THE STEFAN PROBLEM WITH A CONVECTIVE BOUNDARY CONDITION*

By

A. D. SOLOMON (Union Carbide Corporation, Oak Ridge)
V. ALEXIADES (The University of Tennessee, Knoxville)
D. G. WILSON (Union Carbide Corporation, Oak Ridge)

Abstract. We study the one-phase Stefan problem on a semi-infinite strip \( x \geq 0 \), with the convective boundary condition \(-K T_x(0, t) = h[T_L - T(0, t)]\). Points of interest include: a) behavior of the surface temperature \( T(0, t) \); b) asymptotic behavior as \( h \to \infty \); c) uniqueness, and d) bounds on the phase change front and total system energy.

Introduction. In this paper we study the following problem:

**Problem I.** Find \( X(t) \) and \( T(x, t) \) such that

\[
\begin{align*}
X(t) & \text{ is Lipschitz-continuous for } t \geq 0; \quad (1.1) \\
X'(t) & \text{ is continuous for } t > 0; \quad (1.2) \\
T(x, t) & \text{ is continuous for } t > 0 \text{ and } 0 < x < X(t); \quad (1.3) \\
T_t(x, t), T_{xx}(x, t) & \text{ are continuous for } t > 0 \text{ and } 0 < x < X(t); \quad (1.4) \\
-\infty < \lim \inf_{x, t \to 0} T(x, t), \lim \sup_{x, t \to 0} T(x, t) & < \infty; \quad (1.5) \\
T_s(x, t) & \text{ is continuous for } t > 0, 0 \leq x \leq X(t); \quad (1.6) \\
X(t) \text{ and } T(x, t) & \text{ obey the conditions} \\
T_s(x, t) & = \alpha T_{xx}(x, t), t > 0, 0 < x < X(t), \quad (1.7) \\
T(x, t) & \equiv T_{cr}, t > 0, x \geq X(t), \quad (1.8) \\
X(0) & = 0, \quad (1.9) \\
\rho H X'(t) & = -K T_s(X(t), t), \quad (1.10) \\
-\infty < \lim \inf T_s(0, t), \lim \sup T_s(0, t) & < \infty; \quad (1.11) \\
\end{align*}
\]

Here \( \alpha, \rho, H, K, h \) are positive constants, \( T_s \) and \( T_{cr} \) are constants, and \( T_L > T_{cr} \).

Eqs. (1.7-1.11) describe melting of a material due to convective heat transfer from a fluid with ambient temperature \( T_L \) flowing across the face at \( x = 0 \). The parameters are:

\( \alpha = \text{material thermal diffusivity (m}^2\text{/s}) \),
\( K = \text{material thermal conductivity (KJ/m - s - °C}) \),

\[ \rho = \text{material density (Kg/m}^3\text{)}, \]
\[ H = \text{latent heat of melting (KJ/Kg)}, \]
\[ T_{cr} = \text{material melting temperature (°C)}, \]
\[ T_f = \text{ambient fluid temperature (°C)}, \]
\[ h = \text{fluid to material surface heat transfer coefficient (KJ/m}^2\text{ – s – °C).} \]

We will also use
\[ \Delta T = T_f - T_{cr} (°C), \]
and the material specific heat
\[ c = \text{specific heat (KJ/Kg – °C)}. \]

Of course \( \alpha = K/c\rho \). The melting front at time \( t \) is at \( x = X(t) \) while \( T(x, t) \) is the temperature at position \( x \) and time \( t \).

It is known \([3]\) that a solution to Problem I exists. While a number of papers in the heat transfer literature are devoted to various approximations pertinent to this problem \([5, 8-10]\), the only studies of the qualitative behavior of its solution concern existence and smoothness \([7, 12]\), in addition to \([3]\). We will study the qualitative behavior of a solution, focusing on questions pertinent to the melting (or solidification) problem from which it arises. These include

1. **Question 1.** How do \( T(x, t), X(t) \) behave at \( t = 0? \)
2. **Question 2.** How does the surface temperature \( T(0, t) \) vary with \( t? \)
3. **Question 3.** What happens as \( h \to \infty? \)

On physical grounds it would be expected that the surface temperature \( T(0, t) \) would tend to \( T_{cr} \) as \( t \to 0^+ \), and to the fluid temperature \( T_f \) as \( t \to \infty \). Similarly, \((1.10)\) and \((1.11)\) would lead us to conjecture that \( X'(0^+) \) exists and is given by
\[ X'(0^+) = h(T_f - T_{cr})/\rho H. \]

The situation whereby \( h \to \infty \) could arise from a greater flow rate for the transfer fluid at \( x = 0 \) \([6]\), in which case we would expect that \( T(0, t) \to T_f \); in this case we would also anticipate that the solution to problem I should tend to that of the problem with \((1.11)\) replaced by
\[ T(0, t) = T_f, \quad t > 0. \] \quad (1.12)

This latter problem \((1.1)-(1.10), (1.12)\), will be referred to as Problem II, and its exact solution is given by
\[ X_x(t) = 2\lambda\sqrt{\alpha t}, \quad t \geq 0, \] \quad (1.13)
\[ T_x(x, t) = T_f - \Delta T \operatorname{erf}(x/2\sqrt{\alpha t})/\operatorname{erf} \lambda \] \quad (1.14)

with \( \lambda \) the root of
\[ \lambda e^{\lambda^2} \operatorname{erf} \lambda = \text{St}/\sqrt{\pi}. \] \quad (1.15)

Here \( \text{St} \) is the so-called “Stefan number”, indicating the ratio of sensible to latent heat \([11]\), and given by
\[ \text{St} = c\Delta T/H. \]
Our aim is to establish these claims. To do this we use a number of moment-type relations as well as the maximum principle. These are discussed in Sec. 2. In Sec. 3 we address questions 1 and 2; what happens as \( h \to \infty \) is examined in Sec. 4. We close in Sec. 5, with upper and lower bounds on the total heat in the material. In the Appendix we prove a form of the maximum principle which we use.

2. Preliminaries. The maximum principle for the heat equation is normally used in two forms [4]. The first asserts that a solution to the heat equation cannot attain its greatest or least value at an interior point \( P_0 \) of a domain unless it equals that value at all points influencing \( P_0 \). The second, due to Friedman, concerns the behavior of a nontangent temperature derivative at a boundary point. As stated in [4] it presents some difficulty due to the assumed “strong-sphere” property of the boundary. For this reason we use the following version of the maximum principle suggested by a result of Vyborny [13].

**Theorem 1. Corner Point Maximum Principle.** Let \( D \) be a simply connected domain in the \( x, t \) plane and \( P_0 = (x_0, t_0) \) a point of its boundary. Let \( N \) be the disk

\[
N = \{(x, t) \mid (x - x_0)^2 + (t - t_0)^2 < \delta^2\}.
\]

Set

\[
G^0 = D \cap N \cap \{(x, t) \mid t < t_0\}, \quad \bar{G}^0 = \bar{G}^0 - \partial D.
\]

Suppose that \( u \in C(\bar{D}), u_x, u_t, u_{xx} \in C(D) \), and

\[
Lu = u_t - \alpha u_{xx} \leq 0 \tag{2.1}
\]

in \( D \). Furthermore, let

\[
u(P) < u(P_0) \quad \text{for} \quad P \in \bar{G}^0, \tag{2.2}
\]

\[
u(P) \leq u(P_0) \quad \text{for} \quad P \in \partial D \cap N, \tag{2.3}
\]

and suppose that \( \partial D \cap N \) is a \( C^1 \) curve representable as \( x = X(t) \). Then

\[
\lim_{P \to P_0} \frac{u(P) - u(P_0)}{|P - P_0|} < 0
\]

where \( P \) tends to be \( P_0 \) in any nontangential direction.

The proof of this theorem is given in the Appendix.

**Corollary 1.** If all of the conditions of Theorem 1 hold except for (2.2), (2.3), and if

\[
\lim_{P \to P_0} \frac{(u(P) - u(P_0))}{|P - P_0|} \geq 0,
\]

then either

a) there exist points \( P = (x, t) \) in \( G^0 \) arbitrarily close to \( (x_0, t_0) \), with \( t \leq t_0 \), for which \( u(P) \geq u(P_0) \)

or

b) there exist points \( P \) on \( \partial D \cap N \) arbitrarily close to \( P_0 \) for which \( u(P) > u(P_0) \).

Reversing the inequalities in Theorem 1 and Corollary 1 yields the corresponding corner-point minimum principle.
We will use a number of integral relations satisfied by a solution $X(t)$, $T(x,t)$ to Problem I. From the continuity of $T_x(x,t)$ in any region $t \geq \tau > 0$ we find
\begin{align*}
\int_{\tau}^{t} \rho H [X(t) - X(\tau)] d\tau' + \int_{0}^{X(t)} \rho c x [T(x,t) - T_{cr}] dx - \int_{0}^{X(t)} \rho c x [T(x,\tau) - T_{cr}] dx,
\end{align*}
\begin{align*}
\int_{0}^{X(t)} \rho c x [T(x,t) - T_{cr}] dx - \int_{0}^{X(t)} \rho c x [T(x,\tau) - T_{cr}] dx,
\end{align*}
\begin{align*}
+ (\rho H/2) \left[ X^2(t) - X^2(\tau) \right] = K \int_{\tau}^{t} \left[ T(0, t') - T_{cr} \right] dt',
\end{align*}
\begin{align*}
\int_{0}^{X(t)} \left( \rho c/2 \right) \left[ T(x,t) - T_{cr} \right]^2 dx - \int_{0}^{X(t)} \left( \rho c/2 \right) \left[ T(x,\tau) - T_{cr} \right]^2 dx,
\end{align*}
\begin{align*}
+ \int_{0}^{t} \int_{0}^{X(t')} K T_x(x,t')^2 dx dt' = \int_{\tau}^{t} h[T(0, t')][T(0, t') - T_{cr}] dt'.
\end{align*}

For example, (2.4) is derived as follows. Let $\theta$ be any value between 0 and 1/2. Consider the closed domain
\begin{align*}
D_0 = \{(x',t') | \tau \leq t' \leq t, \theta X(t') \leq x' \leq (1-\theta)X(t') \}
\end{align*}
By the conditions (1.1)–(1.11) we find
\begin{align*}
(d/dt') \int_{(1-\theta)X(t')}^{X(t')} \rho c [T(x', t') - T_{cr}] dx' = \theta X(t') \rho c [T[(1-\theta)X(t'), t'] - T_{cr}]
- \theta X(t') \rho c [T[\theta X(t'), t] - T_{cr}] + K T_x[[(1-\theta)X(t'), t'] - K T_x[\theta X(t'), t']).
\end{align*}
Integrating this equation with respect to $t'$ on $[\tau, t]$ and letting $\theta \to 0$ yields (2.4). Relations (2.5), (2.6) are derived similarly. From the boundedness of $T(x, t')$ and the fact that $X(t) \to 0$ as $t \to 0^+$, we conclude that in (2.4)
\begin{align*}
\int_{0}^{X(t)} \rho c [T(x, \tau) - T_{cr}] dx \to 0
\end{align*}
as $\tau \to 0$, whence
\begin{align*}
\int_{0}^{t} h[T(0, t') dt' = \rho H X(t) + \int_{0}^{X(t)} \rho c [T(x, t) - T_{cr}] dx,
\end{align*}
which is the overall heat balance relation on $[0, t]$. In the same way (2.5) implies
\begin{align*}
\int_{0}^{X(t)} \rho c x [T(x, t) - T_{cr}] dx + (\rho H/2) X(t)^2 = K \int_{0}^{t} [T(0, t') - T_{cr}] dt'.
\end{align*}
Consider (2.6). By elementary calculus
\begin{align*}
[T_L - T(0, t')][T(0, t') - T_{cr}] \leq (1/4) (T_L - T_{cr})^2.
\end{align*}
Hence

\[
\int_0^{X(t)} \frac{(c\rho/2)}{(c\rho/2)} (T(x, t) - T_{cr})^2dx - \int_0^{X(t)} \frac{(c\rho/2)}{(c\rho/2)} (T(x, \tau) - T_{cr})^2dx
\]
\[+ \int_0^t \int_0^{X(t')} K T_x(x, t')^2 dx' dt' \leq \frac{[h(T_L - T_{cr})^2/4]}{t - \tau}. \tag{2.9}
\]

Letting \( \tau \to 0 \) we conclude that

\[
\int_0^{X(t)} \frac{(c\rho/2)}{(c\rho/2)} (T(x, t) - T_{cr})^2dx + \int_0^t \int_0^{X(t')} K T_x(x, t')^2 dx' dt' \leq \frac{[h(T_L - T_{cr})^2/4]}{t}, \tag{2.11}
\]

and, in particular, that

\[
\int_0^t \int_0^{X(t')} K T_x(x, t')^2 dx' dt' \leq t[h(T_L - T_{cr})^2/4].
\]

### 3. The qualitative behavior of a solution

We now address the qualitative behavior of a solution to Problem I for a fixed \( h > 0 \).

**Theorem 2.** The phase boundary \( X(t) \) solving Problem I is always positive: \( X(t) > 0 \) for \( t > 0 \).

*Proof.* Since \( X(t) \geq 0 \) for all \( t \geq 0 \), a point \( t_0 > 0 \) for which \( X(t_0) = 0 \) must be a zero of the (continuous) derivative \( X'(t) \). However, we would then have \( T(0, t_0) = T_{cr} \), whence

\[
0 = \rho H X'(t_0) = -K T_x[X(t_0), t_0] = -K T_x(0, t_0) = h(T_L - T_{cr}) \neq 0
\]

and the theorem is proved.

**Theorem 3.** \( T_{cr} \leq T(x, t) < T_L \) for \( t > 0 \) and \( 0 \leq x \leq X(t) \).

Our proof rests upon the following lemma, which asserts that \( T(x, t) \) cannot be bounded away from \( T_{cr} \) in a neighborhood of the origin.

**Lemma 1.** Let \( t_0 > 0 \) be given. There is no function \( x = x^*(t) \) satisfying the following conditions on \( (0, t_0] \): a) \( 0 \leq x^*(t) < X(t) \); b) \( 0 < \omega \leq |T[x^*(t), t] - T_{cr}| \) for some \( \omega \).

*Proof of Lemma 1.* Roughly speaking, we will see that if \( T(x, t) \) is bounded away from \( T_{cr} \), then \( T_x(x, t) \) must grow in a manner inconsistent with the bound (2.11).

For suppose that \( x^*(t) \) satisfies (a) and (b), and let \( t \in (0, t_0] \). Since \( T_x(x, t) \) is continuous on \([0, X(t)]\),

\[
|T_{cr} - T[x^*(t), t]| \leq \int_{x^*(t)}^{X(t)} |T_x(x, t)| dx \leq \{X(t) \int_0^{X(t)} T_x(x, t)^2 dx\}^{1/2}
\]

or by (b),

\[
\omega^2 \leq X(t) \int_0^{X(t)} T_x(x, t)^2 dx.
\]

Integration over \([t/2, t]\) for any \( t \leq t_0 \) yields

\[
(\omega^2 t/2) \leq \int_{t/2}^t X(t') \int_0^{X(t')} T_x(x, t')^2 dx' dt'.
\]
By the generalized mean value theorem,
\[(\omega^2 t/2) \leq X(t^*) \int_{t/2}^{t} \int_{0}^{X(t')} T_x(x, t')^2 \, dx \, dt'\]
where \(t^* \in [t/2, t]\). Now by (2.11) we obtain
\[\frac{1}{2} J X(t^*)^{1/2} \leq \int_{t/2}^{t} \int_{0}^{X(t')} T_x(x, t')^2 \, dx \, dt'\]
This contradicts \(X(t^*) \to 0\) as \(t \to 0\) and thus proves the lemma.

Proof of Theorem 3. We begin by showing that \(T(x, t)\) must be less than \(T_L\) for all points \((x, t)\) with \(t > 0, 0 \leq x \leq X(t)\). Suppose that \(T(x_0, t_0) > T_L\) for some \(t_0 > 0\) and \(x_0 \in [0, X(t_0)]\).

Claim: For each \(t_1 \in (0, t_0)\) there is some \(x^* = x^*(t_1) \in [0, X(t)]\) such that \(T(x^*(t_1), t_1) > T(x_0, t_0)\).

Since this directly contradicts Lemma 1 we need only establish this claim. Fix \(t_1 \in (0, t_0)\) and let
\[S = \{t: t \in [t_1, t_0], T(0, t) > T_L\}.\]
If \(x_0 = 0\) then \(t_0 \in S\) and \(S\) is not empty. If \(x_0 > 0\) then, by the strong maximum principle [4] applied to \(D_1 = \{(x, t): 0 < x < X(t), t_1 \leq t \leq t_0\}\), \(T(x, t)\) must exceed \(T(x_0, t_0) > T_L\) somewhere on its parabolic boundary. If this occurs at some point \((x, t_1)\), \(x \in [0, X(t_1)]\) the claim is proved. If it occurs on \(x = 0\), i.e., for some \((0, t^*)\) with \(t^* \in [t_1, t_0]\), then \(t^* \in S\), so again \(S\) is not empty. Let
\[t^{**} = \inf S.\]
Suppose \(t^{**} > t_1\). Then \(T(0, t_1) < T_L\) while \(T(0, t^{**}) = T_L\), whence \(-K T_x(0, t^{**}) = 0\) and by Corollary 1 to the corner point maximum principle either there exist points \((x, t)\) arbitrarily close to \((0, t^{**})\) with \(t \leq t^{**}\) for which \(T(x, t) \geq T_L\), or there exists some \(t < t^{**}\) for which \(T(0, t) > T_L\). Either possibility violates the definition of \(t^{**}\) and thus \(t^{**} = t_1\). Hence \(T(0, t_1) \geq T_L\), and \(x^*(t_1) = 0\). Thus our claim is proved and \(T(x, t) < T_L\). The proof that \(T(x, t) \geq T_{cr}\) is carried out in a similar way, as we see by assuming that
\[T(x_0, t_0) \leq T_{cr} - \omega\]
for \(\omega > 0\) and some point \((x_0, t_0), x_0 \in [0, X(t_0)]\).

By the strong maximum principle we now have:

Corollary 2. \(T(x, t) > T_{cr}\) for \(t > 0, x \in (0, X(t))\).

This result implies that at any point \((X(t), t)\), \(T(x, t)\) assumes a strictly minimum value relative to points to its left. Hence by the corner point minimum principle \(\rho H X'(t) = -K T_x(X(t), t) > 0\) and we have

Corollary 3. \(X'(t) > 0\) for \(t > 0\).

We will now use the moment-type relations of Sec. 2 to derive upper bounds on \(X(t)\).
Theorem 4. For any $t > 0$ the phase boundary of Problem I obeys the conditions:

$$X(t) \leq f_1(t) = h t \Delta T / \rho H,$$

$$X(t) \leq f_2(t) = \{2Kt \Delta T / \rho H\}^{1/2},$$

$$X(t) \leq \begin{cases} f_1(t), & t \leq t^* \\ f_2(t), & t \geq t^* \end{cases}$$

with

$$t^* = 2K \rho H / h^2 \Delta T.$$

Proof: From Theorem 3, $T(x, t) \geq T_{cr}$, whence (2.7) implies (3.1). Similarly $T(0, t) < T_L$ whence from (2.8), $\frac{1}{2} \rho H X(t)^2 \leq K t \Delta T$, or (3.2) is proved.

By a straightforward calculation we see that $f_1(t) \leq f_2(t)$ for $t \leq t^*$, and $f_1(t) \geq f_2(t)$ for $t \geq t^*$, whence (3.3) holds.

Note that the bound (3.3) indicates an initial linear growth in the phase front, followed by growth as $t^{1/2}$.

Theorem 5. $T(x, t)$ is nondecreasing in $t$; that is, $T(x, t + \Delta t) \geq T(x, t)$ for all $x \in [0, X(t)]$, $\Delta t > 0$.

Proof. The concept of the proof is to show that the first forward difference of $T(x, t)$ in $t$, namely

$$v(x, t, \Delta t) = T(x, t + \Delta t) - T(x, t)$$

is never negative for any choice of $\Delta t > 0$. To do this we note first that $v(x, t, \Delta t)$ is defined and satisfies the heat equation for $t > 0, x \in [0, X(t)]$. Moreover, by Corollary 2 and 3,

$$v(X(t), t, \Delta t) > 0.$$

At $x = 0$

$$K v_x(0, t, \Delta t) = h v(0, t, \Delta t).$$

Suppose now that $v(x, t, \Delta t)$ is negative at some point $(x_0, t_0)$:

$$v(x_0, t_0, \Delta t) \leq -\omega < 0.$$

By an identical argument to that used in proving Theorem 3 we conclude that for each $0 < t < t_0$ there is a point $x^* = x^*(t)$ for which

$$v(x^*(t), t, \Delta t) \leq -\omega.$$

Hence

$$|v(x^*(t), t, \Delta t) - v(X(t), t, \Delta t)| > \omega$$

for all $t \in (0, t_0)$. However, we may now apply the argument used in proving Lemma 1 to show that this violates (2.11) and the theorem is proved.

Corollary 4. $T(x, t) \rightarrow T_{cr}$ as $x, t \rightarrow 0^+$.

Proof. By the Theorem, $T(x, t)$ is nonincreasing for $t \rightarrow 0^+$. But then by Lemma 1 it cannot be bounded away from $T_{cr}$, whence it must tend to $T_{cr}$ as $x, t \rightarrow 0^+$. Thus, we can
now extend $T(x, t)$ continuously to $(0, 0^{+})$ and define it for $t \geq 0$, $x \in [0, X(t)]$ as a continuous function.

An immediate implication of Theorem 5 is that for $t > 0$, $x \in (0, X(t))$,

$$
\varepsilon T_{xx}(x, t) = T_{l}(x, t) \geq 0.
$$

(3.5)

This in turn implies the following theorem.

**Theorem 6.** Let $q(x, t) = -K T_{x}(x, t)$ for $t > 0$, $x \in [0, X(t)]$. Then

$$
\rho H X'(t) \leq q(x, t) \leq h[T_{L} - T(0, t)] \leq h\Delta T.
$$

(3.6)

**Proof.** For any $\theta \in (0, 1/2)$

$$
T_{x}(x, t) - T_{x}(\theta X(t), t) = \int_{\theta X(t)}^{x} T_{xx}(x', t) \, dx' \geq 0
$$

whence

$$
-K T_{x}(x, t) \leq -K T_{x}(\theta X(t), t);
$$

letting $\theta \to 0$ and using the continuity of $T_{x}(x, t)$ on the closed $x$-interval, we have

$$
q(x, t) = -K T_{x}(x, t) \leq -K T_{x}(0, t) = h[T_{L} - T(0, t)].
$$

The second inequality of (3.6) is proved in the same manner.

The key difficulty in understanding the convective boundary condition lies in the variability of the surface temperature $T(0, t)$. We will now obtain a bound on it describing its long-term behavior.

**Theorem 7.** For any $t > 0$, the surface temperature $T(0, t)$ of Problem I obeys

$$
0 < T_{l} - T(0, t) \leq (1 + St)(2K\rho H\Delta T)^{1/2}/ht^{1/2}.
$$

(3.7)

**Proof.** From the heat balance relation (2.7) and the fact that $T(x, t) \in [T_{r}, T_{l}]$,

$$
\int_{0}^{t} h[T_{L} - T(0, t')]dt' \leq H\rho X(t)[1 + St],
$$

and using the upper bound (3.2)

$$
X(t) \leq \{2Kt\Delta T/\rho H\}^{1/2}
$$

we find

$$
\int_{0}^{t} h[T_{L} - T(0, t')]dt' \leq \{2K\rho Ht\Delta T\}^{1/2}[1 + St].
$$

However, $T(0, t)$ is nondecreasing for increasing $t$, whence

$$
h(t[T_{L} - T(0, t)]) \leq \int_{0}^{t} h[T_{L} - T(0, t')]dt'
$$

and (3.7) holds true.

**Corollary 5.** $T(0, t) \to T_{l}$ as $t \to \infty$.

We will now further examine the behavior of $X(t)$, $T(x, t)$ at the origin.
Theorem 8. \( X(t) \) has a right-hand derivative at \( t = 0 \), given by

\[
X'(0^+) = \frac{h \Delta T}{\rho H}.
\] (3.8)

Proof: From (2.7)

\[
\frac{X(t)}{t} = \frac{h}{\rho H} \left( T_L - T(0, t') \right) dt' - \frac{c}{tH} \int_0^{x(t)} (T(x, t) - T_{cr}) dx.
\]

Since \( T(0, t) \) is continuous for \( t \geq 0 \),

\[
\frac{1}{t} \int_0^t (T_L - T(0, t')) dt' = T_L - T(0, \theta t), \quad \theta \in [0, 1].
\]

Moreover,

\[
\frac{c}{tH} \int_0^{x(t)} (T(x, t) - T_{cr}) dx = \frac{cX(t)}{Ht} \int_0^{x(t)} (1/X(t)) \left( \frac{1}{t} \int_0^{x(t)} (T(x, t) - T_{cr}) dx \right)
\]

\[
= \left( \frac{c}{H} \right) \frac{X(t)}{t} \left( T(x^*(t), t) - T_{cr} \right)
\]

for \( x^*(t) \in [0, X(t)] \). Hence

\[
\frac{X(t)}{t} \left( 1 + \frac{c}{H} [T(x^*(t), t) - T_{cr}] \right) = \left( \frac{h}{\rho H} \right) \left( T_L - T(0, \theta t) \right).
\]

Letting \( t \to 0 \) yields the asserted result.

Corollary 6. \( T_l(0, 0^+) \) exists and equals \( (h \Delta T)^2 / \rho H K \).

Proof: For any \( t > 0 \),

\[
(T(0, t) - T_{cr}/t) = (T(0, t) - T(X(t), t)/t
\]

\[
= -(X(t)/t) T_x(x^*(t), t), \quad 0 \leq x^*(t) \leq X(t).
\]

Moreover, from (3.6)

\[
(\rho H X'(t)/K) \leq - T_x(x^*(t), t) \leq (h/K) (T_L - T(0, t)),
\]

and as \( t \to 0 \) this implies

\[
-T_x(0^+, 0^+) = (h \Delta T / K),
\]

whence

\[
T_l(0, 0^+) = ((h \Delta T)^2 / \rho H K).
\]

The bound (3.6) on \( |T_x| \) is the principal tool needed for proving uniqueness of the solution, using the approach of Douglas [2]. Because of the direct nature of the proof we will merely state

Theorem 9. The solution to problem I is unique.

4. Dependence on the heat transfer coefficient. We now address the question of how the solution to Problem I depends upon \( h \). Indeed, from (3.7) of Theorem 7 we can state

Theorem 10. As \( h \to \infty \), \( T(0, t) \to T_L \) in a pointwise manner for all \( t \to 0 \).
Similarly we may assert:

**Theorem 11.** The solution to Problem I depends monotonically on $h$. In particular, if $(T^1(x, t), X_1(t))$ and $(T^2(x, t), X_2(t))$ are the solutions to Problem I for $h = h_1, h_2$, respectively and if $h_1 < h_2$, then $X_2(t) > X_1(t)$ for $t > 0$ and $T^2(x, t) > T^1(x, t)$ wherever they are both defined.

**Proof:** From Theorem 8, Corollary 6 and the maximum principle, there is some $t_0 > 0$ such that our assertion is true when $t < t_0$. This is seen by considering the difference

$$v(x, t) = T^2(x, t) - T^1(x, t).$$

at points where they are both defined. Let

$$t^* = \sup\{t \mid T^2(0, t) > T^1(0, t)\}, \quad t^{**} = \sup\{t \mid X_2(t) > X_1(t)\}. $$

By definition

$$v_x(0, t) = (h_1 - h_2)(T_L - T^2(0, t)) + h_1 V(0, t).$$

Suppose that $t^*, t^{**} < \infty$.

**Claim 1.** $t^* \neq t^{**}$.

Suppose that $t^* = t^{**}$. Then

a) $X_1(t^*) = X_2(t^*)$,

b) $X_1'(t^*) \geq X_2'(t^*)$,

c) $v(X_1(t^*), t^*) = 0$,

while for $t < t^*, 0 \leq x < X_1(t), v(x, t) > 0$. But by (b), $v_x(X_1(t^*), t^*) > 0$, which would contradict the corner minimum principle since

$$v(x, t) > v(X_1(t^*), t^*) \quad \text{for} \quad t < t^*, 0 \leq x < X_1(t).$$

**Claim 2.** $t^* < t^{**}$ is impossible.

On $[0, t^*], X_2(t) > X_1(t)$ whence $v(X_1(t), t) > 0$. Hence we must have $v(0, t^*) = 0$ with $v(0, t) > 0$ for $t < t^*$. But then $v(0, t^*)$ is a minimum value up to time $t^*$ whence $v_x(0, t^*) > 0$, which contradicts

$$v_x(0, t^*) = (h_1 - h_2)(T_L - T^2(0, t^*)) < 0.$$

**Claim 3.** $t^{**} < t^*$ is impossible, since

$$T^2(X_2(t^{**}), t^{**}) = T^1(X_1(t^{**}), t^{**}) = T_c.$$ 

Thus Theorem 11 is proved.

The solution to Problem II (see Sec. 1) is given explicitly by [1]

$$X_\infty(t) = 2\lambda_\sqrt{\lambda t} \tag{4.1}$$

$$T^\infty(x, t) = T_L - (\Delta T/erf_\lambda) \text{erf}(x/2\sqrt{\lambda t}) \tag{4.2}$$

where $\lambda$ is the root of

$$\lambda e^{2}\text{erf}_\lambda = St/\sqrt{\pi}. \tag{4.3}$$

We claim that this solution constitutes an upper bound for that of Problem I, namely

**Theorem 12.** Let $h > 0$, and let $X_h(t), T^h(x, t)$ denote the solution to Problem I for this value of the heat transfer coefficient. Then $X_\infty(t) > X_h(t)$ for all $t > 0$, and $T^\infty(x, t) > T^h(x, t)$ for all $(x, t)$ for which both functions are defined.
Proof: We note first that since \( X_h(t) \leq (ht\Delta T/\rho H) \), we find \( X_\infty(t) > X_h(t) \) for
\[
0 < t < t_0 = (4\lambda^2 K_\rho H^2/\epsilon h^2 \Delta T^2).
\]
Moreover, for \( t < t_0 \), \( T^\infty(0, t) = T_L > T^h(0, t) \) and
\[
T^\infty(X_h(t), t) > T^h(X_h(t), t) = T_{cr}.
\]
Let \( t < t_0 \) and \( x \in [0, X_h(t)] \). Then by the mean value theorem
\[
T^\infty(x, t) - T^h(x, t) = (T^\infty(x, t) - T_{cr}) - (T^h(x, t) - T_{cr})
= (x - X_\infty(t)) T^\infty(x', t) - (x - X_h(t)) T^h(x'', t)
\]
for \( x', x'' \in (x, X_h(t)) \). But then from (4.2)
\[
T^\infty(x, t) - T^h(x, t) = [(X_\infty(t) - x) \Delta T/\sqrt{(\pi xt)} \text{erf} \lambda + (X_h(t) - x) \Delta T/(hx^2/4a)]
\geq (X_\infty(t) - x) \Delta T e^{-x^2/4at} - (X_h(t) - x)(h\Delta T/K)
\geq (X_h(t) - x) \Delta T e^{-x^2/\pi(\pi x)\text{erf} \lambda} - (h\Delta T/K)
> 0
\]
for \( t < t_1 = (e^{-\lambda^2 K/\epsilon h^2(\pi x)\text{erf} \lambda})^2 \). It is easily seen that \( t_1 < t_0 \). Thus for \( t < t_1 \) the solution to problem II bounds that of problem I.

Let
\[
t^* = \sup\{t: X_\infty(t) > X_h(t)\},
\]
\[
t^{**} = \sup\{t: T^\infty(x, t) > T^h(x, t)\}, \text{ for } 0 \leq x \leq \min (X_\infty(t), X_h(t)).
\]
Let
\[
v(x, t) = T^\infty(x, t) - T^h(x, t)
\]
where both functions are defined. Suppose that \( t^*, t^{**} < \infty \).

Claim 1: It is not possible to have \( t^{**} < t^* \).

Indeed, suppose that \( t^{**} < t^* \). Then \( v(x, t) \) would vanish at some point \((x, t^{**})\) for \( x \in (0, X_h(t^{**})) \) while it is positive on the line \( t = t_1/2 \) and at \( x = 0 \) and \( x = X_h(t) \) for \( t < t^{**} \), violating the maximum principle.

Claim 2: It is not possible to have \( t^* < \infty \).

For at \( t^* \),
\[
X_\infty(t^*) = X_h(t^*), \quad v(X_h(t^*), t^*) = 0,
\]
\[
v(x, t) > 0 \text{ for } t < t^*, \quad X'_h(t^*) \geq X'_\infty(t^*)
\]
whence
\[
v_x(X_h(t^*), t^*) \geq 0,
\]
and by the corner point maximum principle \( v(x, t) \) could not have a minimum at \((X_h(t^*), t^*)\). Thus the claim is proved, and \( t^*, t^{**} \) must be infinite, proving the theorem.

We now assert that as \( h \to \infty \) the solution to Problem I converges to that of Problem II.

Theorem 13. Let \( t > 0 \). Then as \( h \to \infty \),
\[
X_h(t) \to X_\infty(t), \quad T^h(x, t) \to T^\infty(x, t).
\]
The proof rests upon the following observation:

**Lemma 2.** The relation (2.8)

\[
\int_0^{X(t)} cpx[T(x, t) - T_{cr}]dx + (1/2) \rho H X(t)^2 = K \int_0^t [T(0, t') - T_{cr}]dt'
\]

holds for \(X_h, T^h\) as well as for \(X_\infty, T^\infty\).

Indeed, the factor \(x\) in the spatial integral prevents the flux at \(x = 0\) from entering into the equation. Of course

\[T^\infty(0, t') = T_L, \quad t' > 0.\]

**Proof of Theorem 13.** For any \(h > 0\), by Lemma 2,

\[
\int_0^{X_h(t)} cpx[T^\infty(x, t) - T_{cr}]dx + (1/2) \rho H X_h(t)^2 = K \int_0^t [T_L - T_{cr}]dt',
\]

and

\[
\int_0^{X_h(t)} cpx[T^h(x, t) - T_{cr}]dx + (1/2) \rho H X_h(t)^2 = K \int_0^t [T^h(0, t') - T_{cr}]dt'.
\]

Recalling that \(X_\infty(t) > X_h(t)\) and \(T^\infty(x, t) > T^h(x, t)\) and subtracting (4.5) from (4.4), we find, using the estimate (3.7) on \((T_L - T^0, 0)\), that

\[
\int_0^{X_h(t)} cpx[T^\infty(x, t) - T^h(x, t)]dx + \int_0^{X_h(t)} cpx[T^\infty(x, t) - T_{cr}]dx
\]

\[
+ (1/2) \rho H[X_\infty(t)^2 - X_h(t)^2] = K \int_0^t (T_L - T^h(0, t')) dt'
\]

\[
\leq (2K\sqrt{(2K\rho H \Delta T)(1 + St)}/h))^{1/2} (t),
\]

which immediately implies that as \(h \to \infty\)

\[X_h(t) \to X_\infty(t).\]

Consider the family of functions \(\{T^h(x, t)\}\) for \(h \to \infty\). By (3.6) and (3.7), for any \(x \in [0, X_h(t)]\)

\[-T(x, t) \leq h[T_L - T^h(0, t)] \leq (1 + St)(2K\rho H \Delta T)^{1/2}/t^{1/2}.\]

Hence for any \(t > 0\) the functions \(T^h(x, t)\) are equicontinuous; since they are all bounded by \(T^\infty(x, t)\) and monotonically increasing in \(h\) the Arzela-Ascoli lemma implies their uniform convergence on \([0, X_\infty(t)]\) (assuming them extended beyond \(X_h(t)\) as \(T_{cr}\)) to a limiting function. By (4.6) this limit must coincide with \(T^\infty(x, t)\) and the theorem is proved.

Relation (4.6) yields the following interesting observation.

**Theorem 14.** As \(t \to \infty\), \((X_h(t)/X_\infty(t)) \to 1\); that is, the fronts for finite and infinite \(h\) agree asymptotically.

**Proof:** From (4.6),

\[
(1/2)\rho H \left[ X_\infty(t)^2 - X_h(t)^2 \right] \leq (2K\sqrt{(2K\rho H \Delta T)} (1 + St)\sqrt{t}/h.
\]

Division by \(X_\infty(t) = 2\lambda \sqrt{\alpha t}\) yields

\[0 \leq 1 - (X_h(t)/X_\infty(t))^2 \leq \theta/\sqrt{t}\]
for \( \theta = (c(1 + \text{St})/\lambda^2 h \sqrt{(2K\rho\Delta T/H)}) \), or

\[
(1 - (\theta/\sqrt{(t)}))^{1/2} \leq (X_h(t)/X_\infty(t)) < 1
\]

which, letting \( t \to \infty \), implies our result. We note that (4.7) provides a potentially useful bound on \( X_h(t) \),

\[
X_\infty(t) \sqrt{(1 - (\theta/\sqrt{(t)}))} \leq X_h(t) < X_\infty(t).
\]

5. A bound on the total system energy. If the motivation for studying problem I lies in the goal of storing heat in a phase-changing material, then the total heat stored by time \( t \),

\[
Q(t) = h \int_0^t [T_L - T(0, t')] dt'
\]

assumes a special importance. Using the relations (2.7, 2.8) we can obtain useful upper and lower bounds on \( Q(t) \). Thus we assert:

**Theorem 15.** At time \( t > 0 \),

\[
Q(t) \geq F_0(t) = (K\rho H/h) \sqrt{[1 + (2t\Delta T/K\rho H)]} - 1,
\]

\[
Q(t) \leq F_1(t) = K\rho H(1 + \text{St}/2)^2\left[1 + (2t\Delta T^2/K\rho H(1 + \text{St}/2)^2)\right]^{1/2} - 1/h.
\]

Moreover,

\[
0 \leq F_1 - F_0 \leq (x^2 t^2 h^4 c^2 \Delta T^2/K^4 H^2) = (t^2 h^4 \Delta T^2/K^2 p^2 H^2).
\]

**Proof of (5.1).** We note that

\[
\int_0^{x(t)} x \rho c [T(x, t) - T_{cr}] dx \leq X(t) \int_0^{x(t)} \rho c [T(x, t) - T_{cr}] dx
\]

while

\[
\int_0^t [T(0, t') - T_{cr}] dt' = -(1/h)Q(t) + t\Delta T,
\]

whence, after some manipulation, (2.7) implies

\[
[X(t) - (Q(t)/\rho H)]^2 + ((2K/\rho H) [ht\Delta T - Q(t)] - [Q(t)^2/(\rho H)^2]) \leq 0
\]

and since \( X(t) \) is real, we find

\[
(2K/\rho H) [ht\Delta T - Q(t)] \leq [Q(t)^2/(\rho H)^2].
\]

Further manipulation yields the bound (5.1).

To obtain the lower bound we note that (by (3.5))

\[
T(x, t) \leq T_L - [x\Delta T/X(t)],
\]

whence

\[
\int_0^{x(t)} \rho c [T(x, t) - T_{cr}] dx \leq [c\rho X(t)\Delta T/2]
\]

and so

\[
Q(t) \leq \rho H X(t)(1 + (1/2)\text{St}).
\]
But from (2.8),

\[ X(t) \leq \sqrt{[(2K/\rho H)] \{ -[Q(t)/H] + t\Delta T \}} \]

whence (5.3) yields

\[ Q(t) \leq [1 + (1/2)St] \sqrt{2K\rho H[t\Delta T - [Q(t)/H]]} \]

By straightforward manipulations we are then led to (5.2).

It is interesting to note that \( F_1 - F_0 = O(St^2) \). Indeed, if we introduce the nondimensional parameters

\[ F_0 = (\alpha t/L^2), \quad Bi = (hL/K), \]

for \( L \) a representative length, then

\[ F_1 - F_0 \leq (F_0Bi^2St)^2. \]

Thus the bounds (5.1, 5.2) are effective for small values of the parameters \( St \) and/or \( Bi \); they may be used to augment previously derived approximations for the surface temperature and moving boundary history [8].

References


Appendix: Proof of the corner point maximum principle. Let \( h(x, t) = |X(t) - x|^{3/2} + \beta |X(t) - x| \) in \( \bar{G}^o \), with \( \beta = \text{const} > 0 \) to be chosen. Note that

\[ h > 0 \text{ in } \bar{G}^o, \quad h|_{\partial D} \equiv 0 \]

Then

\[ Lh = \pm \{ \frac{3}{2} |x - X(t)|^{1/2} + \beta \} X'(t) - \frac{3}{2} |x - X(t)|^{-1/2} \]

where \( \pm \) correspond to the cases where \( X \) lies to the right or left of \( x \), respectively. We can thus choose \( N \) and \( \beta > 0 \) so small that \( Lh < 0 \) in \( \bar{G}^o \). Let \( v(P) = u(P) + \varepsilon h(P), \varepsilon > 0 \). Then
$Lv \leq \varepsilon Lh < 0$ in $\hat{G}^o$ and $v \in C(\hat{G}^o)$. Thus $v(P)$ attains its maximum value in $G^o$ on the boundary of $G^o$. Now

$$\partial G^o = \partial_0 G^o \cup \partial_1 G^o \cup \partial_2 G^o$$

where

$$\partial_0 G^o = \hat{G}^o \cap \{t = t_0\}, \quad \partial_1 G^o = \partial G^o \cap \partial D, \quad \partial_2 G^o = \hat{G}^o \cap \partial N,$$

Suppose $M = \max_{P \in \partial_0 G^o} v(P)$ is attained at a point $P^* \neq P_0$. There are three possibilities:

a) $P^* \in \partial_0 G^o$. Then at $P^*$, $v_1 \geq 0$, $v_{xx} \leq 0$, whence $Lv \geq 0$, which is not possible since $Lv < 0$ in $\hat{G}^o$. Thus $P^* \notin \partial_0 G^o - P_0$.

b) $P^* \in \partial_1 G^o$. Then $M = v(P^*) = u(P^*) \leq u(P_0) = v(P_0)$, whence $v(P_0)$ would equal $M$ (which is claimed) or exceed $M$ (which is not possible).

c) $P^* \in \partial_2 G^o$. Now $M = v(P^*) = u(P^*) + \varepsilon h(P^*)$. But $u(P^*) < u(P_0)$ and we may choose $\varepsilon$ so small that

$$v(P^*) = u(P^*) + \varepsilon h(P^*) < u(P_0) = v(P_0).$$

Hence in all cases

$$u(P_0) \geq v(P), \ P \in \hat{G}^o.$$  

Thus

$$0 > [v(P) - v(P_0)/|P - P_0|] = [v(P) - u(P_0)/|P - P_0|]$$

$$= [u(P) - u(P_0)/|P - P_0|] + \varepsilon [h(P) - h(P_0)/|P - P_0|]$$

or

$$[u(P) - u(P_0)/|P - P_0|] \leq -\varepsilon [h(P) - h(P_0)/|P - P_0|]$$

whence, by the form of $h$,

$$\lim_{P \to P_0, \ P \in \hat{G}^o} \frac{u(P) - u(P_0)}{|P - P_0|} < 0$$

for $P \to P_0$ in a nontangential direction. Thus the principle is proved.