EXISTENCE OF SOLUTIONS IN A POPULATION DYNAMICS PROBLEM*

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Abstract. In this paper we show the existence of a solution for the Gurtin–MacCamy model in population dynamics with age dependence and diffusion. We also discuss the behavior of this solution.

1. Introduction. We consider the population dynamics model with age dependence and diffusion proposed by Gurtin and MacCamy in a series of papers [1–8]. In this theory $p(a, t, x)$ represents the population distribution of age “$a$” at time “$t$”; $x$ is the space variable. There is a birth law

$$p(0, t, x) = \int_0^\infty \beta_0(a, u(x, t)) p(a, t, x) \, da,$$

where $u(x, t) = \int_0^\infty p(a, t, x) \, da$ is the space population density. The initial age-space distribution $p(a, 0, x) = p_0(a, x)$ is assumed to be nonnegative, and the balance law is:

$$\rho_\mu(a, t, x) + \rho_r(a, t, x) + \mu(a, u(x, t)) p(a, t, x) = -q_x(a, t, x)$$

where $q(a, t, x) = -k p(a, t, x) u_s(x, t)$ is the diffusion velocity that corresponds to the case in which species disperse to avoid crowding.

It is also assumed that the death modulus $\mu(a, u)$ is independent of $a$, and that the birth modulus is $\beta_0(a, u) = \beta(u)e^{-a}$ which models the case in which individuals are more fertile at younger ages. The case in which the birth modulus $\beta$ and the death modulus $\mu$ do not depend on the actual population $u$, have been treated by MacCamy [8] in the case of limited environment.

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We integrate the differential equation from 0 to ∞, with respect to a, assuming that \( p(a, t, x) \) tends to 0 as \( a \) tends to ∞. Also, we multiply it by \( e^{-a \alpha} \) and integrate from 0 to ∞. We arrive at

\[
\begin{align*}
    u_t + \mu(u)u - \beta_0(u)G &= (uu_x)_x, \\
    G_t + [\mu(u) + \alpha + \beta_0(u)]G &= (G u_x)_x,
\end{align*}
\]

with the initial data \( u(x, 0) = u_0(x), G(x, 0) = G_0(x), \) where \( G \) is the auxiliary function

\[
G(x, t) = \int_0^\infty e^{-a \alpha} p(a, t, x) \, da
\]

and

\[
u_0(x) = \int_0^\infty \rho_0(a, x) \, da, \quad G_0(x) = \int_0^\infty e^{-a \alpha} \rho_0(a, x) \, da.
\]

The natural assumption for \( \beta \) and \( \mu \) is that they are positive smooth functions, such that \( \beta(u) \) is decreasing in \( u \) and \( \mu(u) \) is increasing in \( u \). However, in this work we will assume only that both functions are uniformly bounded along with their first derivatives.

This problem is of most interest when the initial datum \( u_0(x) \) has compact support, so we assume \( u_0(x) \geq 0 \) for \( |x| < x_1 \), and \( u_0(x) = 0 \) for \( |x| \geq x_1 \). Also, from the definition of \( u_0 \) and \( G_0 \), we can see that if for some \( x \), \( \rho_0(a, x) = 0 \) almost everywhere in \( a \), then \( u_0(x) = G_0(x) = 0 \). Similarly if \( \rho_0(a, x) > 0 \) in a set of finite measure, then \( 0 < G_0(x) < u_0(x) \).

We introduce the quotient function \( q(x, t) \) defined by \( q(x, t)u(x, t) = G(x, t) \) and \( q_0(x, t)u_0(x, t) = G_0(x) \). With this new function the system becomes

\[
\begin{align*}
    (uu_x)_x + [\beta(u)q - \mu(u)]u &= u_t, \\
    q_t - u_xq_x &= (\beta(u) - \alpha)q - \beta(u)q^2, \\
    u(x, 0) &= u_0(x), \quad q(x, 0) = q_0(x),
\end{align*}
\]

with the assumptions \( u_0(x) \geq 0 \) and \( 0 < \bar{m} \leq q_0(x) \leq 1 \).

It is well known that even for the porous medium equation \( (uu_x)_x = u_t \), with real analytic data, we can not expect to obtain classical solutions for all times, unless the initial data is strictly positive. We introduce a suitable definition of weak solutions for the system (1) and (2), in the following way:

Assume \( u \) and \( q \) are classical solutions of the system. Multiplying (1) by \( q \), (2) by \( u \), and substracting, we arrive at

\[
(uq)_t = \left[ \frac{1}{2}(u^2)_x q \right]_x + (\beta(u) - \alpha - \mu(u)) uq.
\]

We let \( \Omega_T = (-\infty, \infty) \times (0, T) \) and define a test function \( \phi(x, t) \) as a continuously differentiable function in \( \Omega_T \), with compact support in \( \Omega_T \), and that equals 0 near \( T \).
Then multiplying this equation and Eq. (1) by $\phi$ and integrating over $\Omega_T$, we obtain

$$\int_{\Omega_T} \left[ \frac{1}{2} (u^2)_x \phi_x - u \phi_t \right] \, dx \, dt$$

$$= \int_{\Omega_T} \left[ \beta(u) q - \mu(u) \right] u \phi \, dx \, dt + \int_\mathbb{R} u_0(x) \phi(x,0) \, dx,$$

(3)

$$\int_{\Omega_T} \left[ \frac{1}{2} (u^2)_x q \phi_x - u q \phi_t \right] \, dx \, dt$$

$$= \int_{\Omega_T} \left[ \beta(u) - \alpha - \mu(u) \right] u q \phi \, dx \, dt + \int_\mathbb{R} u_0(x) q_0(x) \pi(x,0) \, dx.$$

(4)

Thus, we define a weak solution of problem (1) and (2) as a pair of functions $u$, $q$ such that: $u$ is continuous in $\Omega_T$, $u^2$ is differentiable with respect to $x$ in the sense of distributions, $q$ belongs to $L^2_{\text{loc}}(\Omega_T)$ and $u$ and $q$ satisfy (4) for any function $\phi$ continuously differentiable in $\partial \Omega_T$, with compact support in $\partial \Omega_T$ that equals 0 near $T$.

In this work we answer in the affirmative the question of existence, proposed by Gurtin in [1] as an open problem.

The main difficulty of the proof lies in the fact that (1) is not a uniformly parabolic equation because $u_0(x) = 0$ for $|x|$ larger than $x_1$. Thus, first we obtain approximating solutions $u_\epsilon$ and $q_\epsilon$, with positive initial data $u_\epsilon(x) + \epsilon$ and $q_\epsilon(x)$. Still, it is not possible to solve the system directly even with positive smooth initial data because of the presence of the function $q$ in (1) which, being a solution of a first-order differential equation, does not allow good a priori estimates for the function $u$. Thus, we consider another approximation problem by taking $q_{(n)}$, a "nice" version of $q$, instead of $q$. Once we obtain classical solutions to this last problem, we let $n$ tend to $\infty$ and then let $\epsilon$ tend to 0.

To solve the approximating problems, we integrate the equation for $q$ along the characteristics defined by $-u_x$, obtaining an explicit expression for the solution, together with some estimates in the supremum norm of the difference of two such solutions. Some estimates for the gradient of $q$ are also obtained. Then, we substitute this solution in the first equation and solve for $u$. To this scheme we apply the Schauder Fixed Point Theorem to obtain a solution for the system.

2. Some lemmas and notations. Through this paper differentiation is indicated by subscripts, $\Omega_T = \{(x,t)/x \in \mathbb{R}, \ 0 < t < T \}$, and $\overline{\Omega_T} = \{(x,t)/x \in \mathbb{R}, \ 0 \leq x \leq T \}$. $C^{2,1}(\Omega_T)$ is the Banach Space consisting of functions $u(x,t)$ defined in $\Omega_T$ with continuous second derivatives in $x$, and continuous first derivative in $t$. $C^a(\Omega_T)$ is the space of functions $u(x,t)$ defined in $\Omega$, for which the $a$-norm,

$$|u|_a = \sup_{\Omega} |u| + \sup_{\Omega} \frac{|u(x,t) - (y,s)|}{|x - y|^a + |t - s|^{a/2}}$$
is bounded in $\Omega_r; \ C^{2+\alpha}(\Omega_T)$ is the space of functions $u(x,t)$ for which the $2 + \alpha$ norm $|u|_{2+\alpha} = |u_{xx}|_\alpha + |u_{x}|_\alpha + |u_{t}|_\alpha + |u|_\alpha$ is bounded in $\Omega_T$. $K(a, b, c, \ldots)$ are constants that depend on $a, b, c, \ldots$. The following results will be used in the proof.

**Lemma 1. (The Hölder property).** Let $u \in C^{2,1}(\Omega_T)$ be a solution of

$$\begin{align*}
    uu_{xx} + (u_x)^2 &= h(x,t,u)(u-\varepsilon) = u_t, \\
    u(x,0) &= u_0(x) + \varepsilon,
\end{align*}$$

such that $\varepsilon \leq u \leq M$, $|h| \leq M_1$, $u_x, u_t$ are bounded and $|u_0, u_0| \leq M_0$. Then the $\alpha$-norm of $u$ is bounded by a constant $K_1$ depending only on $\alpha, M, M_0, M_1$ and $T$, i.e.,

$$\|u(x,t) - u(y,s)\|_{2+\alpha} \leq K_1, \quad \text{for any } x, y \in \mathbb{R}, \ s, t \in [0, T].$$

The proof of this lemma is given in [9] and consists in considering the function

$$g(x, y, t, s) = \frac{|u(x, t) - u(y, s)|^{2+\alpha}}{|x - y|^\alpha + |t - s|^{\alpha/2}}$$

at a point of maximum.

**Lemma 2. (Bounds for the gradient of $u$).** Let $u \in C^{2,1}(\Omega_T)$ be a solution of

$$\begin{align*}
    uu_{xx} + (u_x)^2 + (\beta(u)q - \mu(u))(u-\varepsilon) &= u_t, \\
    u(x,0) &= u_0(x) + \varepsilon, \quad q(x,0) = q_0(x),
\end{align*}$$

where $q_0(r)$ is a “smoothed” version of $q$. That is to say, if $k$ is a nonnegative $C^\infty$ function in $\mathbb{R}^2$ such that $k(r) = 0$ for $|r| \geq 1$, $k$ is symmetric in $r$ and $\int k(r) \, dr = 1$, we define $k_n(r) = nk(nr)$ and $q_n(r) = \int k_n(r-s)q(r) \, ds$. We note that if $q \in L^2$, $q_n$ is a $C^\infty$ sequence of functions that converge to $q$ in $L^2$. If $q$ is continuous, $q_n$ converge to $q$ uniformly on compact sets.

To prove that this system has a solution we first study equation (8) in terms of existence, bounds for the gradient of $q$ and bounds for the difference of two solutions. Then we apply the Leray-Schauder Fixed Point Theorem as presented in Friedman [10] to obtain a classical solution of (7) and (8).
The following two lemmas establish existence of solutions for Eq. (8) given the function $u$.

**Lemma 3.** Let $v \in C^{2+\alpha}(\Omega_T), \bar{m} \leq q_0(x) \leq 1$ and $|v|_{2+\alpha} \leq K_4$. Then the problem

$$q_t - v_x q_x = (\beta(v) - \alpha)q - \beta(v)q^2,$$

$$q(x,0) = q_0(x)$$

has a unique solution $q(x, t)$ in $\Omega_T$, given by

$$q(x, t) = e^{\int_0^t \beta(v) e^{-\int_0^s \beta(v) - \alpha} ds \, dt} + q_0(0)$$

where $\beta(v) = \beta(v(x(t), t))$. The integration is along the characteristic curves defined by

$$\frac{\partial X}{\partial t} = -v_X(x, t), \quad X(t, \tilde{t}, \tilde{x}) = \tilde{x}.$$  

**Proof.** We define characteristic functions $X(t; \tilde{t}, \tilde{x})$ by means of (11). Since $v_x$ is Lipschitz continuous there is always a unique local solution of this equation. Further, since $v_x$ is also bounded in $\Omega_T$, this local solution can be made global by extending it to the boundary of $[0, T] \times \mathbb{R}$. The usual techniques of maximum principles can be applied to show that if $q, q_x$ are bounded in $\Omega_T$ and $\bar{m} \leq q_0(x) \leq 1$, then $\bar{m}e^{-\alpha t} \leq q(x, t) \leq 1$. In particular we expect $q$ to be a positive function. Let

$$p(x, t) = q^{-1}(x, t) e^{\int_0^t \beta(v(X(s), s)) - \alpha \, ds}.$$  

If we assume that $q$ is a solution of (9), then $p$ satisfies

$$p_t - v_x p_x = \beta(v, t), \quad p(x, 0) = q_0^{-1}(x),$$

where $\beta(v, t) = \beta(v(x(t), t)) e^{\int_0^t \beta(v(X(s), s)) - \alpha \, ds}$.

Integrating this equation along characteristics from 0 to $t$ we obtain

$$p(\tilde{x}, \tilde{t}) = p(X(0), 0) + \int_0^t p(X(s), s) \, ds.$$  

This, together with (12), gives expression (10) for $q(\tilde{x}, \tilde{t})$. The solution thus obtained is just a formal one, but it can be proved by direct differentiation and by the fact that

$$X_t(t) - v(X(t), t) = 0 \quad \text{for every } t,$$

that (10) is a solution of (9). The uniqueness follows from Haar's lemma.

**4. Estimates for $|p - q|$ and $|q_x|$.**

**Lemma 4.** *(Estimates for $|p - q|$).* Let $v, w \in C^{2+\alpha}(\Omega), |v|_{2+\alpha} \leq K_4, |v - w|_{2+\alpha} \leq \delta$ and $p, q$ solutions of (9) with $v$ and $w$, respectively. That is,

$$q_t - v_x q_x = (\beta(v) - \alpha)q - \beta(v)q^2,$$

$$p_t - w_x p_x = (\beta(w) - \alpha)p - \beta(w)p^2,$$

with $q(x, 0) = p(x, 0) = q_0(x)$.

Then $|p - q| \leq K\delta$ where $K_S$ depends only on $K_4, T$, and $q_0$.  

Proof. Let \( X_1(t; i, \bar{x}) \) and \( X_2(t; i, \bar{x}) \) be the characteristics defined by
\[
\begin{align*}
\frac{\partial X_1}{\partial t} &= -v_x(X_1, t), \quad X_1(i) = \bar{x}, \\
\frac{\partial X_2}{\partial t} &= -w_x(X_2, t), \quad X_2(i) = \bar{x}.
\end{align*}
\]
(18)
Let \( X_3(t) = |X_1(t) - X_2(t)| \). Then subtracting both equations in (18) and using Gronwall’s inequality we obtain
\[
X_3(t) \leq T|v_x - w_x|_{\infty} e^{T\beta} \leq T e^{K_4 \delta}.
\]
Now, since \( p \) and \( q \) are given by the same expression (10) (with \( \beta(v) \) for \( q \), \( \beta(w) \) for \( p \), etc.), subtracting these formulae we arrive at an expression that contains multiples (depending on \( K_4, |\beta|_{\infty}, T, \) etc.) of \( \beta(v) - \beta(w) \). This in turn is bounded by \( \beta', v - w, \) and \( X_1 - X_2 \), and is therefore bounded by a constant \( K \) times \( \delta \).

Lemma 5. (Bounds for the gradient of \( q \)). Let \( u \) be as in Lemma 1 and \( q \) given by (10), with \( \partial X/\partial t = u X X(X, t) \) and \( X(t) = \bar{x} \). Then \( |q_\bar{x}| \leq K_5|u_x|_{\infty} \) and \( |q_i| \leq K_5|u_x|^2_{\infty} \), where \( K_5 = K_5(\epsilon, T, M_1, M_0) \).

Proof. Differentiating the characteristic equations with respect to the parameter \( \bar{x} \), we obtain
\[
\begin{align*}
\frac{\partial X_1}{\partial t} &= -u X X(X, t) X_1, \quad X_1(i) = 1, \\
\frac{\partial X_2}{\partial t} &= -u X X(X, t) X_2, \quad X_2(i) = 1,
\end{align*}
\]
so \( X_{\bar{x}} = e_1[u_{xx}(s(x), s) \, ds]. \) Using the differential equation for \( u_{xx} \) and integrating, we obtain \( |X_{\bar{x}}| \leq (M/\epsilon) e^T(M_1 + 1). \) Then by (15), \( |X_{\bar{x}}| \leq (M/\epsilon) e^T(M_1 + 1)|u_x|_{\infty}. \) Differentiating the expression for \( q(\bar{x}, t) \), we obtain that \( q_\bar{x} \) is bounded by multiples of \( \beta', u_x, X_{\bar{x}}, q_0^{-1} \), and \( q_0 \), so that \( q_\bar{x} \) is bounded by a constant \( K_5(\epsilon, T, M_1, M_0) \) times \( |u_x|_{\infty} \). The same argument is valid for \( |q_i| \).

5. Existence of solutions of the \( \epsilon-n \)-approximating problems.

Theorem 1. Under the given hypotheses for \( u_0, q_0, \beta, \) and \( \mu \), there exists a classical solution in \( \Omega_T \) of
\[
\begin{align*}
&uu_{xx} + u^2_x + (\beta(u) q_{(n)} - \mu(u))(u - \epsilon) = u_1, \\
&q_t - u_x q_x = (\beta(u) - \alpha) q - \beta(u) q^2, \\
u(x, 0) = u_0(x) + \epsilon, q(x, 0) = q_0(x). \quad (19)
\end{align*}
\]

Proof. Let \( V \) be the convex set in \( C_2^a(\Omega_T) \) consisting of all functions \( w \) with \( |w|_{2+a} \leq K_6 \) and \( w \geq \epsilon \). We define \( T: V \to C_2^a(\Omega_T) \) in the following way. Given \( w \in W, |w|_{2+a} \leq K_4 \), by Lemma 3 there exists a unique solution \( q \) of
\[
\begin{align*}
q_t - w_x q_x &= (\beta(w) - x) q - \beta(w) q^2, \\
q(x, 0) &= q_0(x).
\end{align*}
\]
By the maximum principle, \( me^{-\alpha t} \leq q(x, t) \leq 1 \). Thus \( q_{(n)} \in C^\infty(\Omega_T) \) and \( |q_{(n)}|_\sigma \leq 2n \) for any \( \sigma \in (0, 1) \). Now consider the problem
\[
\begin{align*}
E(u) u_{xx} + u^2_x + (\beta(u) q_{(n)} - \mu(u))(u - \epsilon) &= u_1, \\
u(x, 0) &= u_0(x) + \epsilon, \quad (20)
\end{align*}
\]
where \( E(u) \) is a \( C^\infty \) function of \( u \) that equals \( \frac{1}{2} \) for \( u \leq \frac{1}{2} \); it increases up to \( \varepsilon \) in \( (\frac{1}{2}, \varepsilon) \) and equals \( u \) for \( u \geq \varepsilon \). This problem has a unique solution \( u \in C^{2+\alpha}(\Omega_T) \), \( |u|_{2+\alpha} \leq K(\varepsilon, n) \). By the maximum principle \( \varepsilon \leq u \leq M(\varepsilon, n) \); then \( E(u) = u \). Actually \( u \) is the unique solution in \( C^{2+\alpha}(\Omega_T) \) of Eq. (20) that satisfies these conditions. We let \( u = T(w) \) and choose \( K_4 \geq K(\varepsilon, n) \) and \( \sigma > \alpha \). Then \( T \) maps \( V \) into \( V \). Since bounded sets in \( C^{2+\alpha}(\Omega_T) \) are precompact in \( C^{2+\alpha}(\Omega_T) \) with \( 0 < \alpha < \sigma < 1 \), we obtain that \( T(V) \) is precompact.

Next we prove that \( T \) is continuous. We let \( u = T(w)z = T(v) \). If we subtract the corresponding \( q \) and \( p \) we find that \( |q - p|_{\infty} \leq K(K_4)\delta \) when \( |\nu - w|_{2+\alpha} \leq \delta \). Thus \( |q(n) - p(n)|_\alpha \leq K(K_4, n)\delta \). We then set \( S = u - z \) and note that \( S \) satisfies the linear equation
\[
\begin{align*}
us_{xx} + (u_x + z_x)s_x + c(x,t)s - S &= f(x,t), \\
S(x,0) &= 0,
\end{align*}
\]
where
\[
c(x,t) = (\beta(u)q(n) - \mu(u)) + (z - \varepsilon)(p(n)\beta'(\cdot) - \mu'(\cdot)) + z_{xx}
\]
and
\[
f(x,t) = \beta(u)(z - \varepsilon)(q(n) - p(n)).
\]
We also have that \( u \geq \varepsilon, u_x + z_x \) is bounded by \( K_4 \), \( c(x, t) \) is bounded by a multiple of \( K_4 \) and the \( \alpha \)-norm of \( f(x,t) \) is bounded by a multiple of the \( \alpha \)-norm of \( q(n) - p(n) \), i.e., it is bounded by \( K\delta \). Then by Theorem 5.1 in [11] we have \( |S|_{2+\alpha} \leq K(K_4, n, \varepsilon)\delta \). This implies continuity.

Thus \( T \) has a fixed point in \( V \), i.e., there exist functions \( u \) and \( q \) solutions of (20). Since \( 0 < \text{me}^{-\alpha T} \leq q \leq 1 \), by the maximum principle we obtain that \( \varepsilon \leq u \leq M_1 \), where \( M_1 = (M_0 + 1)e^{M_3T} \) and \( M_3 = |\beta|_\infty + |\mu|_\infty \). \( M_1 \) is independent of \( \varepsilon \) and \( n \).

Also
\[
\begin{align*}
&u \in C^{2+\alpha}(\Omega_T), \quad |u|_{2+\alpha} \leq K(n, \varepsilon), \\
&q \in C^{1+\alpha}(\Omega_T), \quad |q|_{1+\alpha} \leq K(n, \varepsilon),
\end{align*}
\]
\[
|q|_{1,\infty}, |q|_1 \leq K(|u_x|_\infty, \varepsilon).
\]

Existence of solutions of the \( \varepsilon \)-approximating problems. In this section we prove existence of solutions of the system:
\[
\begin{align*}
(uu_x)_x + (\beta(u)q - \mu(u))(u - \varepsilon) &= u_t, \quad (21) \\
q_t - u_xq_x &= (\beta(u) - \alpha)q - \beta(u)q^2, \quad (22) \\
u(x,0) &= u_0(x) + \varepsilon, \quad q(x,0) = q_0(x).
\end{align*}
\]

**Theorem 2.** Under the given hypothesis on \( \beta, \mu, u_0, \) and \( q_0 \), there exists a (weak) solution of (21) and (22) for each \( \varepsilon > 0 \). That is to say, there exists a continuous function \( u, (u^2)_x \in L^2_{loc}(\Omega_T) \) and a function \( q, q \in L^2_{loc}(\Omega_T) \), that satisfy
\[
\begin{align*}
\iint_{\Omega_T} \left[ \frac{1}{2} (u^2)_x \phi_x - u\phi_t \right] dx dt \\
= \iint_{\Omega_T} (\beta(u)q - \mu(u))(u - \varepsilon) dx dt + \int_{\mathbb{R}} (u_0(x) + \varepsilon)\phi(x,0) dx
\end{align*}
\]
(23)
and
\[
\iint_{\Omega_T} \left[ \frac{1}{2} (u^2) q_x - u_q \right] dx \, dt = \iint_{\Omega_T} \left[ (\beta(u) - \alpha - \mu(u)) u_q \right] dx \, dt \\
+ \int_{\mathbb{R}} (u_0(x) + \epsilon) q_0(x) \phi(x,0) dx
\] (24)
for every test function \( \phi \) over \( \Omega_T \).

**Proof.** By the previous theorem, for each \( n \), we have \( u^n \in C^{2+\alpha}(\Omega_T) \), \( q^n \in C^1(\Omega_T) \) such that
\[
(u^n u^n)_x + \left[ \beta q^n_{(n)} - \mu^n \right] (u^n - \epsilon) = u^n_n,
\]
\[
q^n_t - u^n x q^n_x = (\beta^n - \alpha^n) q^n - \beta(q^n)^2,
\]
\[
u^n(x,0) = u_0(x) + \epsilon, \quad q^n(x,0) = q_0(x).
\] (26)

Multiplying (21) by \( q^n \), (22) by \( u^n \), and adding, we obtain
\[
(u^n q^n)_t = \frac{1}{2} \left( (u^n)^2 \right)_x q^n + (\beta^n - \alpha^n) u^n q^n \\
+ \beta^n u^n q^n (q^n_{(n)} - q^n) - \epsilon q^n (\beta^n q^n_{(n)} - \mu^n).
\] (27)

If \( \phi \) is a test function we multiply (25) and (26) by \( \phi \) and integrate over \( \Omega_T \) to obtain
\[
\iint_{\Omega_T} \left[ \frac{1}{2} (u^n)^2 \right] q_x - u^n \phi \right] dx \, dt = \iint_{\Omega_T} \left[ \beta q^n_{(n)} - \mu^n \right] (u^n - \epsilon) dx \, dt \\
+ \int_{\mathbb{R}} (u_0(x) + \epsilon) \phi(x,0) dx,
\] (28)
and
\[
\iint_{\Omega_T} \left[ \frac{1}{2} (u^n)^2 \right] q_x - u^n q^n \phi \right] dx \, dt \\
= \iint_{\Omega_T} (\beta^n - \alpha^n) u^n q^n \phi \, dx \, dt + \iint_{\Omega_T} \left[ \beta u^n q^n (q^n_{(n)} - q^n) \phi \right] dx \, dt \\
- \epsilon \iint_{\Omega_T} q^n (\beta^n q^n_{(n)} - \mu^n) \phi \, dx \, dt + \int_{\mathbb{R}} (u_0(x) + \epsilon) q_0(x) \phi(x,0) dx.
\] (29)

Since \( |u^n|_\alpha \leq K_1 \) uniformly in \( n \), there exists a subsequence \( \{u^{n_l}\} \) such that \( \{u^{n_l}\} \) converges uniformly on compact sets to an \( \alpha \)-Hölder continuous function \( u \). Moreover \( |u|_\alpha \leq K_1 \).

By Lemma 5, \( |q^n_x| \leq K_5(\epsilon)|u^n_x|_\infty \), thus \( N_n = \max|q^n_x| \leq K_5(\epsilon)|u^n_x|_\infty \) is also finite. If we put \( c = \frac{1}{2} K_5^{-2} \) in Lemma 2, we obtain
\[
|u^n_x|^2 \leq K_2 + 2 K_3 K_5^2 + \left( \frac{1}{2} K_5^{-2} \right) \left( K_5^2 |u^n_x|^2 \right)
\]
Taking the supremum in the left-hand side we have $|u_n|_{\infty}^2 \leq 2(K_2 + 2K_3K_5^2)$ so that $|q_n|_{\infty}^2 \leq 2K_5(K_2 + 2K_3K_5^2)$. Thus $|q_n|_a$ and thus $|q_n|_a$ can be bounded above by a constant $K_{10}$ depending only on $\epsilon$, $M_1$, $T$, and $M_0$. If we put $v = (u^n)^2$ we have $u^n v_{xx} + (\beta^n q^n - v) = \epsilon(\beta^n q^n - v) - \mu^n u^n$. Since $|u^n|_a \leq K_1$, $u^n \geq \epsilon$ and $|q^n|_a \leq K_{11}$, we obtain that $|v|_{2+a}$ is bounded and consequently $|u^n|_{2+a}$ is bounded by a constant $K_{12}(\epsilon)$. Then for $\alpha' < \alpha$, there exists a subsequence $\{u^{n_k}\}$ that converges to $u$ in $C^{2+a}(\Omega_T)$. In particular $\{u^{n_k}\}$ converges uniformly to $u_\epsilon$ on compact sets. Also $\{q^{n_k}\}$ converges pointwise to $q$ and $q^{n_k}(\beta^n q^{n_k} - \mu^n)$ converges pointwise to $q(\beta q - \mu)$. Therefore all the integrals converge in (28) and (29) and we have a weak solution $u = u_\epsilon$, $q = q_\epsilon$ of the problem (21), (22), that is to say, $u_\epsilon$ and $q_\epsilon$ satisfy

$$
\iint_{\Omega_T} \left[ \frac{1}{2} (u_\epsilon)_{x} \phi_\epsilon - u_\epsilon \phi_t \right] dx \, dt
= \iint_{\Omega_T} \left[ (\beta(u_\epsilon) q_\epsilon - \mu(u_\epsilon))(u_\epsilon - \epsilon) \right] \phi dx \, dt + \int_{\Omega} \left( u_0(x) + \epsilon \right) \phi(x,0) \, dx
$$

(30)

and

$$
\iint_{\Omega_T} \left[ \frac{1}{2} (u_\epsilon)_{x} q_\epsilon \phi_\epsilon - u_\epsilon q_\epsilon \phi_t \right] dx \, dt
= \iint_{\Omega_T} \left[ (\beta(u_\epsilon) - \alpha - \mu(u_\epsilon))(u_\epsilon - \epsilon) q_\epsilon \phi \right] dx \, dt + \int_{\Omega} \left( u_0(x) + \epsilon \right) q_0(x) \phi(x,0) \, dx
$$

(31)

for every test function $\phi$ over $\Omega_T$.

Since $\alpha \in (0,1)$ is arbitrary instead of $\alpha'$ we can write again $\alpha$. Moreover $u_\epsilon$ and $q_\epsilon$ satisfy: $u_\epsilon \in C^{2+\alpha}(\Omega_T)$, $|u_\epsilon|_{2+a} \leq K_{12}(\epsilon)$, $q_\epsilon \in C^{\alpha}(\Omega_T)$, $|q_\epsilon|_a \leq K_{11}(\epsilon)$. In particular $u_\epsilon$ is a classical solution of (6) or (21).

6. Convergence of the $u_\epsilon$, $q_\epsilon$ solutions. The main theorem.

**Theorem 3.** Under the given hypotheses on $\beta$, $\mu$, $u_0$, and $q_0$, there exists a weak solution of problem (1) and (2), that is, there is a continuous function $u$, $(u^2)_x \in L^2_{\text{loc}}(\Omega_T)$, and a bounded function $q$, $q \in L^2_{\text{loc}}(\Omega_T)$, that satisfy (3) and (4) for every test function $\phi$ over $\Omega_T$.

**Proof.** Again by the Hölder property, $|u_\epsilon|_a \leq K_1$, so we can extract a subsequence $\{u_{\epsilon_k}\}$ that converges uniformly on compact sets to an $\alpha$-Hölder continuous function $u$. Moreover, $|u|_a \leq K_1$. Since $\{u_{\epsilon_k}\}$ is uniformly bounded by 1, we can extract a subsequence converging weakly to a function $u$ in $L^2_{\text{loc}}(\Omega_T)$. We call these sequences again $\{u_k\}$ and $\{q_k\}$ and will prove that a subsequence $\{(u^2_{k_\alpha})_x\}$ of $\{(u^2_{k_\alpha})_x\}$ converges pointwise to $(u^2)_x$, so that all the integrals in (30) and (31) will converge as $\epsilon_{k_\alpha} \to 0$ proving that $u$ and $q$ are solutions of (1) and (2).
Let $Q_2$ be a rectangle of length $\frac{1}{2}$ and width $\frac{1}{16}$ centered at $(x, t) = (0, 1)$, $Q_2 = 2Q_3$ and $Q_1 = 2Q_2$. Let $f_1(x, t), f_2(x, t)$ be bounded in $Q_1$ with $|f_1, f_2| \leq K_2$. If $z$ is a continuous weak solution in $Q_1$ of

$$\frac{d}{dx} (wz_x + f_1(x, t)) + f_2(x, t)z = z,$$

with $\frac{1}{2} \leq w \leq \frac{3}{2}$ and $|z|_{L^2(Q_2)} \leq K_2$, then by Aronson–Serrin [13] (Theorems 2 and 3) there exist constants $C_1, C_2$ depending only on $K_2, K_2$ such that $|z| \leq C_1$ and $|z|_{a} \leq C_3$ in $Q_3$, for some $\sigma > 0$. We use these facts to prove the next lemma.

**Lemma 6 (Regularity).** If $0 < u_k(x_0, t_0) = \delta < M$, then

$$\left| \left( u_k^2 \right)_x (x_0, t_0) \right| \leq C_4 (u_k(x_0, t_0))^{1/4}$$

and

$$\left| \left( u_k^2 \right)_x \right|_{a} = C_5 (K_2, M, M_1, t_0, D_0)$$

in a neighborhood $D_0$ of $(x_0, t_0)$.

**Proof.** We define a linear change of variables

$$x(\xi) = x_0 + b\delta^{(2 + a)/2}a\xi,$$

$$t(s) = t_0 + b^2\delta^2/\delta(x, t, u_k)(x, t),$$

where $b = \min\{(8K_1)^{-1/a}, t_0, M_1^{-2}\}$, that maps $Q_1$ onto a neighborhood $D_0$ of $(x_0, t_0)$.

Let $w(\xi, s) = \delta^{-1} u_k(x(\xi), t(s))$ for $(\xi, s) \in Q_1$. Then $w$ satisfies: $ww_{\xi} + w_{s}^2 + b^2\delta^{2/a}h(x, t, u_k)(w - \delta^{-1}) = w$. We put $h = b^2\delta^{2/a}h(x, t, u_k)$ and $z = (w_2)^{1/2}$. Differentiating with respect to $\xi$ we get

$$z_{\xi} = \left[ wz_{\xi} + 2h(w^2 - \delta^{-1}w) \right]_{\xi}.$$

By the Hölder property of $u_k$ and by the choice of $b$, $|u_k(x(\xi), t(s)) - u_k(x_0, t_0)| \leq \delta/4$. Thus dividing by $\delta$, we obtain $|w(\xi, s) - 1| \leq \frac{1}{4}$, and consequently $\frac{1}{2} \leq w \leq \frac{3}{2}$.

Since $w$ is a solution of (7) in $Q_1$, we also have

$$\int_Q 1/2 \left( w^2 \right)_{\xi} G_{\xi} - w G_{s} \, d\xi \, ds = \int_{Q_1} h(w - \delta^{-1}) G \, d\xi \, ds$$

(32)

for every test function $G$ that vanishes on the (total) boundary of $Q_1$. Let $g(\xi, s)$ be a $C^\infty$ function that takes values between 0 and 1, that is identically 1 in $Q_2$ and with derivatives $g_{\xi}, g_{s}$ bounded by 4. Then the function $G = w^2 - g^2$ is an admissible function in (32). This gives

$$\int_Q g^2((w^2)_{\xi})^2 \, d\xi \, ds \leq C_3 (K_1, M_1, t_0).$$

In particular,

$$|z|_{L^2(Q_2)} = \int_{Q_2} ((w^2)_{\xi})^2 \, d\xi \, ds \leq C_3.$$
Thus by the Aronson–Serrin theorems, \(|z| = |(w^2)_x| \leq C_1\) and \(|z|_\sigma = |(w^2)_x|_\sigma \leq C_2\) in \(Q_3\). Hence,

\[
\left|\left(\frac{u_k^2}{x}(x_0, t_0)\right)\right| = 2b^{-1}8(3\alpha-2)/2\alpha(\xi_0, 0, 1) \leq \frac{2C_1^{1/4}}{b} = C_4(\frac{\xi_0(x_0, t_0)}{b})^{1/4}
\]

for \(\alpha \geq 4/5\). Also we can assume that \(\sigma\) is smaller than \((3\alpha-2)/2\alpha\). Then

\[
\left|\left(\frac{u_k^2}{x}(x_1, t_1) - \left(\frac{u_k^2}{x}(x_2, t_2)\right)\right|\right| \leq \frac{2b^{-1}8(3\alpha-2)/2\alpha(\xi_0, 0, 1)}{b^{\sigma(2+\alpha)/\alpha}}(\xi_1 - \xi_2)/\alpha + |s_1 - s_2|^{2/2}) \leq 2b^{-1}8(3\alpha-2)/2\alpha C_2 \leq 2b^{-1}8C_2 M^2 = C_5(K_2, M_1, M, t_0, D_0).
\]

In particular, \(C_4\) and \(C_5\) do not depend on \(\delta\).

**Differentiability of \(u^2\).** Now we will prove that \(u^2\) is differentiable with respect to \(x\), for every \(t > 0\). We divide the proof in two cases:

(i) If \(u(x_0, t_0) = 0\), since \(u\) is \(\alpha\)-Hölder continuous in \(x\), \(|u(x, t)| \leq K_1|x - x_0|^\alpha\) and

\[
\left|\frac{u^2(x_0, t_0) - u^2(x_0, t_0)}{h}\right| \leq K_1 h^{2\alpha - 1} \to 0
\]

as \(h \to 0\) for \(\alpha > 1/2\).

(ii) If \(u(x_0, t_0) = \delta > 0\), by the uniform continuity of \(u\) and the uniform convergence of \(\{u_k\}\), there is an integer \(k_0\) and a neighborhood \(D_0\) of \((x_0, t_0)\) in which

\[
\frac{1}{2} \delta \leq u_k(x, t) \leq \frac{3}{4} \delta\text{ for } k \geq k_0.
\]

By the previous lemma, \(|(u_k^2)_x|_x \leq C_5\) in a subdomain \(D_{01}\). Thus we can extract a subsequence \(\{u_{k,n}\}\) such that \(\{(u_{k,n}^2)_x\}\) converges uniformly in a compact subdomain of \(D_{01}\) containing \((x_0, t_0)\).

Since \(\{(u_{k,n}^2)_x\}\) converges uniformly to \(u^2\) we conclude that \(\left(\frac{u^2}{x}(x_0, t_0)\right)_x = \lim_{k \to \infty} (u_{k,n}^2)_x\), for every \(x\) in that subdomain. Therefore \((u^2)_x\) exists everywhere.

**Completion of the proof.** Now we fix \(R\) and \(t_1\). We prove that there exists a subsequence \(\{u_{k,n}\}\) of \(\{u_k\}\) such that \(\{(u_{k,n}^2)_x\}\) converges pointwise to \((u^2)_x\) in \(D_{R,t}\).

Let \(n \geq 2\) and \(B_n = \{(x, t) \in D_{R,t} : u(x, t) \geq 1/n\}\). For \((x_0, t_0) \in B_n\), with \(u(x_0, t_0) = \delta \geq 1/n\) as before, there is an integer \(k_0\) and a neighborhood \(D_0\) in which \(\frac{1}{3}\delta \leq u_k(x, t) \leq \frac{3}{4}\delta\text{ for } k \geq k_0\). Then by the regularity lemma there is a \(D_{01} \leq D_0\) such that \(|(u_k^2)_x|_x \leq C_5\) in \(D_{01}\). By compactness we can cover \(B_n\) with finitely many of these neighborhoods and take \(C_6\) to be the maximum of the \(C_5\)’s. Then \(|(u_k^2)_x|_x \leq C_5\) in \(B_n\), so there exists a subsequence \(\{u_{k,n}\}\) such that \(\{(u_{k,n}^2)_x\}\) converges uniformly in \(B_n\). Then, out of this sequence and by the same arguments we extract a subsequence \(\{u_{k,n+1}\}\) such that \(\{(u_{k,n+1}^2)_x\}\) converges uniformly in \(B_{n+1}\). We repeat for every \(n\) and pick a diagonal subsequence \(\{u_{k,k}\}\). We claim that \(\{(u_{k,k}^2)_x\}\) converges pointwise to \((u^2)_x\) in \(D_{R,t}\). Let \(\eta > 0\) be given. If \(u(x_0, t_0) = 0\), by the differentiability of \(u^2\), \(\{(u_{k,k}^2)_x(x_0, t_0)\}\) and \(\{(u_{k,k}^2)_x(x_0, t_0)\}\) are \(\delta_k \geq \epsilon_k > 0\), by the regularity lemma 6 and since \(\delta_k \to 0\), \(|(u_{k,k}^2)_x(x_0, t_0)| \leq C_4 \delta_k^{1/4} \leq \eta\) for \(k\) large. If \(u(x_0, t_0) > 0\) we pick a positive integer \(r\) such that \(u(x_0, t_0) \geq 1/r\). Then \((x_0, t_0) \in B_r\) and since \(\{u_{k,k}\}\) is a
subsequence of \( \{ u_k, r \} \) for \( k > r \), and \( \{(u^2_k, r)\} \) converges uniformly in \( B \) to \( (u^2)_r \), we have \( \{(u^2_k, r)(x_0, t_0) - (u^2)_r(x_0, t_0)\} \leq \eta \) for \( k \) large, and thus \( \{(u^2_k, r)(x_0, t_0) - (u^2)_r(x_0, t_0)\} \leq \eta \) for \( k \) large.

We now finish the proof of the main theorem. We let \( t \) go to 0 and \( R \) go to \( \infty \) and pick a diagonal subsequence, that we call again \( \{ u_k \} \). Then the corresponding \( \{ q_k \} \) converges weakly to \( q \), \( \{(u^2_k, r)\} \) converges pointwise to \( (u^2)_r \), \( \{ u_k \} \) converges uniformly on compact sets to \( u \), and \( \varepsilon_k(B(u_k)q_k - \mu(u_k)) \) converges to 0. In this form all the integrals converge in (30), (31) and we obtain a weak solution of problem (1) and (2).

The solution we have obtained has the following property: If \( u_0(x) > 0 \) in \( (x_1, x_2) \) and \( u_0(x) = 0 \) otherwise, then the set \( P = \{(x, t)/u(x, t) > 0\} \) is bounded by a decreasing curve \( x = \xi_1(t) \) through \( (x_1, 0) \) and an increasing curve \( x = \xi_2(t) \) through \( (x_2, 0) \). We prove this assertion by means of the next two lemmas.

**Lemma 7.** Let \( t_0 \in [0, T) \). If \( u(x_0, t_0) = \eta > 0 \), then there is a positive constant \( K \), independent of \( \varepsilon \), such that \( u(x_0, t) \geq K \eta^2 \) for every \( t \geq t_0 \). In particular, if \( u_0(x_0) > 0 \) then \( u(x_0, t) > 0 \) for every \( t \geq 0 \).

**Proof.** Let \( L_1[z] = (zz_x)_x - Nz - z \), \( N \geq 2M_1 \), and assume we have a bounded solution \( v \) in \( \Omega_T \) of

\[
(vw_x)_x - Nv = v, \quad v(x, 0) = u_0(x) + \varepsilon. \tag{32}
\]

Then \( L_1[u] \leq 0 = L_1[v] \) and \( v(x, 0) \leq u(x, 0) \), so by the maximum principle, \( v(x, t) \leq u(x, t) \) in \( \Omega_T \). We can obtain solutions of (32) by considering the transformation

\[
\frac{w(\tau, \tau\tau)}{1 - N\tau} = \frac{1}{N} \log(1 - N\tau).
\]

Then (32) transforms into

\[
w_\tau = (ww_x)_x, \quad w(x, 0) = u_0(x) + \varepsilon. \tag{33}
\]

Let \( w \) be the unique solution of (33) (that exists by Theorem 2 in [12]). This solution is obtained as the limit of solutions to the boundary value problems:

\[
(w''w_x)_x = w_x, \quad w'(x, 0) = u_0(x) + \varepsilon, \quad w''(\pm n, t) = u_0(\pm n) + \varepsilon.
\]

Let \( a = x_0 - \eta/2M_0, b = x_0 + \eta/2M_0 \). Since \( u_0 \) is Lipshitz with constant \( M_0 \), \( u_0(x) \geq \eta/2 \) in \( [a, b] \). We consider

\[
\bar{w} = \frac{-\eta}{(b - a)^2 + 4\eta^2}(x - a)(x - b) \quad \text{and} \quad L_2[z] = (zz_x)_x - z\tau.
\]

Then \( L_2[w] \leq 0 \leq L_2[\bar{w}] \), \( \bar{w}(x, 0) \leq \eta/2 \leq w^n(x, 0) \) and \( \bar{w}(a, \tau) = 0 \leq w^n(a, \tau) \). Further, \( \bar{w}(b, \tau) = 0 \leq w^n(b, \tau) \), so \( w \leq w^n \) in \( [a, b] \times [0, \tau) \) for any \( \tau < 1/N \).

At \( x = x_0 \) we obtain

\[
\bar{w}(x_0, \tau) = -\frac{\eta(b - a)^2}{2(b - a)^2 + 8\eta^2} \leq w^n(x_0, \tau).
\]

Since this holds for every \( n \), the same is true for \( w \). Thus

\[
u_\varepsilon(x_0, \tau) \geq \frac{\eta(b - a)^2(1 - N\tau)}{2(b - a)^2 + 8\eta^2} \geq \frac{\eta^2}{2} \cdot \frac{Ne^{-NT}}{N + 2M_0^2},
\]

and the result follows with \( K = Ne^{-NT}/2(N + 2M_0^2) \).
Lemma 8. For $t_0 \in [0, T)$, the support of $u(x, t_0)$ is an interval $[\xi_1(t_0), \xi_2(t_0)]$.

Proof. Suppose not. Then there exists a point $(x_0, t_0)$ for which $u(x_0, t_0) = 0$ and $u(x, t)$ is positive for some values of $x$ before $x_0$ and after $x_0$. Pick a point $a > x_0$ with $u(a, t_0) = \eta > 0$. By Lemma 1, $x_0 > x_2$ and $u(x_0, t) = 0$ for $t \leq t_0$. By the continuity of $u$ and the uniform convergence of $\{u_{\varepsilon_k}\}$ for $k$ larger than a certain $k_0$, $u_{\varepsilon_k}(x_0, t) \leq \eta/4$ and there is a neighborhood $D_0$ containing $(a, t_0)$ in which $u_{\varepsilon_k}(x, t) > \eta/4$. We also pick $k_0$ large such that $\varepsilon_k \leq \eta/4$. Then in the domain $[x_0, \infty) \times [0, t_0)$, $u_{\varepsilon_k} \leq \eta/4$ on the lower and left boundaries, and $u_{\varepsilon_k}$ tends to $\varepsilon_k \leq \eta/4$ as $|x| \to \infty$, so by the maximum principle, $u$ cannot be larger than $\eta/2$. This contradiction proves the lemma.

References