THE SOLITARY WAVE WITH SURFACE TENSION*

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Recently Brooke Benjamin [1] has drawn attention to the paper of Korteweg and de Vries [2] in which they analysed the effect of surface tension on solitary waves. If $\tau = T/\rho g h^2$ where T is the coefficient of surface tension for the liquid, ρ its density, g the gravity force per unit mass and h the depth of the liquid at rest, they show that solitary waves of elevation exist when $\tau < \frac{1}{3}$ which are supercritical with speeds $C > (gh)^{1/2}$. Also, when $\tau > \frac{1}{3}$ solitary waves still exist but are waves of depression rather than waves of elevation, with subcritical speeds $C < (gh)^{1/2}$. Brooke Benjamin [1] remarks "that in the exceptional case $\tau = \frac{1}{3}$, there is no solitary wave according to the approximation used—the question whether the complete hydrodynamic problem has any such solution in this case remains open."

The three dimensional equations for water waves are not considered here. Instead some information about the case $\tau = \frac{1}{3}$ is obtained by a different procedure. Equations for propagation of waves in incompressible liquids have been derived by Green, Laws and Naghdi [3] using a direct two dimensional method. Similar equations have also been obtained by Green and Naghdi [4] from the three dimensional equations by an approximation. These wave propagation equations take account of surface tension and an equation for the study of solitary type waves with surface tension was given in [3], equation (5.7). In non-dimensional form this equation is

$$\frac{1}{3}k\eta'^2 + 2\tau(1+\eta)\left[(1+\eta'^2)^{-1/2} - 1\right] = \eta^2(k - 1 - \eta) \tag{1}$$

where $h\eta$ is the wave height, η is a function of $z = (x - Ct)/h$, a prime denotes differentiation with respect to z, and

$$\tau = T/\rho g h^2, \qquad k = C^2/gh = F^2. \tag{2}$$

Previously [3] Eq. (1) was integrated when $\tau = 0$. When $\tau \neq 0$ the character of the solutions of (1) may be studied in the (η'^2, η) phase plane. When $\tau < \frac{1}{3}$, $k > 1$ there are

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solitary waves of elevation at supercritical speeds, with maximum wave heights independent of τ, in agreement with [1, 2]. The form of the waves and the value of the maximum heights differ from those in [1, 2]. Here the maximum heights are $h(F^2 - 1)$ in comparison with $2h(F - 1)$, but the two values are approximately the same within the order of approximation used in the derivation of the K. de V. equation. When $\tau > \frac{1}{3}$, $k < 1$ there are solitary waves of depression with subcritical speeds and minimum heights independent of τ. This also agrees with [1, 2] although the form of waves and their minimum height are somewhat different. Finally the open question raised in [1] about the exceptional case $\tau = \frac{1}{3}$ can be partly answered on the basis of equation (1). There are solitary waves with $k > 1$, i.e. at supercritical speeds, with maximum heights again $h(F^2 - 1)$. These waves have the special property that at the peak of the wave the radius of curvature is zero while the slope is still zero.

References

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