BURGERS FLOW PAST AN ARBITRARY ELLIPSE*

By

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Summary. This paper examines a linearization of the Navier-Stokes equation due to Burgers in which vorticity is transported by the velocity field corresponding to continuous potential flow. The governing equations are solved exactly for the two dimensional steady flow past an ellipse of arbitrary aspect ratio. The requirement of no slip along the surface of the ellipse results in an infinite algebraic system of linear equations for coefficients appearing in the solution. The system is truncated at a point which gives reliable results for Reynolds numbers R in the range $0 < R \leqslant 5$.

Predictions of the Burgers approximation regarding separation, drag and boundary layer behaviour are investigated. In particular, Burgers linearization gives drag coefficients which are closer to observed experimental values than those obtained from Oseen's approximation. In the special case of flow past a circular cylinder, Burgers approximation predicts a boundary layer whose thickness is roughly proportional to $R^{-1/2}$. This is in agreement with the nonlinear theory despite the fact that the Burgers calculations are carried out using only moderate values of the Reynolds number. In the matter of separation, it is shown that standing eddies form on the downstream side of a circular cylinder at $R = 1.12$. Interestingly enough, this is the same value predicted by Skinner [1] using singular perturbation techniques on the full nonlinear problem (see Van Dyke [2]).

The linearizations due to Oseen and Burgers both give spatially uniform approximations to the flow past a finite obstacle. The main difference is that vorticity is transported around the obstacle in Burgers flow rather than through it. The results of this paper suggest that Burgers approximation provides a qualitatively accurate model of the flow near an obstacle at low Reynolds numbers.

1. Introduction. Owing to the formidable nature of the Navier-Stokes equation, the history of fluid mechanics research is filled with simplifying approximations to this nonlinear problem. Included among these is a class of approximations which replaces the nonlinear inertial term $(\vec{v} \cdot \vec{\nabla})\vec{v}$ by a linear one $(\vec{v}_0 \cdot \vec{\nabla})\vec{v}$ where $\vec{v}_0$ is given. Of particular interest are the Stokes and Oseen approximations where $\vec{v}_0$ is constant. These have

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contributed significantly to the understanding of basic fluid dynamics behaviour, especially in the low Reynolds number regime where they are related asymptotically to the Navier-Stokes solution.

A linearization which has received considerably less attention is Burgers approximation [3] in which $\vec{v}_0$ is taken to be the continuous potential flow around the body. As noted by Dryden et al. [4], the vorticity equation in this case is identical to the temperature equation used by Boussinesq in his study of the conduction of heat from a hot body placed in an irrotational fluid. In Burgers approximation the convective velocity field $\vec{v}_0$ follows the surface of the body in its immediate neighborhood and approaches the velocity of a uniform stream at a great distance from the body. Burgers flow is asymptotically equivalent to Oseen flow far from the body, but in its immediate neighborhood Burgers flow models the exact flow more accurately. Of course Burgers approximation suffers from the defect that the convective velocity vector $\vec{v}_0$ does not tend to $\vec{0}$ as one approaches the surface of the obstacle. Furthermore at moderate values of the Reynolds number R, separation occurs on the downstream side of a bluff obstacle and the resultant velocity field $\vec{v}$ no longer resembles $\vec{v}_0$. This limits the suitability of Burgers approximation to small values of R. Nevertheless Burgers linearization provides a spatially uniform approximation to the solution of the Navier-Stokes equation and it is the purpose of this paper to investigate the extent to which it improves upon the Oseen approximation. Attention is restricted to two dimensional steady flow past an ellipse and its various special cases.

2. Mathematical formulation. The nondimensional Navier-Stokes equation has the form

$$R(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \nabla^2\vec{v} \tag{2.1}$$

where R, $\vec{v}$, p are Reynolds number, fluid velocity and pressure respectively. The velocity also satisfies the continuity equation

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{2.2}$$

which is guaranteed by the introduction of a stream function $\Psi(x, y)$ defined by

$$\vec{v} = \operatorname{curl}\{\Psi\hat{k}\}. \tag{2.3}$$

The problem for the stream function corresponding to (2.1) is given by

$$\left[\nabla^2 + RD(\Psi)\right]\omega = 0, \tag{2.4}$$

$$\nabla^2\Psi = -\omega, \tag{2.5}$$

where $\omega(x, y)$ is the magnitude of the vorticity vector and

$$D(\Psi) = \frac{\partial\Psi}{\partial x}\cdot\frac{\partial}{\partial y} - \frac{\partial\Psi}{\partial y}\cdot\frac{\partial}{\partial x}.$$

Consider a finite obstacle whose boundary is denoted by $\mathscr{B}$. Let Ψ_0 be the stream function for the continuous potential flow around $\mathscr{B}$ which approaches a uniform stream at infinity. The boundary value problem for $\Psi_0(x, y)$ is given by

$$\nabla^2\Psi_0 = 0, \tag{2.6}$$

$$\Psi_0|_{\mathscr{B}} = 0, \qquad \Psi_0 \sim y \quad \text{as } x^2 + y^2 \to \infty. \tag{2.7}$$

Burgers linearization is defined in the following way:

$$\left[\nabla^2 + RD(\Psi_0)\right]\omega = 0, \tag{2.8}$$

$$\nabla^2\Psi = -\omega, \tag{2.9}$$

$$\Psi|_{\mathscr{B}} = \left.\frac{\partial\Psi}{\partial n}\right|_{\mathscr{B}} = 0, \qquad \Psi \sim y \quad \text{as } x^2 + y^2 \to \infty. \tag{2.10}$$

Equation (2.8) is called the vorticity equation and (2.9) is Poisson's equation. The Oseen linearization is obtained by substituting $\Psi_0(x, y) = y$ in (2.8).

3. Solution of the vorticity equation. The exponential decay of the vorticity in Oseen flow suggests that we attempt a solution of (2.8) having the form

$$\omega(x, y) = F(x, y)\exp\left[f_0(x, y)\right]. \tag{3.1}$$

The substitution of (3.1) into (2.8) gives a second order partial differential equation for $F(x, y)$ with non-constant coefficients involving derivatives of Ψ_0 and f_0. If we set the coefficients of $\partial F/\partial x$ and $\partial F/\partial y$ to zero, we obtain the following:

$$\frac{\partial f_0}{\partial x} = \frac{1}{2}R\frac{\partial\Psi_0}{\partial y}, \qquad \frac{\partial f_0}{\partial y} = -\frac{1}{2}R\frac{\partial\Psi_0}{\partial x}. \tag{3.2}$$

These are the Cauchy-Riemann equations. They suggest that we choose $f_0(x, y)$ to be $\frac{1}{2}R$ times the velocity potential of the irrotational flow past $\mathscr{B}$. The resulting equation for $F(x, y)$ is given by

$$\nabla^2 F - \tfrac{1}{4}R^2 q(x, y)^2 F = 0 \tag{3.3}$$

where

$$q(x, y) = \left[\left(\frac{\partial\Psi_0}{\partial x}\right)^2 + \left(\frac{\partial\Psi_0}{\partial y}\right)^2\right]^{1/2}$$

is the magnitude of the potential flow velocity. In Oseen flow we have $q(x, y) = 1$.

Consider a curvilinear coordinate system (ξ, η) defined by

$$x + iy = f(\xi + i\eta) \tag{3.4}$$

where f is an entire function. The metric coefficient $\alpha(\xi, \eta)$ for this transformation is defined by

$$\alpha^2 = \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 = \left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2. \tag{3.5}$$

The vorticity in (ξ, η) coordinates is expressible in the form

$$\omega(\xi, \eta) = F(\xi, \eta)\exp\left[1/2R\phi(\xi, \eta)\right] \tag{3.6}$$

where ϕ is the harmonic conjugate of Ψ_0. From (3.3) the equation for $F(\xi, \eta)$ is given by

$$\frac{\partial^2 F}{\partial\xi^2} + \frac{\partial^2 F}{\partial\eta^2} - 1/4R^2\rho(\xi, \eta)^2 F = 0 \tag{3.7}$$

where

$$\rho(\xi, \eta) = \left[\left(\frac{\partial\Psi_0}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi_0}{\partial\eta}\right)^2\right]^{1/2} = \alpha(\xi, \eta)q(\xi, \eta).$$

Equation (3.7) is separable provided ρ^2 is expressible in the form

$$\rho(\xi, \eta)^2 = g(\xi) + h(\eta) \tag{3.8}$$

where g and h are arbitrary functions. Under this assumption we have

$$F(\xi, \eta) = E(\xi)H(\eta) \tag{3.9}$$

where

$$E'' - \left[\sigma + 1/4R^2 g(\xi)\right] E = 0, \tag{3.10}$$

$$H'' + \left[\sigma - 1/4R^2 h(\eta)\right] H = 0 \tag{3.11}$$

and σ is the separation eigenvalue.

The vorticity equation can be solved exactly when the obstacle is an ellipse whose major axis is parallel to the flow at infinity. The appropriate coordinate system (ξ, η) is defined by

$$x + iy = a\cosh(\xi + i\eta) = a\cosh\xi\cos\eta + ia\sinh\xi\sin\eta. \tag{3.12}$$

The curve $\xi = \xi_0$ is an ellipse with major axis of length $2a\cosh\xi_0$ and minor axis $2a\sinh\xi_0$. In order that the unit of length be the semi-major axis, we choose

$$a = \operatorname{sech}\xi_0. \tag{3.13}$$

It is convenient to make the transformation

$$\zeta = \xi - \xi_0. \tag{3.14}$$

Thus from (3.12), the coordinate system (ζ, η) is defined by

$$x + iy = a\cosh(\zeta + \xi_0 + i\eta) \tag{3.15}$$

and the ellipse is given by $\zeta = 0$. The metric coefficient for this transformation is $\alpha(\zeta, \eta)$ where

$$\alpha^2 = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 = 1/2a^2\left[\cosh 2(\zeta + \xi_0) - \cos 2\eta\right]. \tag{3.16}$$

The potential flow stream function $\Psi_0(\zeta, \eta)$ satisfies the boundary value problem

$$\nabla^2\Psi_0 = \alpha^{-2}\left\{\frac{\partial^2\Psi_0}{\partial\zeta^2} + \frac{\partial^2\Psi_0}{\partial\eta^2}\right\} = 0 \tag{3.17}$$

with

$$\Psi_0(0, \eta) = 0,$$

$$\Psi_0(\zeta, \eta) \sim \frac{1}{2}ae^{\xi_0+\zeta}\sin\eta = Ae^{\zeta}\sin\eta \text{ as } \zeta \to \infty$$

where

$$A = \frac{1}{2}ae^{\xi_0}.$$

The solution is

$$\begin{aligned} \Psi_0(\zeta, \eta) &= 2A\sinh\zeta\sin\eta, \\ \phi(\zeta, \eta) &= 2A\cosh\zeta\cos\eta \end{aligned} \tag{3.18}$$

from which we obtain

$$\rho(\zeta, \eta)^2 = 2A^2(\cosh 2\zeta - \cos 2\eta). \tag{3.19}$$

Thus the separability condition (3.8) is satisfied and the corresponding separated equations are, from (3.10) and (3.11),

$$E''(\zeta) - \left[\sigma + \frac{1}{2}A^2R^2\cosh 2\zeta\right]E(\zeta) = 0, \tag{3.20}$$

$$H''(\zeta) + \left[\sigma + \frac{1}{2}A^2R^2\cos 2\eta\right]H(\eta) = 0. \tag{3.21}$$

These are respectively the modified and the conventional Mathieu equations whose solutions are discussed at length in McLachlan [5].

In a streaming two dimensional flow past a symmetric body, the vorticity must be odd in η and periodic with period 2π. It must decay exponentially as $\zeta \to \infty$, except possibly in the wake $\eta = 0$. The periodicity condition determines the eigenvalues σ_n in (3.21). The corresponding odd eigenfunctions are denoted by $se_n(\eta, -\frac{1}{4}A^2R^2)$. In (3.20) the eigenfunctions which decay exponentially are denoted by $Gek_n(\zeta, -\frac{1}{4}A^2R^2)$. Thus from (3.6), (3.9) and (3.18) the vorticity function has the form

$$\begin{aligned}\omega(\zeta, \eta) = &-\exp[AR\cosh\zeta\cos\eta] \\ &\cdot \sum_{n=1}^{\infty} W_n Gek_n\left(\zeta, -\frac{1}{4}A^2R^2\right) se_n\left(\eta, -\frac{1}{4}A^2R^2\right)\end{aligned} \tag{3.22}$$

where the minus sign is included for convenience and the coefficients W_n are constants to be determined. The asymptotic behaviour of the vorticity is

$$\omega(\zeta, \eta) \sim W(\eta) \cdot \exp\left[-\frac{1}{2}\zeta - \frac{1}{2}ARe^{\zeta}(1 - \cos\eta)\right] \tag{3.23}$$

where $W(0) = 0$, $W'(0) \neq 0$ and $W(\eta)$ has period 2π.

The results for the Burgers flow past the unit circle ($r = 1$, $-\pi < \theta \leqslant \pi$) are recovered by letting $\xi_0 \to \infty$ and observing that

$$\begin{aligned} A &\to 1, \\ \zeta &\to \ln r, \\ \eta &\to \theta. \end{aligned} \tag{3.24}$$

If the ellipse is oriented so that its major axis is perpendicular to the flow at infinity, we use the coordinate system (λ, η) defined by

$$x + iy = a\sinh(\lambda + i\eta). \tag{3.25}$$

The ellipse is given by $\lambda = \lambda_0$. As before we define a modified coordinate system (μ, η) by letting $\mu = \lambda - \lambda_0$. The metric coefficient is $\alpha(\mu, \eta)$ where

$$\alpha^2 = 1/2a^2[\cosh 2(\mu + \lambda_0) + \cos 2\eta]. \tag{3.26}$$

The potential flow stream function $\Psi_0(\mu, \eta)$ satisfies a boundary value problem identical to that given in (3.17) and so the solution of the vorticity equation proceeds exactly as

before. The vorticity for this case is obtained from (3.22) by making the following adjustments:

$$\begin{aligned} \zeta &\to \mu, \\ A &= 1/2e^{\lambda_0}\operatorname{sech}\lambda_0. \end{aligned} \tag{3.27}$$

4. Solution of Poisson's equation. Poisson's equation (2.9) can be solved using a Green's function approach. Equation (3.22) gives the form of the vorticity function and so the problem reduces to the following:

$$\nabla^2\Psi \equiv [\alpha(\zeta,\eta)]^{-2}\left\{\frac{\partial^2\Psi}{\partial\zeta^2} + \frac{\partial^2\Psi}{\partial\eta^2}\right\} = -\omega(\zeta,\eta), \tag{4.1}$$

$$\Psi(0,\eta) = 0, \qquad \Psi(\zeta,\eta) \sim Ae^{\zeta}\sin\eta \quad \text{as } \zeta \to \infty.$$

The appropriate Green's function is defined by

$$\begin{aligned} \nabla^2 G(\zeta,\eta|\zeta',\eta') &= -[\alpha(\zeta,\eta)]^{-2}\delta(\zeta-\zeta')\delta(\eta-\eta'), \\ G(0,\eta|\zeta',\eta') = 0, \qquad G(\infty,\eta|\zeta',\eta') &= O(1). \end{aligned} \tag{4.2}$$

The solution of (4.2) is givenby

$$G(\zeta,\eta|\zeta',\eta') = -\frac{1}{4\pi}\ln\left[\frac{\cosh(\zeta-\zeta') - \cos(\eta-\eta')}{\cosh(\zeta+\zeta') - \cos(\eta-\eta')}\right]. \tag{4.3}$$

Proceeding in the usual way, we have

$$\Psi(\zeta',\eta') = \iint_{\mathscr{A}} [G\nabla^2\Psi - \Psi\nabla^2 G]\alpha^2\,d\zeta\,d\eta + \iint_{\mathscr{A}} \omega G\alpha^2\,d\zeta\,d\eta \tag{4.4}$$

where $\mathscr{A}$ is the fluid region between the obstacle $\zeta = 0$ and the curve $\zeta = \Gamma$ with Γ being large.

Green's Identity transforms the first integral in (4.4) into contour integrals around the boundary curves. The integral around $\zeta = 0$ vanishes because of the boundary conditions on G and Ψ. Thus we have

$$\iint_{\mathscr{A}} [G\nabla^2\Psi - \Psi\nabla^2 G]\alpha^2\,d\zeta\,d\eta = \int_{-\pi}^{\pi}\left(G\frac{\partial\Psi}{\partial\zeta} - \Psi\frac{\partial G}{\partial\zeta}\right)_{\zeta=\Gamma} d\eta. \tag{4.5}$$

Asymptotic expressions for G, Ψ and their derivatives are given below:

$$\Psi \sim Ae^{\zeta}\sin\eta + O(1), \tag{4.6}$$

$$\frac{\partial\Psi}{\partial\zeta} \sim Ae^{\zeta}\sin\eta + O(e^{-\zeta}), \tag{4.7}$$

$$G \sim \frac{1}{2\pi}\zeta' + \frac{1}{\pi}\sinh\zeta'\cdot e^{-\zeta}\cos(\eta-\eta') + O(e^{-2\zeta}), \tag{4.8}$$

$$\frac{\partial G}{\partial\zeta} \sim -\frac{1}{\pi}\sinh\zeta'\cdot e^{-\zeta}\cos(\eta-\eta') + O(e^{-2\zeta}). \tag{4.9}$$

Substituting these into (4.5), we obtain the following result:

$$\lim_{\Gamma\to\infty}\int_{-\pi}^{\pi}\left(G\frac{\partial\Psi}{\partial\zeta}-\Psi\frac{\partial G}{\partial\zeta}\right)_{\zeta=\Gamma}d\eta = 2A\sinh\zeta'\sin\eta' = \Psi_0(\zeta',\eta'). \tag{4.10}$$

Thus in (4.4) we have

$$\Psi(\zeta',\eta') = \Psi_0(\zeta',\eta') + \int_{-\pi}^{\pi}\int_0^{\infty}\alpha^2(\zeta,\eta)\omega(\zeta,\eta)G(\zeta,\eta|\zeta',\eta')\,d\zeta\,d\eta. \tag{4.11}$$

5. Determination of vorticity coefficients. The only boundary condition from (2.10) which remains to be satisfied is the no-slip condition. Its invocation yields unique values for the vorticity coefficients $\{W_n\}$. Thus in (4.11) we require

$$\frac{\partial\Psi}{\partial\zeta'}(0,\eta') = 0 \tag{5.1}$$

which implies that

$$0 = 2A\sin\eta' + \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty}\alpha^2\omega\,\frac{\sinh\zeta}{\cosh\zeta-\cos(\eta-\eta')}\,d\zeta\,d\eta. \tag{5.2}$$

From Gradshteyn and Ryzhik [6], we have

$$\frac{\sinh\zeta}{\cosh\zeta-\cos(\eta-\eta')} = 1 + 2\sum_{k=1}^{\infty}e^{-k\zeta}\cos k(\eta-\eta'). \tag{5.3}$$

But $\alpha^2(\zeta,\eta)\omega(\zeta,\eta)$ is odd in η and so the even part of (5.3) makes no contribution to (5.2). After substituting (5.3) into (5.2) and equating coefficients of $\sin k\eta'$ to zero ($k = 1,2,3,\ldots$), we have

$$\int_{-\pi}^{\pi}\int_0^{\infty}\alpha^2\omega e^{-k\zeta}\sin k\eta\,d\zeta\,d\eta = -2A\pi\delta_{k1}, \qquad k = 1,2,3,\ldots. \tag{5.4}$$

The substitution of (3.22) into (5.4) yields the following infinite linear system for the unknowns $\{W_n\}$:

$$\sum_{n=1}^{\infty}W_nC_{nk} = A\pi\delta_{k1}, \qquad k = 1,2,3,\ldots \tag{5.5}$$

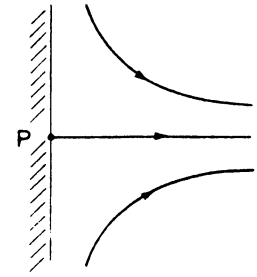
where

$$C_{nk} = \int_0^{\pi}\int_0^{\infty}\exp[-k\zeta + AR\cosh\zeta\cos\eta]\alpha(\zeta,\eta)^2 Gek_n(\zeta,-1/4A^2R^2)$$
$$\cdot se_n(\eta,-1/4A^2R^2)\sin k\eta\,d\zeta\,d\eta. \tag{5.6}$$

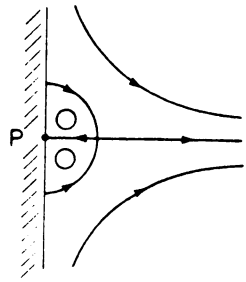
By truncating (5.5) at $n = k = 8$, we were able to obtain good results in the range $0 \leqslant R \leqslant 5$. The vorticity series (3.22) was observed to converge rather slowly in the vicinity of the ellipse. The rate of convergence was a maximum in the case of the circle and we therefore investigated separation phenomena for this geometry only.

6. Separation. Separation occurs on the downstream side of the circular cylinder provided the Reynolds number exceeds a critical value R_c defined as the value of R for which

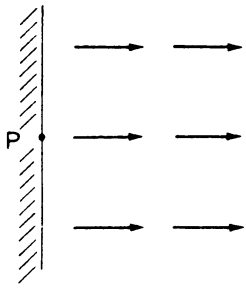
$$\frac{\partial\omega}{\partial\eta}(0,0) = 0. \tag{6.1}$$



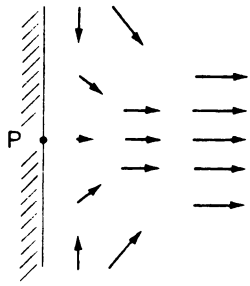
(a) Streamlines at the rear stagnation point P of the cylinder prior to separation.



(b) Streamlines after separation.



(c) Convective velocity field at the rear stagnation point of the cylinder in Oseen flow.



(d) Convective velocity field in Burgers flow.

FIG. 1.

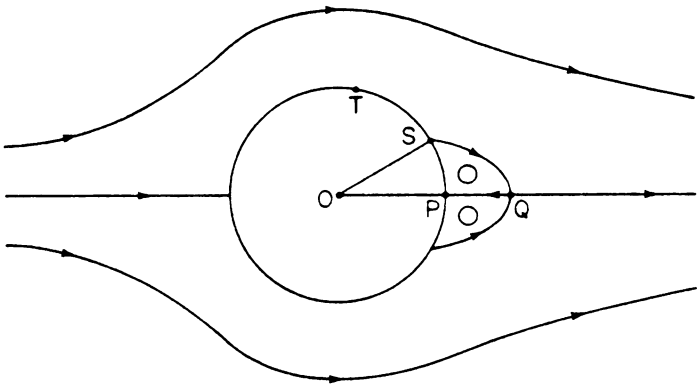


FIG. 2. Burgers flow past a circular cylinder at $R = 2.0$.

For Burgers flow we find $R_c = 1.12$ which is a new result. Yamada [7] has shown that $R_c = 1.51$ for Oseen flow and Underwood [8] has obtained $R_c = 2.88$ from a numerical solution of the full nonlinear equation.

The Burgers result should be less than the numerical value. The convective velocity field in Burgers flow is potential flow past the cylinder and this violates the no-slip condition at the cylinder's surface. The velocity field which solves the full Navier-Stokes equation satisfies this condition. Thus convection effects near the cylinder are more dominant in Burgers flow than in Navier-Stokes flow and any phenomena related to convection, such as separation, should occur at lower Reynolds numbers in Burgers flow.

The fact that the Burgers result is less than the Oseen value also can be explained. Separation begins at the rear stagnation point P of the cylinder where locally the flow appears as in Fig. 1(a). At the onset of separation two eddies of circulating fluid form about P. (We refer to this pair of eddies as a separation vortex.) The direction of motion along the axis of symmetry inside the vortex is opposed to that outside (Fig. 1(b)). In Oseen flow the convective velocity field is constant in magnitude and perpendicular to the cylinder boundary in the vicinity of P as shown in Fig. 1(c). Oseen convection therefore deters the establishment of reverse flow at P because it directly opposes the direction of fluid motion along the axis of symmetry inside the vortex. In contrast the convective velocity field in Burgers flow vanishes at the point P and is small in magnitude near P (Fig. 1(d)). Burgers convection does not oppose the establishment of a vortex about P to the same degree that Oseen convection does and separation therefore initiates in Burgers flow at a lower Reynolds number.

As R increases beyond $R_c = 1.12$ in Burgers flow, the separation vortex grows in size. When $R = 2.0$ the flow appears as in Fig. 2. The length of the vortex is $PQ = 0.53$ where OP is the unit of length, and $< SOP = 34.8°$. Point T, whose θ-coordinate is 83°, marks the location where the fluid pressure along the boundary is a minimum. The flow from T to S is against an adverse pressure gradient.

7. Calculation of drag coefficients. Several authors (Imai [9], Kawaguti [10], Dennis and Dunwoody [11]) have shown that an obstacle's drag coefficient can be obtained from the term of $O(1)$ in the asymptotic expansion of the stream function. This obviates the calculation of stress components which often are difficult to obtain accurately. The task therefore is to find the leading terms in the asymptotic expansion of (4.11).

We first obtain a series representation for the Green's function in (4.3). From Magnus et al. [12], we have

$$\sum_{k=1}^{\infty} \frac{1}{k} e^{-kt} \cos kx = 1/2t - 1/2 \ln(2\cosh t - 2\cos x), \qquad t > 0. \tag{7.1}$$

Manipulation of (7.1) yields the following results:

$$G(\zeta, \eta | \zeta', \eta') = \frac{1}{2\pi}\zeta + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1}{k} e^{-k\zeta'} \sinh k\zeta \cos k(\eta - \eta'), \qquad \zeta < \zeta', \tag{7.2}$$

$$= \frac{1}{2\pi}\zeta' + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1}{k} e^{-k\zeta} \sinh k\zeta' \cos k(\eta - \eta'), \qquad \zeta > \zeta'. \tag{7.3}$$

Substituting these into (4.11) and simplifying, we obtain

$$\Psi(\zeta', \eta') = \Psi_0(\zeta', \eta') + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-k\zeta'}\sin k\eta' \int_{-\pi}^{\pi}\int_0^{\zeta'}\alpha^2\omega\sinh k\zeta \sin k\eta \, d\zeta \, d\eta + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sinh k\zeta' \sin k\eta' \int_{-\pi}^{\pi}\int_{\zeta'}^{\infty}\alpha^2\omega e^{-k\zeta}\sin k\eta \, d\zeta \, d\eta. \tag{7.4}$$

An expression for the second integral can be obtained from (5.4):

$$\int_{-\pi}^{\pi}\int_{\zeta'}^{\infty}\alpha^2\omega e^{-k\zeta}\sin k\eta \, d\zeta \, d\eta = -2A\pi\delta_{k1} - \int_{-\pi}^{\pi}\int_0^{\zeta'}\alpha^2\omega e^{-k\zeta}\sin k\eta \, d\zeta \, d\eta. \tag{7.5}$$

Thus a series representation for the stream function is, from (7.4),

$$\Psi(\zeta', \eta') = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-k\zeta'}\sin k\eta' \int_0^{\pi}\int_0^{\zeta'}\alpha^2\omega e^{k\zeta}\sin k\eta \, d\zeta \, d\eta - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{k\zeta'}\sin k\eta' \int_0^{\pi}\int_0^{\zeta'}\alpha^2\omega e^{-k\zeta}\sin k\eta \, d\zeta \, d\eta. \tag{7.6}$$

This expression can be written in the form

$$\Psi(\zeta', \eta') = Ae^{\zeta'}\sin\eta' - \frac{1}{\pi}\sin\eta'\left[\frac{\int_0^{\pi}\int_0^{\zeta'}\alpha^2\omega e^{-\zeta}\sin\eta \, d\zeta \, d\eta + \pi A}{e^{-\zeta'}}\right] - \frac{1}{\pi}\sum_{k=2}^{\infty}\frac{1}{k}\sin k\eta'\left[\frac{\int_0^{\pi}\int_0^{\zeta'}\alpha^2\omega e^{-k\zeta}\sin k\eta \, d\zeta \, d\eta}{e^{-k\zeta'}}\right] + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin k\eta'\left[\frac{\int_0^{\pi}\int_0^{\zeta'}\alpha^2\omega e^{k\zeta}\sin k\eta \, d\zeta \, d\eta}{e^{k\zeta'}}\right]. \tag{7.7}$$

It can be seen from (5.4) that the first two bracked expressions in (7.7) are indeterminate forms $(0/0)$ as $\zeta' \to \infty$. The third expression is of the form (∞/∞) in the limit. By invoking l'Hôpital's rule we obtain contributions from each expression to the $O(1)$ term in the asymptotic expansion of Ψ. The result is

$$\Psi(\zeta', \eta') \sim Ae^{\zeta'}\sin\eta' + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin k\eta'\left\{\lim_{\zeta\to\infty}\int_0^{\pi}\alpha^2\omega\sin k\eta \, d\eta\right\} + O(e^{-\zeta'}). \tag{7.8}$$

The limit can be computed and the series summed. The details are included in the Appendix. The expansion has the form

$$\Psi(\zeta', \eta') \sim Ae^{\zeta'}\sin\eta' - 1/2C_D\left(\pm 1 - \frac{\eta'}{\pi}\right) + O(e^{-\zeta'}) \tag{7.9}$$

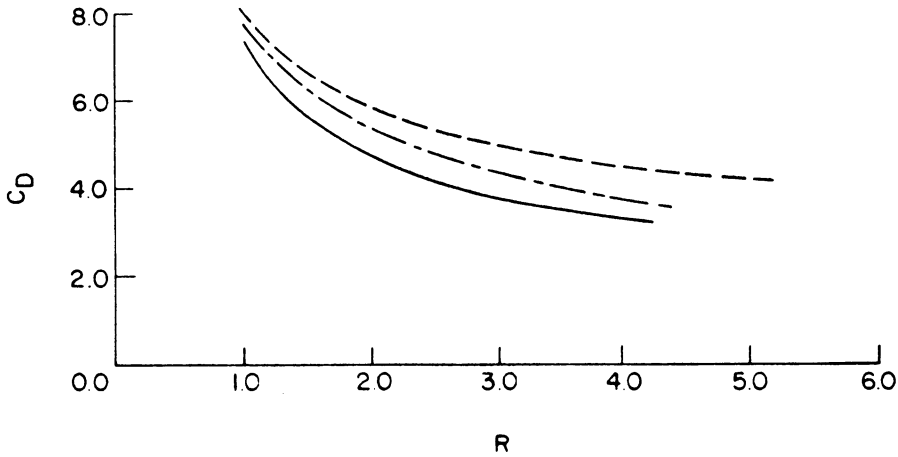


FIG. 3. Drag coefficient vs. Reynolds number for flow past a circular cylinder. ----------, Oseen flow; —·—·—·—, Burgers flow; ————, experiment (Tritton [13]).

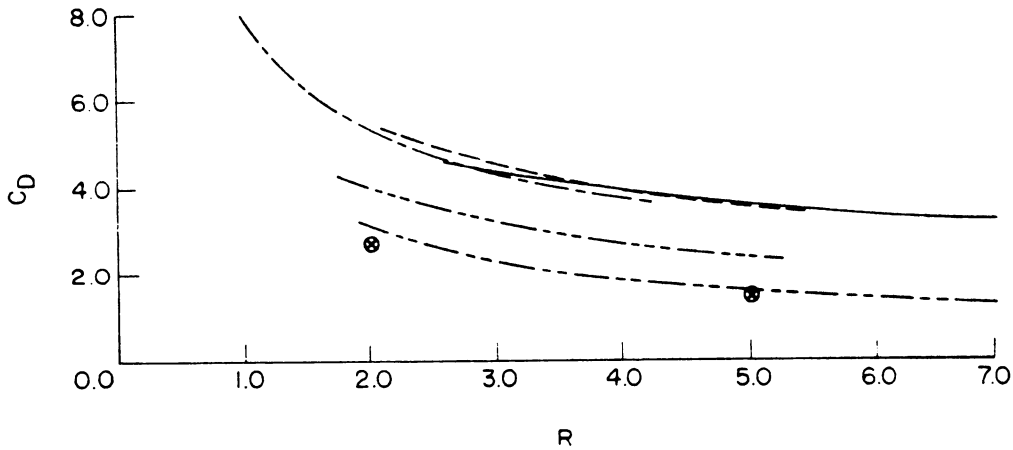


FIG. 4. Drag coefficient vs. Reynolds number for Burgers flow past a variety of geometries. ————, flat plate perpendicular to flow; ---------, ellipse (aspect ratio 3 : 1) perpendicular to flow; —·—·—·—·—, circle; —·---·—·---, ellipse (aspect ratio 3 : 1) parallel to flow; —·—·—·—·—·—, flat plate parallel to flow; ⊗, numerical solution (Dennis & Dunwoody [11]) of the full nonlinear equation for flow past a flat plate.

where, in the second term, the plus sign is chosen when $0 < \eta' \leqslant \pi$ and the minus sign when $-\pi < \eta' < 0$. This term is analytic along $\eta' = \pi$, but suffers a finite jump discontinuity along $\eta' = 0$ which coincides with the wake. Dennis and Dunwoody [11] comment that the term must be present in order to give non-zero drag. Kawaguti [10] shows that the constant C_D is the drag coefficient for the obstacle in the flow. Results concerning the Burgers drag coefficients are summarized in Figs. 3 and 4.

8. Boundary layer thickness. The procedure outlined in Sec. 7 can be extended to obtain higher order terms in the expansion of Ψ. When expressed in polar coordinates (r, θ), this expansion has the form

$$\Psi \sim r\sin\theta - 1/2C_D\left(\pm 1 - \frac{\theta}{\pi}\right) + \sum_{n=1}^{\infty} \Phi_n r^{-n} \sin n\theta + \chi(r, \theta) \tag{8.1}$$

where the Φ_n $(n = 1, 2, 3\ldots)$ are constants and $\chi(r, \theta)$ is exponentially small in r. Except along the line $\theta = 0$, the algebraic part of the expansion is harmonic and constitutes the potential flow far from the obstacle.

Because of the tedious nature of the calculation, only the Φ_1-term was computed. In Table 1, the coefficients C_D and Φ_1 are given for both Oseen and Burgers flows past a circular cylinder.

TABLE 1

	(a) Oseen flow			(b) Burgers flow	
R	C_D	$-\Phi_1$	R	C_d	$-\Phi_1$
1.51	6.6236	0.5995	1.12	7.3018	1.1251
2.0	5.8494	0.5777	2.0	5.3418	1.1396
3.0	4.9776	0.5485	2.83	4.4654	1.1439
4.0	4.4962	0.5302	3.46	4.0426	1.1486
5.0	4.1836	0.5171	4.0	3.7526	1.1497

Since both Oseen and Burgers flows are spatially uniform approximations to the exact solution of the Navier-Stokes equation, they predict a boundary layer surrounding the obstacle. The outer edge of the boundary layer is defined to be the curve along which the algebraic part of the asymptotic expansion of Ψ vanishes. The curve so defined determines the displacement body which the potential flow far from the cylinder "sees". The displacement body includes the cylinder, its separation vortex and the surrounding boundary layer.

By setting equal to zero the first few terms of the expansion (8.1), we obtain an approximation to the displacement body for the flow in question. A typical example is given in Fig. 5. The displacement body is semi-infinite with its width at infinity numerically equal to the drag coefficient. Since the boundary of the circular cylinder is given by $r = 1$, the thickness of the boundary layer is easily calculated.

Boundary layer thicknesses were obtained for the ten flows considered in Table 1 and an expression of the form $\Lambda(\theta)/R^{\gamma(\theta)}$ was curve-fitted to the data. The results are given in

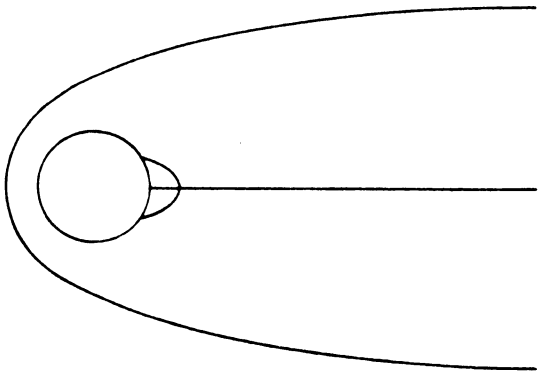


FIG. 5. Displacement body for Burgers flow past a circular cylinder at $R = 2.0$.

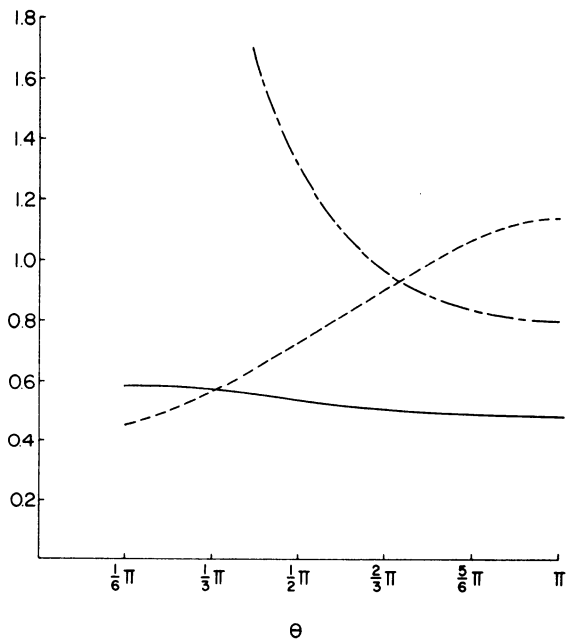


FIG. 6. Graphs of the functions obtained using a least-squares curve-fitting procedure when the boundary layer thickness is assumed to have the form $\Lambda(\theta)/R^{\gamma(\theta)}$. The flow is past a circular cylinder with $\theta = \pi$ denoting the upstream axis of symmetry. —·—·—·—·—·—, $\Lambda(\theta)$ for both Burgers and Oseen flows; ————, $\gamma(\theta)$ for Burgers flow; ----------, $\gamma(\theta)$ for Oseen flow.

Fig. 6. The function $\Lambda(\theta)$ is essentially the same in both cases. In Burgers flow the function $\gamma(\theta)$ hovers about the value 0.5 in the range $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$.

Analytic studies of the boundary layer on a semi-infinite flat plate using the full nonlinear equations show that the thickness is proportional to $R^{-1/2}$. Our work, although not conclusive, suggests a similar result for a circular cylinder using a linear model. The apparent agreement between this prediction of Burgers flow and that of nonlinear analysis suggests that Burgers approximation provides a qualitatively accurate model of the flow near an obstacle at low Reynolds numbers.

Appendix. The expansion of Ψ in (7.8) contains the expression

$$\lim_{\zeta \to \infty} \int_0^{\pi} \alpha^2(\zeta, \eta)\omega(\zeta, \eta) \sin k\eta \, d\eta \tag{A1}$$

which can be computed once the asymptotic behaviour of α^2 and ω is known. From (3.16) we have

$$\alpha^2(\zeta, \eta) \sim A^2 e^{2\zeta}. \tag{A2}$$

From McLachlan [5], we have

$$Gek_n(\zeta, q) \sim c_n\left(\frac{1}{2}\pi\right)^{1/2} |q|^{-1/4} \exp\left[-\frac{1}{2}\zeta - |q|^{1/2} e^{\zeta}\right] \tag{A3}$$

where $q < 0$.

The asymptotic form of the vorticity is therefore given by

$$\omega(\zeta, \eta) \sim -\left(\frac{\pi}{AR}\right)^{1/2} \exp[-1/2\zeta - 1/2ARe^{\zeta}(1 - \cos\eta)] \cdot \sum_{n=1}^{\infty} c_n W_n se_n(\eta, -1/4A^2R^2). \tag{A4}$$

Thus the integral in (A1) simplifies to

$$\int_0^{\pi} \alpha^2\omega \sin k\eta \, d\eta \sim -\left(\frac{\pi A^3}{R}\right)^{1/2} \exp[-1/2(ARe^{\zeta} - 3\zeta)] \cdot \sum_{n=1}^{\infty} c_n W_n \int_0^{\pi} \exp[1/2ARe^{\zeta}\cos\eta] \sin k\eta se_n(\eta, -1/4A^2R^2) \, d\eta. \tag{A5}$$

From McLachlan [5], we have

$$se_n(\eta, -1/4A^2R^2) = \sum_{m=0}^{\infty} S_m^{(n)} \sin(2m + \varepsilon_n)\eta \tag{A6}$$

where $\varepsilon_n = 1$ if n is odd, $= 2$ if n is even. Therefore we have

$$se_n(\eta, -1/4A^2R^2)\sin k\eta$$
$$= 1/2 \sum_{m=0}^{\infty} S_m^{(n)}[\cos(2m + \varepsilon_n - k)\eta - \cos(2m + \varepsilon_n + k)\eta]. \quad \text{(A7)}$$

The substitution of (A7) into (A5) yields integrals of the form

$$\int_0^{\pi} \exp[u\cos\theta]\cos n\theta\, d\theta = \pi I_n(u) \quad \text{(A8)}$$

where I_n is the modified Bessel function of the first kind. After evaluating the integral in (A5) and replacing the Bessel functions by their asymptotic forms, we have

$$\int_0^{\pi} \alpha^2\omega \sin k\eta\, d\eta \sim -\frac{2\pi}{R^2} k \sum_{n=1}^{\infty} c_n W_n \sum_{m=0}^{\infty} (2m + \varepsilon_n) S_m^{(n)} + O(e^{-\zeta}). \quad \text{(A9)}$$

Taking the limit in (A1) and substituting back into (7.8), we obtain

$$\Psi(\zeta', \eta') \sim Ae^{\zeta'} \sin\eta' - \frac{1}{\pi} C_D \sum_{k=1}^{\infty} \frac{1}{k} \sin k\eta' + O(e^{-\zeta'}) \quad \text{(A10)}$$

where

$$C_D = \frac{4\pi}{R^2} \sum_{n=1}^{\infty} c_n W_n \sum_{m=0}^{\infty} (2m + \varepsilon_n) S_m^{(n)}. \quad \text{(A11)}$$

But

$$\sum_{k=1}^{\infty} \frac{1}{k} \sin k\eta' = 1/2(\pm\pi - \eta') \quad \text{(A12)}$$

where the plus sign is chosen when $0 < \eta' \leqslant \pi$ and the minus sign when $-\pi < \eta' < 0$. Thus the asymptotic form of the stream function for Burgers flow past an elliptical cylinder is given by (7.9).

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