 2\right) \frac{1}{\rho_1^2}\right| - \left|\frac{1}{1 - \varepsilon} + \frac{1}{1 - k\varepsilon} - \left(1 + \frac{2}{\rho_2^2}\right)\right| \vartheta^2 > 0.$$

For future reference we observe that (4.4) implies that for some $k \leq 2\tilde{k}$

$$|(Ax, Bx) - (Bx, Ax)| \leq k \|A^{1/2}x\| \|Ax\|, \quad x \in \mathcal{D}(A). \tag{4.5}$$

If $B: \mathcal{D}(A^{3/2}) \rightarrow \mathcal{D}(A)$, this can be replaced by the commutator condition

$$|(x, (AB - BA)x)| \leq k \|A^{1/2}x\| \|Ax\|, \quad x \in \mathcal{D}(A^{3/2}). \tag{4.6}$$

The proof of Theorem 4.1 involves, as before in Theorem 3.1, showing that the three operators

$$A^{1/2}P(\lambda, A, B)^{-1}A^{1/2}, \quad \lambda A^{1/2}P(\lambda, A, B)^{-1}, \quad \lambda^2 P(\lambda, A, B)^{-1} \tag{4.7}$$

are uniformly bounded in the sector

$$\sum \left(\tan^{-1} \vartheta + \frac{\pi}{2} \right) = \left\{ \lambda \in \mathbb{C} \mid |\arg \lambda| \leq \frac{\pi}{2} + \tan^{-1} \vartheta \right\}. \tag{4.8}$$

To this end, we establish

LEMMA 4.2. Given (i)–(iii), (4.5), there exist positive constants c_1, c_2 such that

$$\|P(\lambda, A, B)x\|^2 \geq c_1 |\lambda|^4 \|x\|^2 + c_2 \|Ax\|^2, \quad x \in \mathcal{D}(A), \lambda \in \sum \left(\tan^{-1} \vartheta + \frac{\pi}{2} \right) \tag{4.9}$$

if ϑ is sufficiently small.

Proof. Let $\lambda = \xi + i\eta$. Then for $x \in \mathcal{L}(A)$

$$\begin{aligned}
 (P(\lambda, A, B)x, P(\lambda, A, B)x) &= ((\lambda^2 I + \lambda B + A)x, (\lambda^2 I + \lambda B + A)x) \\
 &= |\lambda|^4 \|x\|^2 + |\lambda|^2 \|Bx\|^2 + \|Ax\|^2 + 2\xi |\lambda|^2 (x, Bx) + 2(\xi^2 - \eta^2)(x, Ax) \\
 &\quad + \xi[(Bx, Ax) + (Ax, Bx)] + i\eta[-(Ax, Bx) + (Bx, Ax)] \\
 &\equiv T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7.
 \end{aligned} \tag{4.10}$$

We divide the discussion into the treatment of two cases.

Case I. $\text{Re } \lambda = \xi \geq 0$. In this case, T_1, T_2, T_3, T_4 are all non-negative. For T_5 we have, using (4.2) twice,

$$2(\xi^2 - \eta^2)(x, Ax) \geq \left(\frac{2\xi^2}{\rho_2^2} - \frac{2\eta^2}{\rho_1^2} \right) \|Bx\|^2,$$

while for T_6 , with ε such that $0 < k\varepsilon < 1$,

$$|\xi[(Bx, Ax) + (Ax, Bx)]| \leq 2\xi \|Ax\| \|Bx\| \leq \frac{\xi^2}{1 - k\varepsilon} \|Bx\|^2 + (1 - k\varepsilon) \|Ax\|^2, \tag{4.11}$$

and for T_7 , with $\tilde{\varepsilon} > 0$, using (4.5),

$$\begin{aligned}
 |i\eta[-(Ax, Bx) + (Bx, Ax)]| &\leq k|\eta| \|A^{1/2}x\| \|Ax\| \leq \frac{k}{4\tilde{\varepsilon}} \eta^2 \|A^{1/2}x\|^2 + k\tilde{\varepsilon} \|Ax\|^2 \\
 &\leq \frac{k}{4\tilde{\varepsilon}\rho_1^2} \eta^2 \|Bx\|^2 + k\tilde{\varepsilon} \|Ax\|^2.
 \end{aligned} \tag{4.12}$$

Combining these, we see that (4.10) satisfies

$$\begin{aligned}
 \sum_{i=1}^7 T_i &\geq \sum_{i=1}^4 T_i + \left(\frac{2\xi^2}{\rho_2^2} - \frac{2\eta^2}{\rho_1^2} \right) \|Bx\|^2 - \frac{\xi^2}{1 - k\varepsilon} \|Bx\|^2 \\
 &\quad - (1 - k\varepsilon) \|Ax\|^2 - \frac{k}{4\tilde{\varepsilon}\rho_1^2} \eta^2 \|Bx\|^2 - k\tilde{\varepsilon} \|Ax\|^2 \\
 &\geq |\lambda|^4 \|x\|^2 + \left\{ \left(1 + \frac{2}{\rho_2^2} - \frac{1}{1 - k\varepsilon} \right) \xi^2 + \left(1 - \frac{2}{\rho_1^2} - \frac{k}{4\tilde{\varepsilon}\rho_1^2} \right) \eta^2 \right\} \|Bx\|^2 \\
 &\quad + k(\varepsilon - \tilde{\varepsilon}) \|Ax\|^2.
 \end{aligned} \tag{4.13}$$

Let ε be such that $0 < k\varepsilon < 1$ and (i), (ii) are satisfied. Clearly, if $0 < \tilde{\varepsilon} < \varepsilon$ and $\tilde{\varepsilon}$ is sufficiently close to ε , the coefficient of $\|Bx\|^2$ is nonnegative.

Case II. $\text{Re } \lambda = \xi < 0, \lambda \in \sum (\tan^{-1} \vartheta + (\pi/2))$, i.e., $-\xi \leq \vartheta |\eta|$. In this case only T_1, T_2, T_3 are obviously non-negative. We have, for $0 < \tilde{\varepsilon} < \varepsilon < 1, 0 < k\varepsilon < 1$,

$$|T_4| = 2|\xi| |\lambda|^2 (x, Bx) \leq (1 - \varepsilon) |\lambda|^4 \|x\|^2 + \frac{\xi^2}{1 - \varepsilon} \|Bx\|^2,$$

$$T_5 \geq \frac{2\xi^2}{\rho_2^2} - \frac{2\eta^2}{\rho_1^2} \|Bx\|^2,$$

$$|T_6| \leq \frac{\xi^2}{1 - k\varepsilon} \|Bx\|^2 + (1 - k\varepsilon) \|Ax\|^2 \quad (\text{cf. (4.11)}),$$

$$|T_7| \leq \frac{k}{4\tilde{\varepsilon}\rho_1^2} \eta^2 \|Bx\|^2 + k\tilde{\varepsilon} \|Ax\|^2 \quad (\text{cf. (4.12)}).$$

Hence

$$\begin{aligned}
 \sum_{i=1}^7 T_i &\geq \sum_{i=1}^3 T_i - (1 - \varepsilon) |\lambda|^4 \|x\|^2 - \frac{\xi^2}{1 - \varepsilon} \|Bx\|^2 \\
 &\quad + \frac{2\xi^2}{\rho_2^2} - \frac{2\eta^2}{\rho_1^2} \|Bx\|^2 - \frac{\xi^2}{1 - k\varepsilon} \|Bx\|^2 - (1 - k\varepsilon) \|Ax\|^2 - \frac{k\eta^2}{4\tilde{\varepsilon}\rho_1^2} \|Bx\|^2 - k\tilde{\varepsilon} \|Ax\|^2 \\
 &= \varepsilon |\lambda|^4 \|x\|^2 + \left[\left(1 - \frac{2}{\rho_1^2} - \frac{k}{4\tilde{\varepsilon}\rho_1^2} \right) \eta^2 + \left(1 - \frac{1}{1 - \varepsilon} + \frac{2}{\rho_2^2} - \frac{1}{1 - k\varepsilon} \right) \xi^2 \right] \\
 &\quad \times \|Bx\|^2 + k(\varepsilon - \tilde{\varepsilon}) \|Ax\|^2 \geq \varepsilon |\lambda|^4 \|x\|^2 \\
 &\quad + \left[\left(1 - \frac{2}{\rho_1^2} - \frac{k}{4\tilde{\varepsilon}\rho_1^2} \right) - \left(\frac{1}{1 - \varepsilon} + \frac{1}{1 - k\varepsilon} - \left(1 + \frac{2}{\rho_2^2} \right) \right) \eta^2 \right] \|Bx\|^2 \\
 &\quad \quad \quad + k(\varepsilon - \tilde{\varepsilon}) \|Ax\|^2 \geq \varepsilon |\lambda|^4 \|x\|^2 + k(\varepsilon - \tilde{\varepsilon}) \|Ax\|^2 \quad (4.14)
 \end{aligned}$$

if $\lambda \in \sum (\tan^{-1} \vartheta + (\pi/2))$, (i)–(iii) hold, $0 < \tilde{\varepsilon} < \varepsilon$ and $\tilde{\varepsilon}$ is sufficiently close to ε .

Combining (4.13), (4.14) and taking $c_1 = \min(1, \varepsilon) = \varepsilon$, $c_2 = k(\varepsilon - \tilde{\varepsilon})$, we see that (4.9) holds and the proof of Lemma 4.2 is complete.

Proof of Theorem 4.1. As we noted before Lemma 4.2, we must establish that the operators (4.7) are uniformly bounded in the sector (4.8). For the third operator in (4.7), the result is now immediate.

$$\|\lambda^{-2} P(\lambda, A, B)x\|^2 = \|P(\lambda, A, B)(\lambda^{-2}x)\|^2 \geq c_1 |\lambda|^4 \|\lambda^{-2}x\|^2 + c_2 \|A(\lambda^{-2}x)\|^2 \geq c_1 \|x\|^2. \quad (4.15)$$

To treat the first operator, we note that with the transformation

$$\zeta = 1/\lambda,$$

which may be seen to leave any sector (4.8) invariant, and with

$$B_1 = A^{-1/2}BA^{-1/2}, \quad A_1 = A^{-1},$$

we have

$$A^{-1/2}P(\lambda, A, B)A^{-1/2} = \lambda^2 P(\zeta, A_1, B_1). \quad (4.16)$$

We verify that $P(\zeta, A_1, B_1)$ satisfies the hypotheses of Lemma 4.2. From the second inequality in (4.4) it follows that

$$\|A^{-1/2}B\| = \|(BA^{-1/2})^*\| = \|BA^{-1/2}\| \leq \rho_2 \Rightarrow \|A^{-1/2}Bx\|^2 \leq \rho_2^2 \|x\|^2. \quad (4.17)$$

The first inequality gives, in the same way,

$$\|B^{-1}A^{1/2}\| = \|(A^{1/2}B^{-1})^*\| = \|A^{1/2}B^{-1}\| \leq 1/\rho_1$$

so that, with $y = A^{-1/2}Bx$,

$$\|B^{-1}A^{1/2}y\|^2 \leq \frac{1}{\rho_1^2} \|y\|^2 \Rightarrow \|A^{-1/2}Bx\|^2 \geq \rho_1^2 \|x\|^2. \quad (4.18)$$

Combining (4.17) and (4.18), we have

$$\rho_1^2 \|x\|^2 \leq \|A^{-1/2}Bx\|^2 \leq \rho_2^2 \|x\|^2. \quad (4.19)$$

Since $A^{-1/2}B = A^{-1/2}BA^{-1/2}A = B_1 A_1^{-1/2}$, we see that $B_1 A_1^{-1/2}$ is bounded on the domain of $A^{1/2}$ and, since that domain is dense in X , $B_1 A_1^{-1/2}$ extends to a bounded operator on X , which we still call $B_1 A_1^{-1/2}$. Then we note that, just as we obtained (4.5), we have, readjusting k in (4.5) if necessary,

$$|(A_1 x, B_1 x) - (B_1 x, A_1 x)| \leq k \|A_1^{1/2} x\| \|A_1 x\| \tag{4.20}$$

Proceeding as in Lemma 4.2, we see that if the inequalities (i)-(iii) are satisfied we have

$$\|P(\zeta, A_1, B_1)x\|^2 \geq c_1 \|\zeta\|^4 \|x\|^2 + c_2 \|A_1 x\|^2$$

so that

$$\|\lambda^2 P(\zeta, A_1, B_1)x\|^2 \geq c_1 \|x\|^2 \quad \left(\text{recall } \lambda = \frac{1}{\zeta}\right).$$

But, as we have noted in (4.16), this is the same as

$$\|(A^{-1/2}P(\lambda, A, B)A^{-1/2})x\|^2 \geq c_1 \|x\|^2$$

and we conclude that the first operator in (4.7) is bounded uniformly in the sector $\sum (\tan^{-1} \vartheta + \pi/2)$.

Now consider the second operator in (4.7). We have seen (cf. (4.9)) that

$$\|P(\lambda, A, B)x\|^2 \geq c_1 |\lambda|^4 \|x\|^2 + c_2 \|Ax\|^2. \tag{4.21}$$

Let us note that the hypothesis that A is self-adjoint and positive implies the same for $A^{1/2}$. Then it is easy to establish that for $\lambda \in \sum (\tan^{-1} \vartheta + \pi/2)$ the range of $\lambda I + A^{1/2}$ is the whole space X . With

$$x = (A^{-1/2} + \lambda^{-1}I)y$$

(4.21) gives

$$\|P(\lambda, A, B)(A^{-1/2} + \lambda^{-1}I)y\| \geq c_1 |\lambda|^4 \|(A^{-1/2} + \lambda^{-1}I)y\|^2 + c_2 \|A(A^{-1/2} + \lambda^{-1}I)y\|^2$$

so that

$$\begin{aligned} \|\lambda^{-1}P(\lambda, A, B)A^{-1/2}(\lambda I + A^{1/2})y\| &\geq c_1 |\lambda|^2 \|(I + \lambda A^{-1/2})y\|^2 + c_2 \|(A^{1/2} + \lambda^{-1}A)y\|^2 \\ &\geq \tilde{c}_1 |\lambda|^2 \|y\|^2 + \tilde{c}_2 \|A^{1/2}y\|^2 \geq \hat{c} \|(\lambda I + A^{1/2})y\|^2, \quad \hat{c} > 0, \end{aligned} \tag{4.22}$$

provided we can establish that there are positive numbers \hat{c}_1, \hat{c}_2 such that for $\lambda \in \sum (\tan^{-1} \vartheta + \pi/2)$,

$$\|(I + \lambda A^{-1/2})y\|^2 \geq \hat{c}_1 \|y\|^2, \quad y \in X, \tag{4.23}$$

$$\|(A^{1/2} + \lambda^{-1}A)y\|^2 \geq \hat{c}_2 \|A^{1/2}y\|^2, \quad y \in \mathcal{D}(A^{1/2}). \tag{4.24}$$

This can again be done with the aid of the spectral representation for A which, being self-adjoint and positive, can be written as

$$A = \int_0^\infty \mu dE(\mu),$$

where $E(\mu)$ is the spectral family for A . Then

$$\|(I + \lambda A^{-1/2})y\|^2 = \int_0^\infty |1 + \lambda \mu^{-1/2}|^2 d\|E(\mu)y\|^2.$$

Since $\mu^{-1/2} > 0$, $\lambda\mu^{-1/2} \in \sum (\tan^{-1} \vartheta + \pi/2)$ and

$$|1 + \lambda\mu^{-1/2}|^2 \geq \cos^2(\tan^{-1}(\vartheta)) = \frac{1}{1 + \vartheta^2}.$$

Then

$$\|(I + \lambda A^{-1/2})y\|^2 \geq \int_0^\infty \left(\frac{1}{1 + \vartheta^2}\right) d\|E(\mu)y\|^2 = \frac{1}{1 + \vartheta^2} \|y\|^2$$

and we can take $\hat{c}_1 = 1/(1 + \vartheta^2)$ in (4.23). In the case of (4.24) similar considerations lead to examination of the ratio

$$\left| \frac{\mu^{1/2} + \lambda^{-1}\mu}{\mu^{1/2}} \right|^2 = |1 + \lambda^{-1}\mu^{1/2}|^2 \geq \frac{1}{1 + \vartheta^2}$$

because $\lambda^{-1}\mu^{1/2} \in \sum (\tan^{-1} \vartheta + (\pi/2))$. Thus (4.24) is satisfied also with $\hat{c}_2 = 1/(1 + \vartheta^2)$.

Since $\mathcal{R}(\lambda I + A^{1/2}) = X$, (4.22) shows that $A^{1/2}P(\lambda, A, B)^{-1}\lambda$ is uniformly bounded for $\lambda \in \sum (\tan^{-1} \vartheta + (\pi/2))$. Since this sector is invariant under conjugation, we conclude that $A^{1/2}P(\bar{\lambda}, A, B)^{-1}\bar{\lambda} = [\lambda P(\lambda, A, B)^{-1}A^{1/2}]^*$ is uniformly bounded for that $\lambda \in \sum (\tan^{-1} \vartheta + (\pi/2))$ and then $\lambda P(\lambda, A, B)^{-1}A^{1/2}$ is uniformly bounded in that sector.

At this point we have established the uniform boundedness of the three operators (4.7) in $\sum (\tan^{-1} \vartheta + (\pi/2))$. From our earlier observations we conclude that \mathcal{L}_B generates a holomorphic semigroup if our conditions are satisfied, completing the proof of the theorem.

COROLLARY 4.1. Let C_1, C_2 be two compact operators on X . Let B satisfy the assumptions of Theorem 3.1 or 4.1 so that \mathcal{L}_B generates a holomorphic semigroup. Then the operator

$$\begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B + C_1A^{1/2} + A^{1/2}C_2 \end{pmatrix}$$

also generates a holomorphic semigroup.

Proof: From a result in [10], the operator

$$\begin{pmatrix} I & 0 \\ C_1 & I \end{pmatrix} \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & C_1A^{1/2} - B \end{pmatrix}$$

generates a holomorphic semigroup. By the same theorem,

$$\begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & C_1A^{1/2} - B \end{pmatrix} \begin{pmatrix} I & -C_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B + C_1A^{1/2} + A^{1/2}C_2 \end{pmatrix}$$

also generates a holomorphic semigroup.

Examples. Let $X = L^2(0, l)$ and

$$A = d^4/dx^4 \quad \text{with } D(A) = \{w \in H^4(0, l) \mid w(0) = w(l) = w''(0) = w''(l) = 0\}.$$

Let B be defined by

$$Bw = -\frac{d}{dx} \left[(\rho_1 + kb_1(x)) \frac{dw}{dx} \right] + kb_2(x)w, \quad \rho_1 > 0$$

where b_1, b_2 are sufficiently smooth functions on $[0, l]$ and $k \geq 0$, with

$$D(B) = \{w \in H^2(0, l) \mid w(0) = w(l) = 0\}.$$

Then B is positive, self-adjoint and satisfies

$$\rho_1^2 A \leq B^2 \leq \rho_2^2 A$$

for some $\rho_2 \geq \rho_1$.

If $b_1(x)$ and $b_2(x)$ are not identically zero then A and B do not commute in general. We have, for $w \in D(A^{3/2})$,

$$\begin{aligned} (AB - BA)w &= \frac{d^4}{dx^4} \left\{ -\frac{d}{dx} \left[(\rho_1 + kb_1(x)) \frac{dw}{dx} \right] + kb_2(x)w \right\} \\ &\quad + \left\{ \frac{d}{dx} \left[(\rho_1 + kb_1(x)) \frac{d^5 w}{dx^5} - kb_2 \frac{d^4 w}{dx^4} \right] \right\} \\ &= k \left[-4b_1'(x) \frac{d^5 w}{dx^5} - 10b_1''(x) \frac{d^4 w}{dx^4} - (10b_1^{(3)}(x) - 4b_2'(x)) \frac{d^3 w}{dx^3} \right. \\ &\quad \left. - (5b_1^{(4)}(x) - 6b_2''(x)) \frac{d^2 w}{dx^2} - (b_1^{(5)}(x) - 4b_2^{(3)}(x)) \frac{dw}{dx} \right]. \end{aligned}$$

Since $A^{3/2} = -d^6/dx^6$, from interpolation one easily sees that the commutator condition (4.6) is satisfied provided that $b_1^{(5)}(x)$ and $b_2^{(3)}(x)$ are continuous functions on $[0, l]$ and that k is appropriate.

For any $w \in X$, define $C_1, C_2: X \rightarrow X$ by

$$(C_i w)(x) = \int_0^l c_i(x, \xi)w(\xi) d\xi, \quad i = 1, 2$$

where the kernels $c_1(x, \xi), c_2(x, \xi)$ are Hilbert-Schmidt, and $(\partial^2/\partial x^2)c_2(x, \xi)$ is continuous in x for almost all $\xi \in [0, l]$. By Corollary 4.1, we know that

$$\begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B + C_1 A^{1/2} + A^{1/2} C_2 \end{pmatrix}$$

generates a holomorphic semigroup.

REFERENCES

[1] R. P. Boas, *A trigonometric moment problem*, J. London Math. Soc. **14**, 242–244 (1939)
 [2] N. Dunford and J. T. Schwartz, *Linear operators*, Interscience, New York, 1957
 [3] Y. C. Fung, *Foundations of solid mechanics*, Prentice-Hall, Englewood Cliffs, 1965
 [4] W. C. Hurty and M. F. Rubinstein, *Dynamics of structures*, Prentice-Hall, Englewood Cliffs, 1964
 [5] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966
 [6] J. L. Lions, *Optimal control of systems governed by partial differential equations*, Springer-Verlag, New York, 1971
 [7] R. S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc. **90**, 123–254 (1959)
 [8] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations*, SIAM Review **20**, 639–739 (1978)
 [9] R. H. Scanlon and R. Rosenbaum, *Introduction to the theory of aircraft vibration and flutter*, Macmillan, New York, 1951
 [10] J. Zabczyk, *On decomposition of generators*, SIAM J. Cont. Opt. **16**, 523–534 (1978)