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ON THE STABILITY OF LINEAR NONCONSERVATIVE SYSTEMS*

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Abstract. For a system of linear second-order differential equations, a stability criterion is derived which gives a simple relation between eigenvalues of two of the coefficient matrices and an estimate of the lower bound $|\lambda|_{\min}$ of the eigenvalue for the nonlinear eigenvalue problem of the total system. Estimations of $|\lambda|_{\min}$ are given, and applications of the stability criterion are shown by numerical examples.

1. Introduction. Linear mechanical systems are naturally described by a matrix differential equation of the form

$$M\ddot{X} + (D + G)\dot{X} + (K + N)X = 0, \tag{1}$$

where M, D, and K are Hermitian matrices (often real symmetric), G and N are skew-Hermitian (often real skew-symmetric), X is a vector containing the generalized coordinates, and a dot means differentiation with respect to time t. If $D \neq 0$ and/or $N \neq 0$, system (1) is called nonconservative.

The standard tool for stability investigations is to apply the Routh-Hurwitz criterion, see [1], [2], to the system (1) involving the characteristic polynomial of degree $2n$. This procedure is somewhat cumbersome, and therefore several authors have discussed the stability of system (1) expressed by the properties of the M, D, G, K, and N matrices. One way to do this is by means of the Rayleigh quotients, see [3], [4], another is to use the direct method of Liapunov, see [5], [6], [7], and a third method is using Gerschgorin's theorem as shown in [8].

Nevertheless, there are not many stability theorems concerning nonconservative systems involving the N matrix. Here we would like to state a sufficient criterion of stability for certain systems (1), which contain the N matrix. Our procedure is a kind of energy consideration, finding the first integral of (1), and is related to a method applied in [9], where $N = 0$.

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2. A stability theorem for nonconservative systems.

THEOREM. If

1) $\lambda = 0$ is not an eigenvalue of system (1), ($\det(K + N) \neq 0$),

2) M and K are both positive semi-definite and D is positive definite,

3)
$$n_{\max} < |\lambda|_{\min} d_{\min}, \tag{2}$$

then the system (1) is asymptotically stable.

Here $d_{\min}$ and $n_{\max}$ denote, respectively, the minimum and maximum eigenvalues of D and iN (iN is Hermitian). $|\lambda|_{\min}$ is the smallest numerical value of all eigenvalues of system (1).

Particularly when $N = 0$, conditions 1) and 2) imply K positive definite and the theorem reduces to a well-known condition for asymptotic stability. Therefore mainly the case $N \neq 0$ is of interest. In order to prove the theorem, we multiply (1) by $\dot{\bar{X}}'$, where $\bar{X}$ denotes the complex conjugate and X' the transpose. Integrating from 0 to t, some simple calculations lead to the equation

$$\dot{\bar{X}}'M\dot{X} + \bar{X}'KX + 2\int_0^t \dot{\bar{X}}'D\dot{X}\,du + \int_0^t (\dot{\bar{X}}'NX - \bar{X}'N\dot{X})\,du = k, \tag{3}$$

where $k = \dot{\bar{X}}'(0)M\dot{X}(0) + \bar{X}'(0)KX(0)$ is a constant.

For every eigenvalue $\lambda = \alpha + i\beta$ of system (1) there exists—independent of the multiplicity of the eigenvalue—a solution to (1) of the form

$$X = X_0 e^{\lambda t}, \tag{4}$$

where X_0 is the corresponding eigenvector to λ. Since (4) has to fulfil (3), we get

$$\begin{aligned} &|\lambda|^2 \exp(2\alpha t)\,\bar{X}_0'MX_0 + \exp(2\alpha t)\,\bar{X}_0'KX_0 \\ &\quad + \left[2|\lambda|^2 \bar{X}_0'DX_0 - 2\beta \bar{X}_0'iNX_0\right] \int_0^t \exp(2\alpha u)\,du = k. \end{aligned} \tag{5}$$

Using normalized eigenvectors X_0, that means $\bar{X}_0'X_0 = 1$, we recognize in Eq. (5) the Rayleigh quotients of M, K, D, and iN. These can be estimated in accordance with condition 2) of the theorem as

$$\bar{X}_0'MX_0 \geqslant 0, \quad \bar{X}_0'KX_0 \geqslant 0, \quad \bar{X}_0'DX_0 \geqslant d_{\min} > 0, \quad \bar{X}_0'iNX_0 \leqslant n_{\max}. \tag{6}$$

This leads, together with inequality (2), to

$$\bar{X}_0'iNX_0 \leqslant n_{\max} < |\lambda|_{\min} d_{\min} \leqslant |\lambda|_{\min} \bar{X}_0'DX_0. \tag{7}$$

Multiplying by $2|\lambda|$, (remember condition 1)), we get

$$2|\lambda|\bar{X}_0'iNX_0 < 2|\lambda|\,|\lambda|_{\min} \bar{X}_0'DX_0. \tag{8}$$

Using $|\beta| \leqslant |\lambda|$ and $|\lambda|_{\min} \leqslant |\lambda|$, we have

$$2|\beta|\bar{X}_0'iNX_0 < 2|\lambda|^2 \bar{X}_0'DX_0. \tag{9}$$

Therefore the bracket in (5) is always positive:

$$2|\lambda|^2 \bar{X}_0'DX_0 - 2\beta \bar{X}_0'iNX_0 > 0. \tag{10}$$

Now we assume that the eigenvalue λ has a real part $\alpha \geqslant 0$. Applying (6) and (10) to Eq. (5) shows that the left-hand side of (5) goes to infinity as t becomes infinite, since the integral does so, whereas the right-hand side of (5) is a constant. Therefore the assumption $\alpha \geqslant 0$ must be wrong.

3. Estimation of $|\lambda|_{\min}$. The usefulness of the stability theorem depends on whether we can make good estimates of $|\lambda|_{\min}$. In the following, m, k, d, g, and n denote the Rayleigh quotients of M, K, D, iG, and iN. The indices max or min denote, respectively, maximum and minimum value of the quotient, this value being identical with, respectively, the largest or smallest eigenvalue of the associated matrix. We shall now estimate $|\lambda|_{\min}$ in three different ways.

(a) *Perturbation theory*. For any conservative system

$$M\ddot{X} - KX = 0 \tag{12}$$

with both M and K positive definite, we have

$$|\lambda|_{\min} \geqslant \sqrt{k_{\min}/m_{\max}}\,. \tag{13}$$

Introducing small disturbances in the form of matrices D, G, and N proportional to small parameters, a perturbation calculation of first order shows a relation for the nonconservative system

$$|\lambda|_{\min} \geqslant \sqrt{k_{\min}/m_{\max}} - g_{\max}/2, \tag{14}$$

which is consistent with (13) for $G = 0$.

(b) *Norm estimate*. In the general case a nonconservative system—for simplicity written with the unit matrix I as mass matrix—

$$I\ddot{X} + B\dot{X} + CX = 0 \tag{15}$$

is equivalent to a first-order system

$$\dot{Y} = AY, \quad A = \begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix}, \quad Y = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}. \tag{16}$$

Therefore we have

$$|\lambda| \geqslant \|A^{-1}\|^{-1}, \tag{17}$$

where $\|A^{-1}\|$ denotes an arbitrary norm of the inverse

$$A^{-1} = \begin{bmatrix} -C^{-1}B & -C^{-1} \\ I & 0 \end{bmatrix}. \tag{18}$$

Using, for example, the simple Euclidean norm, it follows that

$$|\lambda|_{\min} \geqslant \left(\|C^{-1}B\|^2 + \|C^{-1}\|^2 + \|I\|^2\right)^{-1/2}. \tag{19}$$

(c) *Rayleigh polynomial*. Finally, the eigenvalue λ is one of the roots of the equation

$$m\lambda^2 + (d + ig)\lambda + k + in = 0, \tag{20}$$

where m, d, g, k, and n are the earlier mentioned Rayleigh quotients formed with the eigenvector belonging to λ. It is easy to show that

$$|\lambda| \geqslant -|d + ig|/2m + \sqrt{|d + ig|^2/4m^2 + |k + in|/m} \tag{21}$$

and therefore

$$|\lambda|_{\min} \geqslant -\sqrt{(d_{\max}^2 + g_{\max}^2)}/2m_{\min} + \sqrt{d_{\min}^2/4m_{\max}^2 + k_{\min}/m_{\max}}\,. \tag{22}$$

4. Examples. (a) A massless shaft with elastic coefficient k_e carries a single mass m_0 and rotates with constant angular velocity ω. Assuming external damping with coefficient d_e and internal damping with coefficient d_i, the equation of motion for the center of mass moving in a plane perpendicular to the shaft is

$$\begin{bmatrix} m_0 & 0 \\ 0 & m_0 \end{bmatrix} \ddot{X} + \begin{bmatrix} d_e + d_i & 0 \\ 0 & d_e + d_i \end{bmatrix} \dot{X} + \left\{ \begin{bmatrix} k_e & 0 \\ 0 & k_e \end{bmatrix} + \begin{bmatrix} 0 & d_i\omega \\ -d_i\omega & 0 \end{bmatrix} \right\} X = 0. \tag{23}$$

If the damping coefficients d_e and d_i are small, we can use (13)

$$|\lambda|_{\min} \geqslant \sqrt{k_e/m_0} = \omega_0, \tag{24}$$

where ω_0 is the normal critical speed of the shaft. Furthermore

$$n_{\max} = d_i\omega, \qquad d_{\min} = d_e + d_i \tag{25}$$

and therefore the stability theorem states that (23) is asymptotically stable if

$$\omega < \omega_0(1 + d_e/d_i), \tag{26}$$

which is the actual stability limit of the shaft found in the literature, see [10].

(b) For the nonconservative system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{X} + \begin{bmatrix} 4 & -1.5 \\ 1 & 4 \end{bmatrix} \dot{X} + \begin{bmatrix} 2 & 0.5 \\ -0.5 & 3 \end{bmatrix} X = 0, \tag{27}$$

we find

$$C^{-1} = \begin{bmatrix} 0.48 & -0.08 \\ 0.08 & 0.32 \end{bmatrix}, \qquad C^{-1}B = \begin{bmatrix} 1.84 & -1.04 \\ 0.64 & 1.16 \end{bmatrix}. \tag{28}$$

If we use estimation (19), we get $|\lambda|_{\min} = 0.34163$. With $n_{\max} = 0.5$ and $d_{\min} = 3.75$ the theorem shows that (27) is asymptotically stable. We can confirm this by computing the eigenvalues as $\lambda = -0.54119 \pm i0.36833$ and $\lambda = -3.45881 \pm i1.61882$.

(c) Instead of using matrix norms to estimate $|\lambda|_{\min}$, we can use Eq. (22) derived from the Rayleigh polynomial. For the system given by (27) we get $d_{\max} = 4.25$, $d_{\min} = 3.75$, $g_{\max} = 1.25$, $m_{\max} = m_{\min} = 1$ and $k_{\min} = 2$, which gives $|\lambda|_{\min} \geqslant 0.13353$, a considerably inferior estimate than the one we obtained by using (19). But again the stability theorem gives that (27) is asymptotically stable.

5. Conclusion. Nonconservative systems (1) are asymptotically stable if all eigenvalues have negative real parts. Sufficient conditions for this are that $\det(K + N) \neq 0$, the M and K matrices are nonnegative definite, the D matrix is positive definite and inequality

(2) holds. The properties of the D and N matrices enter explicitly in the criterion (2) via $d_{\min}$ and $n_{\max}$, respectively the smallest and largest eigenvalues of the matrices D and iN, while the M, G, and K matrices enter implicitly via $|\lambda|_{\min}$ for the total system as seen, for example, in the perturbation estimation (14). The usefulness of the theorem depends mainly on the estimation of $|\lambda|_{\min}$. It must be admitted that the theorem—because of its specially simple construction—leads to pessimistic stability limits for many systems with parameters.

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