STABILITY CONDITIONS FOR LINEAR NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS*

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Abstract. We derive new sufficient conditions for uniform asymptotic stability of the zero solution of linear non-autonomous delay differential equations. The equations considered include scalar equations of the form

\[ x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - T_i) \]

where \( c(t), b_i(t) \) are continuous for \( t \geq 0 \) and \( T_i \) is a positive number (\( i = 1, 2, \ldots, n \)), and also systems of the form

\[ x'(t) = B(t)x(t - T) - C(t)x(t) \]

where \( B(t) \) and \( C(t) \) are \( n \times n \) matrices. The results are found by using the method of Lyapunov functionals.

1. Scalar equations with a single delay. The purpose of this paper is to derive some new sufficient conditions for stability of linear delay differential equations. We first consider the scalar equation

\[ x'(t) = b(t)x(t - T) - c(t)x(t) \] (1)

where \( b \) and \( c \) are given continuous functions and \( T \) is a positive constant. Extensions to scalar equations with several delays and to systems of equations are given in Secs. 2 and 3.

The simplest available sufficient condition for asymptotic stability is contained in the following theorem of Hale [5, page 108].

**Theorem 1.** Suppose that \( b \) and \( c \) are bounded continuous functions on \( \mathbb{R} \) and satisfy

(i) \( c(t) \geq \delta > 0 \) for all \( t \), and

(ii) \( |b(t)| \leq \theta \delta \) for all \( t \), and for some \( \theta, 0 \leq \theta < 1 \).

Then, the zero solution of (1) is uniformly asymptotically stable.

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In this result, the function \( c \) is required to dominate the function \( |b| \) in the very strong sense that the supremum of \( |b| \) must be less than the infimum of \( c \). Some such condition is needed, since if \( b \) and \( c \) are constants and \( b \geq 0 \), then \( b < c \) is necessary for stability. In the theorems that we give here the hypotheses on \( b \) and \( c \) are less stringent. For example, when \( b \) and \( c \) are periodic with period \( T \), the hypothesis \( |b(t)| < c(t) \) suffices. This can also be shown to hold in more general circumstances by applying a stability theorem of Dyson and Villella-Bressan [4].

Our results are obtained by using certain simple Lyapunov functionals \( V(t, \phi) \) rather than the autonomous functionals \( V(\phi) \) used in proving Thm. 1. Although the theory of Lyapunov functionals has been extensively developed for autonomous equations, for example by Carvalho, Infante and Walker [3], a similar development is still lacking for non-autonomous equations.

Our first result for Eq. (1) is contained in the following theorem.

**Theorem 2.** Suppose that \( b \) and \( c \) are continuous and assume that the following conditions are satisfied:

(a) Given \( \eta > 0 \) there exists \( \tau > 0 \) such that

\[
\int_{t}^{t+\tau} |b(s)|ds < \eta \quad \text{for } t \geq 0
\]

(and consequently for some \( B > 0 \))

\[
\int_{-\infty}^{t} |b(t + T + \theta)|d\theta \leq B < \infty,
\]

\( t \geq 0 \).

(b) There exist \( a > 0 \) and \( q > 0 \) such that

\[
2c(t) - a|b(t)| - |b(t + T)|/a \geq q \quad \text{for } t \geq 0.
\]

Then the zero solution of (1) is uniformly asymptotically stable.

**Proof.** The proof consists in applying the Lyapunov theorem for functional differential equations given in Sec. 4 with a Lyapunov function \( V: \mathbb{R} \times C \to C \) of the form

\[
V(t, \phi) = a\phi^2(0) + \int_{-T}^{0} K(t + \theta)\phi^2(\theta) d\theta
\]

where \( K \) is a continuous function, \( K: \mathbb{R} \to \mathbb{R} \), to be chosen later. Let \( x(s, \phi) \) denote the solution of (1) satisfying \( x_s = \phi \) and, for simplicity, let \( x(t) \) denote the value of \( x(s, \phi) \) at \( t \). Then

\[
\dot{V}(t, \phi) = \lim_{h \to 0} \frac{1}{h} \left[ V(t + h, x_{t+h}(t, \phi)) - V(t, \phi) \right]
\]

\[
= \frac{d}{dt} ax^2(t)
\]

\[
+ \lim_{h \to 0} \frac{1}{h} \left\{ \int_{-T+h}^{h} K(t + \theta)x^2(t + \theta) d\theta - \int_{-T}^{0} K(t + \theta)x^2(t + \theta) d\theta \right\}
\]

\[
= 2ax(t)x'(t) + K(t)x^2(t) - K(t - T)x^2(t - T).
\]
Since \( x \) satisfies (1), we have

\[
\dot{V}(t, \phi) = \left[ K(t) - 2ac(t) \right] \phi^2(0) + 2ab(t)\phi(0)\phi(-T) - K(t - T)\phi^2(-T). \tag{2}
\]

Letting \( K(t) = |b(t + T)| \) in (2), we note that the discriminant of the resulting quadratic form is

\[
4a^2b^2(t) + 4|b(t)||[b(t + T)| - 2ac(t)
\]

\[
= 4|b(t)|\left[a^2|b(t)| + |b(t + T)| - 2ac(t)\right] \leq -4aq|b(t)|,
\tag{3}
\]

the inequality following from condition (b). Now, whenever \(|b(t)| \geq q/8a\), we see from (3) that the quadratic form (2) is negative definite (uniformly for all such \( t \)). Hence, there exists a constant \( \alpha_1 > 0 \), such that \( \dot{V}(t, \phi) \leq -\alpha_1\phi^2(0) \) for all \( t \) where \(|b(t)| \geq q/8a\). However, if \(|b(t)| < q/8a\), we have from (2) with \( K(t) = |b(t + T)|\):

\[
\dot{V}(t, \phi) = \left[ |b(t + T)| - 2ac(t) \right] \phi^2(0) + 2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T)
\]

\[
\leq -aq\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T) \tag{4}
\]

since from (b) we have \(|b(t + T)| - 2ac(t) \leq -aq\). Now, if \( 2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T) > 0 \), then \( 2a|\phi(0)| > |\phi(-T)| \), hence

\[
2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T) < 4a^2|b(t)|\phi^2(0) - |b(t)|\phi^2(-T)
\]

\[
< 4a^2|b(t)|\phi^2(0) < \frac{qa}{2}\phi^2(0).
\]

Using this in (4) we obtain

\[
\dot{V}(t, \phi) \leq -\frac{aq}{2}\phi^2(0), \quad \text{whenever } |b(t)| < q/8a.
\]

Letting \( \alpha = \min(\alpha_1, qa/2) \), we see that \( \dot{V}(t, \phi) \leq -\alpha\phi^2(0) \) for all \( t \geq 0 \) and all \( \phi \in C \). Moreover, the inequalities

\[
a\phi^2(0) \leq V(t, \phi) \leq (B + a)|\phi|_{\infty}^2
\]

follow directly from (a) and the definition of \( V \); and the zero solution is asymptotically stable. This completes the proof of the theorem.

**Remark.** The condition (a) can hold even when \( c(t) - b(t) \geq q > 0 \) and \( c(t) - b(t + T) \geq q > 0 \) fail to hold. In fact if we take \( a = 1 \) and \( q = 1/2 \) in condition (a), we see that it holds for the special case

\[
T = 3, \quad c(t) \equiv 1,
\]

\[
b(t) = \begin{cases} 
\frac{3}{2} [1 - |6n + 3 - t|], & t \in (6n + 2, 6n + 4), n = 0, \pm 1, \pm 2, \ldots, \\
0, & \text{otherwise}.
\end{cases}
\]

However, \( c(6n + 3) - |b(6n + 3)| = -1/2 < 0 \), and \( c(6n) - |b(6n + 3)| = -1/2 < 0 \). Note that the stability conditions of Dyson and Villella-Bressan [4] when applied to Eq. (1) require that \( c(t) - b(t) \geq q > 0 \).
Theorem 2 has some immediate corollaries that are worth stating because they deal with situations that are frequently encountered in applications.

**Corollary 1.** Suppose that $c$ is continuous and $b$ is continuous and periodic of period $T$. Then, if there exists $q > 0$ such that
\[ c(t) - |b(t)| \geq q, \quad t > 0, \]
the zero solution of (1) is uniformly asymptotically stable.

Note that if $b$ and $c$ are constants, then condition (a) with $a = 1$ reduces to $c - |b| > 0$. This is the best possible stability condition regardless of the size of the delay $T$ in this case ([5], page 108). So, in this sense, the condition (a) is also the best possible condition of this type.

**Corollary 2.** Assume that $b$ and $c$ are continuous and that:

(b) There exists $\lambda \in (0, 1)$ such that $|f(t)| < t > 0$,

(c) $c(t) \geq c_1 > 0$, and either $c(t)$ is non-increasing or $|b(t)|$ is non-increasing.

Then the zero solution of (1) is uniformly asymptotically stable.

The above results were obtained by choosing $K(t) = |b(t + T)|$ in (2). If different choices of $K$ are taken, then other stability conditions can be obtained. For example, we shall prove the following theorem by choosing $K(t) = b^2(t + T)$.

**Theorem 3.** The results of Theorem 2 hold provided that

(a') $2ac(t) - b^2(t + T) - a^2 \geq q$, for some $a > 0$, $q > 0$, and

(b') $\int_t^{t+T} b^2(s) \, ds$ is bounded and given $\eta > 0$ there exists $\tau > 0$ such that
\[ \int_t^{t+\tau} |b(s)| \, ds < \eta \]
for $t \geq 0$.

**Proof.** If $K(t) = b^2(t + T)$, then (2) has the form
\[ \dot{V}(t, \phi) = \left[ b^2(t + T) - 2ac(t) \right] \phi^2(0) + 2ab(t) \phi(0) \phi(-T) - b^2(t) \phi^2(-T). \]
Using (a'), we obtain
\[ \dot{V}(t, \phi) \leq -\left( a^2 + q \right) \phi^2(0) + 2ab(t) \phi(0) \phi(-T) - b^2(t) \phi^2(-T) \]
\[ \leq -q \phi^2(0) - \left[ a\phi(0) - b(t) \phi(-T) \right]^2 \]
\[ \leq -q \phi^2(0), \]
for all $\phi \in C$. Moreover, $V(t, \phi) \geq a\phi^2(0)$ and
\[ V(t, \phi) \leq a\phi^2(0) + |\phi|_\infty^2 \int_{-T}^{0} b^2(t + T + \theta) \, d\theta \]
\[ \leq a\phi^2(0) + |\phi|_\infty^2 \int_{t}^{t+T} b^2(s) \, ds. \]
By condition (b'), there is a constant $B$ such that
\[ V(t, \phi) \leq B|\phi|_\infty^2. \]
As for Thm. 2, uniform asymptotic stability follows from Theorem 8, and the theorem is proved.
A special case occurs again when \( b \) is periodic of period \( T \). Then conditions \( (a') \) and \( (b') \) are implied by the single condition
\[
2ac(t) - b^2(t) - a^2 > 0, \quad 0 \leq t \leq T.
\]

As a final example, we examine the consequences of choosing
\[
K(t) = \frac{b^2(t + T)}{c(t + T)},
\]
as was done in [2].

**Theorem 4.** Assume that \( b \) and \( c \) are continuous and that the following conditions hold,
- \( (a'') \) There is a constant \( \lambda \) such that \( b^2(t + T)/c(t + T) < \lambda < 1 \) for \( t \geq 0 \),
- \( (b'') \) \( \int_{t}^{t+T} b^2(s) \, ds \) is bounded and given \( \eta > 0 \) there exists \( \tau > 0 \) such that
  \[
  \int_{t}^{t+\tau} b(s) \, ds < \eta
  \]
  for \( t \geq 0 \).
- \( (c'') \) There is a constant \( c_1 \) such that \( c(t) \geq c_1 > 0 \) for \( t \geq 0 \).
  Then the zero solution of (1) is uniformly asymptotically stable.

**Proof.** If \( K(t) = \frac{b^2(t + T)}{c(t + T)} \), then (2) has the form
\[
\dot{V}(t, \phi) = \left[ \frac{b^2(t + T)}{c(t + T)} - 2ac(t) \right] \phi^2(0) + 2ab(t)\phi(0)\phi(-T) - \frac{b^2(t)}{c(t)} \phi^2(-T).
\]

From \( (a'') \) and \( (c'') \) we get
\[
\dot{V}(t, \phi) \leq (\lambda - 2a)c(t)\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - \frac{b^2(t)}{c(t)} \phi^2(-T)
\]
\[
= - \left[ (2a - \lambda)c^2(t)\phi^2(0) - 2ab(t)c(t)\phi(0)\phi(-T) + b^2(t)\phi^2(-T) \right]/c(t).
\]
Choosing \( a = 1 \), we have, for all \( \phi \in C \),
\[
\dot{V}(t, \phi) \leq -(1 - \lambda)c(t)\phi^2(0) - \left[ c(t)\phi(0) - b(t)\phi(-T) \right]^2/c(t)
\]
\[
\leq -(1 - \lambda)c(t)\phi^2(0) - (1 - \lambda)c_1\phi^2(0).
\]

Moreover,
\[
\phi^2(0) \leq V(t, \phi) \leq |\phi|_\infty \left( 1 + \int_{t}^{t+\tau} \frac{b^2(s)}{c(s)} \, ds \right) \leq B|\phi|_\infty^2,
\]
and the proof is completed.

2. **Scalar equations with several delays.** The analysis of the previous section can be directly generalized to cover equations with several delays of the form
\[
x'(t) = -c(t)x(t) + \sum_{i=1}^{N} b_i(t)x(t - T_i)
\]
where \( T_i > 0 \) is a positive constant \((i = 1, 2, \ldots, N)\). We use the functional
\[
V(t, \phi) = \phi^2(0) + \sum_{i=1}^{N} \int_{-T_i}^{0} K_i(t + \theta)\phi^2(\theta) \, d\theta.
\]
where $K_i$ are continuous functions to be chosen below. A calculation of the same sort as in Sec. 1 yields

$$
\dot{V}(t, \phi) = \left[ -2c(t) + \sum_{i=1}^{N} K_i(t) \right] \phi^2(0)
$$

$$
+ 2\phi(0) \sum_{i=1}^{N} b_i(t) \phi(-T_i) - \sum_{i=1}^{N} K_i(t - T_i) \phi^2(-T_i).
$$

(6)

When $-\dot{V}$ is viewed as a quadratic form in $\phi(0)$ and $\phi(-T_i)$, $i = 1, 2, \ldots, N$, it has the following associated symmetric matrix

$$
M = \begin{bmatrix}
2c(t) - \sum_{i=1}^{N} K_i(t) & -b_1(t) & -b_2(t) & \cdots & -b_N(t) \\
-b_1(t) & K_1(t - T_1) & 0 & \cdots & 0 \\
-b_2(t) & 0 & K_2(t - T_2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-b_N(t) & 0 & 0 & \cdots & K_N(t - T_N)
\end{bmatrix}.
$$

We now choose

$$
K_i(t) = |b_i(t + T_i)|/a_i, \quad i = 1, 2, \ldots, N,
$$

(7)

and note that the principal minors of $M$ are

$$
2c(t) - \sum_i |b_i(t + T_i)|/a_i
$$

$$
\frac{1}{a_1} |b_1(t)| \left[ 2c(t) - a_1 |b_1(t)| - \sum_{i=1}^{N} \frac{1}{a_i} |b_i(t + T_i)| \right]
$$

$$
\cdots
$$

$$
\frac{1}{a_1 \cdots a_N} |b_1(t)| \cdots |b_N(t)| \left[ 2c(t) - \sum_{i=1}^{N} a_i |b_i(t)| - \sum_{i=1}^{N} \frac{1}{a_i} |b_i(t + T_i)| \right].
$$

If $|b_i(t)| \geq \varepsilon > 0$ for all $i$, then the quadratic form $-\dot{V}$ is positive definite whenever there exists $q > 0$ such that

$$
2c(t) - \sum_{i=1}^{N} a_i |b_i(t)| - \sum_{i=1}^{N} \frac{1}{a_i} |b_i(t + T_i)| \geq q > 0.
$$

(8)

Using the arguments of Thm. 2, we can conclude that, if (8) holds, then there exists $\alpha > 0$ such that

$$
\dot{V}(t, \phi) \leq -\alpha \phi^2(0).
$$

Clearly,

$$
|\phi(0)|^2 \leq V(t, \phi) \leq |\phi|_{\infty}^2 \left( 1 + \sum_{i=1}^{N} \int_{-T_i}^{0} \frac{1}{a_i} |b_i(t + T_i + \theta)| \, d\theta \right)
$$

$$
= |\phi|_{\infty}^2 \left( 1 + \sum_{i=1}^{N} \int_{t}^{t+T_i} \frac{1}{a_i} |b_i(s)| \, ds \right),
$$

and we have established the following result.
Theorem 5. Let \( c(t) \) and \( b_i(t) \) be continuous functions satisfying the following conditions:

(i) Given \( \eta > 0 \) there exists \( \tau > 0 \) such that
\[
\int_{t}^{t + \tau} |b_i(s)| \, ds < \eta
\]
for \( i = 1, 2, \ldots, n \) and \( t \geq 0 \).

(ii) \( 2c(t) - \sum_{i=1}^{N} a_i |b_i(t)| - \sum_{i=1}^{N} |b_i(t + T_i)| / a_i \geq q > 0 \), for some constants \( q > 0 \), \( a_i > 0 \), \( i = 1, 2, \ldots, N \) and for \( t \in [0, \infty) \).

Then, the zero solution of (5) is uniformly asymptotically stable.

It is easy to derive analogues of Corollaries 1 and 2 of Sec. 1. We only mention one of these.

Corollary 3. If \( c(t) \) and \( b_i(t) \) are continuous, and \( b_i(t) \) is periodic of period \( T_i \), \( i = 1, 2, \ldots, N \); a sufficient condition for the uniform asymptotic stability of the zero solution of (5) is that there exist \( q > 0 \), with
\[
c(t) - \sum_{i=1}^{N} |b_i(t)| \geq q, \quad t \in [0, \infty).
\]

Other results follow from different choices of the \( K_i \). For example, the choice
\[
K_i(t) = \frac{1}{a_i} b_i^2(t + T_i)
\]
yields the following form for \( \dot{V}(t, \phi) \)
\[
\dot{V}(t, \phi) = \left[ -2c(t) + \sum_{i=1}^{N} \frac{1}{a_i} b_i^2(t + T_i) \right] \phi^2(0) + 2\phi(0) \sum_{i=1}^{N} b_i(t) \phi(-T_i)
\]
\[
- \sum_{i=1}^{N} \frac{1}{a_i} b_i^2(t) \phi^2(-T_i),
\]
and we have the following result.

Theorem 6. The zero solution of Eq. (5) is uniformly asymptotically stable if \( c(t) \) and \( b_i(t), i = 1, 2, \ldots, N \), are continuous and

(i') there exist constants \( q > 0 \), \( a_i > 0 \), \( i = 1, 2, \ldots, N \) with
\[
2c(t) - \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} \frac{1}{a_i} b_i^2(t + T_i) \geq q,
\]

(ii') \( \sum_{i=1}^{N} \int_{t}^{t + T_i} b_i^2(s) \, ds \leq B < \infty \), and given \( \eta > 0 \) there exists \( \tau > 0 \) such that
\[
\int_{t}^{t + \tau} |b_i(s)| \, ds < \eta
\]
for \( i = 1, 2, \ldots, n \) and \( t \geq 0 \).
Proof. Using condition (i') in (9) we note that

\[
\dot{V}(t, \phi) \leq -q\phi^2(0) + \sum_{i=1}^{N} \frac{1}{a_i} \left[ -a_i^2\phi^2(0) + 2a_ib_i(t)\phi(0)\phi(-T_i) - b_i^2(t)\phi^2(-T_i) \right]
\]

\[
\leq -q\phi^2(0) - \sum_{i=1}^{N} \frac{1}{a_i} \left[ a_i\phi(0) - b_i(t)\phi(-T_i) \right]^2 \leq -q\phi^2(0).
\]

The condition (ii') immediately implies that

\[
\kappa(0) \leq \left| b_i(t) \right| \leq 1 + E - \sum_{i=1}^{N} \frac{1}{a_i} \int_{t}^{t+T_i} b_i^2(s) \, ds,
\]

where \( L = N\sum_{i=1}^{N} 1/a_i \), and the proof is completed.

An immediate corollary is the following.

Corollary 4. If \( c(t) \) and \( b_i(t) \) are continuous and if \( b_i(t) \) is periodic with period \( T_i \), \( i = 1, 2, \ldots, N \), then a sufficient condition for the uniform asymptotic stability of the zero solution of (5) is that there exist \( q > 0 \), \( a_i > 0 \), \( i = 1, 2, \ldots, N \), such that

\[
2c(t) - \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} \frac{1}{a_i} b_i^2(t) \geq q.
\]

We note that all of these results can be generalized, at the expense of complicating the stability conditions, by choosing Lyapunov functions of the form

\[
V(t, \phi) = a(0)\phi^2(0) + \sum_{i=1}^{N} \int_{t}^{t+T_i} K_i(t + \theta)\phi^2(\theta) \, d\theta,
\]

with \( a(0) \geq \alpha_0 > 0 \), a continuously differentiable function. The proofs of the corresponding results proceed in the same manner as before with obvious changes in the stability conditions. For example, the conclusions of Thm. 5 hold if condition (ii) of that result is replaced by

\[
2c(t) - \alpha(t) \sum_{i=1}^{N} a_i|b_i(t)| - \frac{1}{\alpha(t)} \sum_{i=1}^{N} \frac{1}{a_i} |b_i(t + T_i)| - \frac{\alpha'(t)}{\alpha(t)} \geq q > 0,
\]

for any function \( \alpha \) of the type described above. All of our results have analogous extensions.

3. Some simple stability criteria for systems. Consider the system

\[
x'(t) = B(t)x(t - T) - C(t)x(t)
\]

where \( x \) is an \( n \)-dimensional vector and \( B \) and \( C \) are continuous functions whose range is in the set of \( n \times n \) matrices. Introducing the functional (the superscript \( T \) denotes the transpose of a matrix):

\[
V(t, \phi) = \phi(0)^T D\phi(0) + \int_{-T}^{0} \phi(\theta)^T K(t + \theta)\phi(\theta) \, d\theta
\]
where \( K(t) \) and \( D \) are \( n \times n \) matrices to be chosen below, and assuming that \( K \) is continuous, we obtain
\[
\dot{V}(t, \phi) = x'(t)^T D x(t) + x(t)^T D x'(t) + x(t)^T K(t) x(t) - x(t - T)^T K(t - T) x(t - T)
\]
\[
= \left[ x(t - T)^T B(t) - x(t)^T C(t) \right]^T D x(t) + x(t)^T D \left[ B(t) x(t - T) - C(t) x(t) \right] + x(t)^T K(t) x(t) - x(t - T)^T K(t - T) x(t - T)
\]
\[
= -\phi(0)^T \left[ C(t)^T D + D C(t) - K(t) \right] \phi(0) + \phi(-T)^T B(t) D \phi(0) + \phi(0)^T D B(t) \phi(-T) - \phi(-T)^T K(t - T) \phi(-T).
\]

If \( D = D^T \), we have
\[
\dot{V}(t, \phi) = -\phi(0)^T \left[ C(t)^T D + D C(t) - K(t) \right] \phi(0)
\]
\[
+ 2\phi(0)^T D B(t) \phi(-T) - \phi(-T)^T K(t - T) \phi(-T).
\]

This quadratic form \(-\dot{V}\) has the associated symmetric matrix
\[
\begin{bmatrix}
C(t)^T D + D C(t) - K(t) & \frac{1}{2} \left( D B(t) + B(t)^T D \right) \\
\frac{1}{2} \left( D B(t) + B(t)^T D \right) & K(t - T)
\end{bmatrix}
\]

Several tests can be applied to establish that this is a positive definite matrix.

As a specific example, choose \( D \) to be positive definite and symmetric, and let \( K(t) = B(t + T)^T B(t + T) \).

Then
\[
\dot{V}(t, \phi) = -\phi(0)^T \left[ C(t)^T D + D C(t) - B(t + T)^T B(t + T) \right] \phi(0)
\]
\[
+ 2\phi(0)^T D B(t) \phi(-T) - \phi(-T)^T B(t)^T B(t) \phi(-T).
\]

and if we impose the condition
\[
C(t)^T D + D C(t) - B(t + T)^T B(t + T) - D^2 \geq \gamma I,
\]
where \( \gamma > 0 \) and \( I \) is the identity, we obtain from (13)
\[
\dot{V}(t, \phi) \leq -\gamma \phi(0)^T \phi(0) - (D \phi(0) + B \phi(-T))^T (D \phi(0) + B \phi(-T))
\]
\[
\leq -\gamma \phi(0)^T \phi(0).
\]

Moreover, since \( D \) is positive definite, there exist constants \( \alpha_1 > 0, \alpha_2 > 0 \) with
\[
\alpha_1 \| \phi(0) \|^2 \leq \phi(0)^T D \phi(0) \leq \alpha_2 \| \phi(0) \|^2,
\]

hence, if \( \int^t_{t+T} \| B(s) \|^2 \, ds \leq \beta < \infty \), we have
\[
\alpha_1 \| \phi(0) \|^2 \leq V(t, \phi) \leq \alpha_2 \| \phi(0) \|^2 + \| \phi \|_\infty \int^t_{t+T} \| B(s) \|^2 \, ds
\]
\[
\leq \| \phi \|_\infty^2 (\alpha_2 + \beta).
\]
Applying Theorem 8 in Sec. 4, we have

**Theorem 7.** Consider the system (10) and assume $B$ and $C$ are continuous matrix valued functions satisfying the conditions

(i) $C(t)^T D + DC(t) - B(t+T)^T B(t+T) - D^2 \geq \gamma I$, for some $\gamma > 0$ and some positive definite matrix $D$, and

(ii) Given $\eta > 0$ there exists $r > 0$ such that

$$\int_{t}^{t+r} \|B(s)\| ds < \eta \text{ for } t \geq 0.$$

Then the zero solution of (10) is uniformly asymptotically stable.

Theorem 7 is an exact extension to systems of Thm. 3 for scalar equations. It is clear that Thm. 6 also has an analogous extension to systems of the form

$$x'(t) = -C(t)x(t) + \sum_{i=1}^{N} B_i(t)x(t - T_i). \quad (14)$$

The stability conditions in this case take the form

$$\sum_{i=1}^{N} \int_{t}^{t+T} \|B_i(s)\|^2 ds \leq \beta < \infty$$

and

$$C(t)^T D + DC(t) - \sum_{i=1}^{N} a_i D^2(t) - \sum_{i=1}^{N} \frac{1}{a_i} B(t + T_i)^T B(t + T_i) \geq \gamma I,$$

for some positive definite symmetric matrix $D$ and some constants $\gamma > 0$, $a_i > 0$, $i = 1, 2, \ldots, N$. If $D$ is taken to be non-constant: $D: [0, \infty) \to$ positive definite $n \times n$ matrices, $D(t)$ continuously differentiable and $D(t) \geq D_0$, where $D_0$ is a constant positive definite matrix, then the second stability condition changes to

$$C(t)^T D(t) + D(t)C(t) - \sum_{i=1}^{N} a_i D^2(t) - D'(t)$$

$$- \sum_{i=1}^{N} \frac{1}{a_i} B(t + T_i)^T B(t + T_i) \geq \gamma I.$$

We finally note that our results are intrinsically different from those of Lewis and Anderson [6] because we allow the possibility that the matrices $B_i(t)$ in (14) have non-zero diagonal terms. The hypotheses in [6] require that all diagonal terms in the $B_i(t)$ be equal to zero. The techniques in [6] can be extended to cover the situation where the $B_i(t)$ have non-zero diagonal terms, and stability criteria which differ from those presented here are obtainable in that way. This has been done and will be described in a forthcoming paper of R. Volz.

**4. A Lyapunov functional result.** In proving the results of the previous sections we have used a version of the Lyapunov asymptotic stability result of Krasovskii that does not require the standard restriction that the right-hand side of the functional equation map
Theorem 8. Suppose that there exist continuous nondecreasing functions \( u, v, w : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( u(0) = v(0) = 0 \) and \( u(s) > 0, v(s) > 0, w(s) > 0 \) for \( s > 0 \). Suppose also that there exists a continuous function \( V : \mathbb{R} \times \mathbb{C} \to \mathbb{R} \) such that

\[
\frac{d}{ds} \left( |\phi(0)| \right) \leq V(t, \phi) \leq \frac{d}{ds} \left( |\phi(0)| \right),
\]

Finally, assume that given positive \( \eta > 0, \gamma > 0 \) there exists \( \tau > 0 \) such that

\[
\int_t^{t+\tau} |F(s, \phi)| \, ds < \eta \quad \text{for all} \quad t > 0 \quad \text{and} \quad |\phi| \leq \gamma, \quad \text{and} \quad x^T D G(t, x) \geq 0 \quad \text{for some positive definite symmetric matrix} \quad D \quad \text{and for} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.
\]

Then the solution \( x = 0 \) of (15) is uniformly asymptotically stable.

Proof. The proof proceeds in the same way as that of Theorem 2.1 in [5, page 105] with the exception of the part of that proof which uses the added assumption that \( f \) in (15) take \( \mathbb{R} \times (\text{bounded sets of} \ \mathbb{C}) \) into bounded sets of \( \mathbb{R}^n \). So, we shall present only that part of the proof.

Let \( \delta_0 > 0 \) be such that \( |\phi| < \delta_0 \) implies \( |x(t, \phi)| < 1 \) for all \( t \geq \sigma \). Assume that there exists a sequence \( \{t_k\} \) such that

\[
s + (2k - 1)r \leq t_k \leq s + 2k(r), \quad k : 1, 2, \ldots
\]

and, with \( |x|_D = (x^T D x)^{1/2} \),

\[
|x(t_k)|_D \geq \delta, \quad \text{for some} \quad \delta > 0.
\]

The proof can be completed as in [5], if we can show that there exists \( \tau > 0 \) such that \( |x(t)|_D > \delta/2 \) for

\[
t \in [t_k - \tau, t_k + \tau].
\]

Now, for \( |\phi| \leq \varepsilon \) choose \( \tau > 0 \) so that

\[
\int_t^{t+\tau} |F(s, \phi)| \, ds < \frac{3\delta^2}{8d}, \quad \text{where} \quad d = \|D\|.
\]

Note that the continuity of \( x(t) \) implies that there exist \( \tau_k > 0 \) with \( |x(t)|_D > \delta/2 \) for \( t \in [t_k - \tau_k, t_k + \tau_k] = I_k \), and for each \( k \), let \( \tau_k > 0 \) be the maximal such \( \tau_k \). We shall show that \( \tau_k \geq \tau \). For, supposing that \( \tau_k < \tau \) we have

\[
\frac{d}{dt} \left( x^T(t) D x(t) \right) = x^T(t) DF(t, x(t)) + F^T(t, x(t)) D x(t)
\]

\[
- \left[ x^T(t) DG(t, x(t)) + GT(t, x(t)) D x(t) \right]
\]

\[
\leq x^T(t) DF(t, x(t)) + F^T(t, x(t)) D x(t).
\]
So, for \( t \in [t_k - \tau, t_k + \tau] \)

\[
|x(t_k)|_D^2 - |x(t)|_D^2 \leq 2d \int_{t_k}^{t_k+\tau} |F(s, x_s)| ds < 3\delta^2/4,
\]

that is \( |x(t)|_D > \delta/2 \), which contradicts the maximalities of \( \tau_k \). So \( \tau_k \geq \tau \), and the proof is completed.

**Remark.** Burton [1] gives extensions of the Lyapunov theorem of Krasovskii for more general equations than (15). In the scalar case \( (n = 1) \), the condition on \( G(t, x) \) reduces to the requirement that \( xG \geq 0 \).

**References**