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INSTABILITIES OF THE GINZBURG-LANDAU EQUATION: PT. II, SECONDARY BIFURCATION*

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Abstract. A perturbation treatment of secondary bifurcation for the Ginzburg-Landau equation is presented. An analytical form for limit cycle instability is determined. This is compared with numerical results and shown to be in good agreement over a wide parameter range.

1. Introduction. The Ginzburg-Landau (G-L) amplitude equation governing the modulation of quasi-monochromatic waves in fluid systems with supercritical dimensionless parameter (e.g., Reynolds number) has been the focus of many recent studies on transition to chaos [1–9]. The general form of this equation is

$$A_t - (\lambda_r + i\lambda_i)A_{xx} = \sigma_r A - (\beta_r + i\beta_i)|A|^2 A \tag{1}$$

where λ_r, λ_i, σ_r, β_r, β_i are real quantities and under a suitable renormalization it may be written as [8]

$$iA_t + (1 - ic_0)A_{xx} = i\rho A - (1 + i\rho)|A|^2 A \tag{2}$$

where

$$0 \leqslant c_0^2 < c_1, \qquad \rho = c_0/c_1. \tag{3}$$

The Stokes solution to (2) is given by

$$A = \exp(it). \tag{4}$$

If this is perturbed by a spatially periodic disturbance of wavelength $L = 2\pi/q$ it is linearly stable unless

$$q^2 < \frac{2(1 - c_0^2/c_1)}{(1 + c_0^2)} \equiv q_0^2. \tag{5}$$

Nevertheless in all instances solutions remain pointwise bounded [8].

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As a first step toward chaos, a spatially dependent limit cycle solution to (2) bifurcates from the Stokes solution as the parameter q is decreased below q_0. For small departures from neutral stability the perturbation solution was constructed in [8]. This was accomplished by introducing the small parameter

$$\varepsilon^2 = q_0^2 - q^2 \tag{6}$$

and postulating the separable form

$$A(x,t) \equiv A_0(x,t;\varepsilon) = \phi(x;\varepsilon)\exp(i\Omega(\varepsilon)t) \tag{7}$$

for the limit cycle. The amplitude ϕ and frequency Ω were then expanded as follows:

$$\phi(x;\varepsilon) = \sum_{n=0} a_n(\varepsilon)\cos(nqx), \tag{8}$$

$$a_k(\varepsilon) = \sum_{n=0} a_{2n+k}^{(k)}\varepsilon^{2n+k} = \sum_{n=0}\left(\alpha_{2n+k}^{(k)} + i\beta_{2n+k}^{(k)}\right)\varepsilon^{2n+k}, \tag{9}$$

$$\Omega(\varepsilon) = 1 + \sum_{n=1}\Omega_{2n}\varepsilon^{2n}. \tag{10}$$

Explicit forms for the coefficients through $O(\varepsilon^4)$ are given in [8]. When these were compared with the exact solutions the agreement was remarkable even well beyond the small ε range. For this reason it was felt that a study of secondary bifurcation could proceed from our knowledge of the perturbation form of the limit cycle solution. In the following we describe this analysis.

2. Perturbation Procedure. To study the linear stability of the periodic solution (7), we introduce in (2) the perturbed form

$$A(x,t) = A_0(x,t;\varepsilon) + \delta\psi(x,t)\exp(i\Omega t) \tag{11}$$

where δ is a formal small parameter. Linearization then yields

$$\psi_t = (c_0 + i)\psi_{xx} + (\rho - i\Omega)\psi + (i - \rho)\left(P\psi^* + |\phi|^2\psi\right), \tag{12}$$

where ψ^* is the complex conjugate of ψ, and

$$P = P_1 + iP_2 = \phi^2. \tag{13}$$

For present purposes, it is convenient to write (12) in real terms. Thus we set

$$\psi = U + iV \tag{14}$$

and use the notation

$$\mathbf{U} = (U, V). \tag{15}$$

Equation (12) may then be written in the form

$$\frac{\partial \mathbf{U}}{\partial t} = L\mathbf{U} = \left(\mathbf{C}\frac{\partial^2}{\partial x^2} + \mathbf{W} + \mathbf{D}\right)\mathbf{U}, \tag{16}$$

where

$$\mathbf{C} = \begin{pmatrix} c_0 & -1 \\ 1 & c_0 \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} \rho & \Omega \\ -\Omega & \rho \end{pmatrix}, \tag{17}$$

$$\mathbf{D} = \begin{pmatrix} -\rho P_1 - P_2 - 2\rho|\phi|^2 & P_1 - \rho P_2 - 2|\phi|^2 \\ P_1 - \rho P_2 + 2|\phi|^2 & \rho P_1 + P_2 - 2\rho|\phi|^2 \end{pmatrix}. \tag{18}$$

For the stability problem under study it will suffice if we consider the eigenvalue problem

$$\lambda \mathbf{U} = \mathbf{L}\mathbf{U}. \tag{19}$$

Next we expand (19) in powers of ε to obtain

$$(\lambda_0 + \varepsilon\lambda_1 + \cdots)(\mathbf{U}_0 + \varepsilon\mathbf{U}_1 + \cdots)$$

$$= \left(\mathbf{C}\frac{\partial^2}{\partial x^2} + \mathbf{W}_0 + \mathbf{D}_0 + \varepsilon\mathbf{D}_1\cos qx + \varepsilon^2\left(\mathbf{W}_2 + \mathbf{D}_2^{(0)} + \mathbf{D}_2^{(2)}\cos 2qx\right) + \cdots\right) \cdot(\mathbf{U}_0 + \varepsilon\mathbf{U}_1 + \cdots), \tag{20}$$

where

$$\mathbf{W}_0 = \begin{pmatrix} \rho & 1 \\ -1 & \rho \end{pmatrix}, \qquad \mathbf{W}_2 = \begin{pmatrix} 0 & \Omega_2 \\ -\Omega_2 & 0 \end{pmatrix}, \tag{21}$$

$$\mathbf{D}_0 = \begin{pmatrix} -3\rho & -1 \\ 3 & -\rho \end{pmatrix}, \tag{22}$$

$$\mathbf{D}_1 = 2\begin{pmatrix} -3\rho\alpha_1^{(1)} - \beta_1^{(1)} & -\alpha_1^{(1)} - \rho\beta_1^{(1)} \\ 3\alpha_1^{(1)} - \rho\beta_1^{(1)} & \beta_1^{(1)} - \rho\alpha_1^{(1)} \end{pmatrix}, \tag{23}$$

$$\mathbf{D}_2^{(0)} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}, \tag{24}$$

$$d_1 = -6\rho\alpha_2^{(0)} - \alpha_1^{(1)}\beta_1^{(1)} - \tfrac{3}{2}\rho\left(\alpha_1^{(1)}\right)^2 - \tfrac{1}{2}\rho\left(\beta_1^{(1)}\right)^2,$$

$$d_2 = -2\alpha_2^{(0)} - \rho\alpha_1^{(1)}\beta_1^{(1)} - \tfrac{1}{2}\left(\alpha_1^{(1)}\right)^2 - \tfrac{3}{2}\left(\beta_1^{(1)}\right)^2,$$

$$d_3 = 6\alpha_2^{(0)} - \rho\alpha_1^{(1)}\beta_1^{(1)} + \tfrac{3}{2}\left(\alpha_1^{(1)}\right)^2 + \tfrac{1}{2}\left(\beta_1^{(1)}\right)^2,$$

$$d_4 = -2\rho\alpha_2^{(0)} + \alpha_1^{(1)}\beta_1^{(1)} - \tfrac{1}{2}\rho\left(\alpha_1^{(1)}\right)^2 - \tfrac{3}{2}\rho\left(\beta_1^{(1)}\right)^2,$$

$$\mathbf{D}_2^{(2)} = \begin{pmatrix} d_5 & d_6 \\ d_7 & d_8 \end{pmatrix}, \tag{25}$$

$$d_5 = -\tfrac{3}{2}\rho\left(\alpha_1^{(1)}\right)^2 - \tfrac{1}{2}\rho\left(\beta_1^{(1)}\right)^2 - 6\rho\alpha_2^{(2)} - \alpha_1^{(1)}\beta_1^{(1)} - 2\beta_2^{(2)},$$

$$d_6 = -\tfrac{1}{2}\left(\alpha_1^{(1)}\right)^2 - \tfrac{3}{2}\left(\beta_1^{(1)}\right)^2 - 2\alpha_2^{(O)} - \rho\alpha_1^{(1)}\beta_1^{(1)} - 2\rho\beta_2^{(2)},$$

$$d_7 = \tfrac{3}{2}\left(\alpha_1^{(1)}\right)^2 + \tfrac{1}{2}\left(\beta_1^{(1)}\right)^2 + 6\alpha_2^{(2)} - 2\rho\beta_2^{(2)} - \rho\alpha_1^{(1)}\beta_1^{(1)},$$

$$d_8 = -\tfrac{1}{2}\rho\left(\alpha_1^{(1)}\right)^2 - \tfrac{3}{2}\rho\left(\beta_1^{(1)}\right)^2 - 2\rho\alpha_2^{(2)} + 2\beta_2^{(2)} + \alpha_1^{(1)}\beta_1^{(1)}.$$

Explicit forms for the α's and β's are given in [8].

At the lowest order we have

$$O(\varepsilon^0): \mathbf{T}\mathbf{U}_0 \stackrel{\text{def}}{=} (\lambda_0 - \mathbf{W}_0 - \mathbf{D}_0)\mathbf{U}_0 = \mathbf{C}\frac{\partial^2}{\partial x^2}\mathbf{U}_0. \tag{26}$$

Equation (26) is the same as that governing the first transition which takes place at wave number q_0 given by (5). The lowest order eigenvalue, λ_0, also plays a role in the present calculation. As can be seen by direct calculation, corresponding to the nth harmonic we obtain the eigenvalues

$$\lambda_0 = -c_0 n^2 q_0^2 - \rho \pm \sqrt{(\rho + c_0 q_0^2)^2 + q_0^2(n^2 - 1)[2 - (n^2 + 1)q_0^2]}\,. \tag{27}$$

As one easily sees, for $n = 0$,

$$\lambda_0 = -2\rho, 0 \tag{28}$$

while for $n = 1$,

$$\lambda_0 = -2(c_0 q_0^2 + \rho), 0. \tag{29}$$

All other eigenvalues are strictly complex, but of more significance to the present calculation is the fact that their real parts become increasingly negative with n. We now show that the eigenvalue (28) located at $\lambda_0 = -2\rho$ for $q = q_0$ moves to the right as q decreases and, as a detailed analysis shows, this triggers the secondary instability.

To start this demonstration, observe that

$$\mathbf{T}\mathbf{U}_0 = 0 \tag{30}$$

yields the eigenvalues (28) and corresponding eigenvectors

$$\mathbf{U}_0 = \begin{pmatrix} -\rho \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{31}$$

If we denote the left null vector of $\mathbf{T}$ by

$$\mathbf{N} = (1, 0), \tag{32}$$

then the next order equation,

$$O(\varepsilon): \mathbf{T}\mathbf{U}_1 = \mathbf{C}\frac{\partial^2}{\partial x^2}\mathbf{U}_1 - (\lambda_1 - \mathbf{D}_1 \cos qx)\mathbf{U}_0, \tag{33}$$

will have solutions only if

$$0 = (\mathbf{N}\cdot\mathbf{C})\frac{\partial^2}{\partial x^2}\mathbf{U}_1 - \lambda_1(\mathbf{N}\cdot\mathbf{U}_0) + (\mathbf{N}\cdot\mathbf{D}_1\cdot\mathbf{U}_0)\cos qx. \tag{34}$$

Clearly this implies $\lambda_1 = 0$ since the other terms are spatially dependent and periodic. From (33) one can see that $\mathbf{U}_1$ is made up of a homogeneous and a particular solution. The homogeneous solution is not needed, hence we may write

$$\mathbf{U}_1 = \mathbf{u}_1 \cos qx, \tag{35}$$

$$\mathbf{u}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tag{36}$$

where

$$\mathbf{u}_1 = \mathbf{M}^{-1} \cdot \mathbf{D}_1 \cdot \mathbf{U}_0 \tag{37}$$

with

$$\mathbf{M} = \begin{pmatrix} q_0^2 c_0 & -q_0^2 \\ q_0^2 - 2 & q_0^2 c_0 - 2\rho \end{pmatrix}. \tag{38}$$

Proceeding one more order

$$O(\varepsilon^2)\colon \mathbf{T}\mathbf{U}_2 = \mathbf{C}\frac{\partial^2}{\partial x^2}\mathbf{U}_2 + \mathbf{D}_1 \cdot \mathbf{u}_1\left(\frac{1}{2} + \frac{1}{2}\cos 2qx\right) + \left(-\lambda_2 + \mathbf{W}_2 + \mathbf{D}_2^{(0)} + \mathbf{D}_2^{(2)}\cos 2qx\right)\mathbf{U}_0, \tag{39}$$

which has solutions only if

$$0 = (\mathbf{N} \cdot \mathbf{C})\frac{\partial^2}{\partial x^2}\mathbf{U}_2 + (\mathbf{N} \cdot \mathbf{D}_1 \cdot \mathbf{u}_1)\left(\frac{1}{2} + \frac{1}{2}\cos 2qx\right) - \lambda_2(\mathbf{N} \cdot \mathbf{U}_0) + (\mathbf{N} \cdot \mathbf{W}_2 \cdot \mathbf{U}_0) + \left(\mathbf{N} \cdot \mathbf{D}_2^{(0)} \cdot \mathbf{U}_0\right) + \left(\mathbf{N} \cdot \mathbf{D}_2^{(2)} \cdot \mathbf{U}_0\right)\cos 2qx. \tag{40}$$

From this it follows that

$$\lambda_2 = \frac{(\mathbf{N} \cdot \mathbf{W}_2 \cdot \mathbf{U}_0) + \frac{1}{2}(\mathbf{N} \cdot \mathbf{D}_1 \cdot \mathbf{u}_1) + \left(\mathbf{N} \cdot \mathbf{D}_2^{(0)} \cdot \mathbf{U}_0\right)}{(\mathbf{N} \cdot \mathbf{U}_0)}. \tag{41}$$

Computations show that in the parameter range of interest, $\lambda_2 > 0$. This indicates that the eigenvalue $\lambda_0 = -2\rho$, after the first transition ($\varepsilon = 0$) increases along the negative real axis. In particular, this gives an approximate secondary transition point, which can be computed from the zero crossing of

$$\lambda \approx -2\rho + \varepsilon^2\lambda_2. \tag{42}$$

For the values $c_1 = 1$, $c_0 = .25$, this gives a transition at wave number $q^2 \approx 1.57$ which is significantly different from the numerical value of $q^2 \approx 1.21$ given in [6] and our more accurate value of $q^2 \approx 1.12$ [9]. More terms in the series expansion (20) are therefore called for. We cast this approach aside as being too laborious and consider an alternative method in the next section.

3. A Galerkin Approach. In view of the extreme accuracy of the limit cycle solution over a wide range of ε, we take

$$\phi \approx \left(a_0 + \varepsilon^2 a_0^{(2)}\right) + \varepsilon a_1^{(1)}\cos qx + \varepsilon^2 a_2^{(0)}\cos 2qx \tag{43}$$

where the coefficients are given explicitly in [8]. This expansion is correct through $O(\varepsilon^2)$ and, in particular, for $\rho = c_0 = .25$ is within 5% of the exact numerical limit cycle for $\varepsilon \approx 1$ [8].

If (43) is inserted into (19) we obtain

$$\lambda\mathbf{U} = L\mathbf{U} \stackrel{\text{def}}{=} \left(\mathbf{A} + \mathbf{C}\frac{\partial^2}{\partial x^2} + \varepsilon\cos qx\,\mathbf{D}_1 + \varepsilon^2\mathbf{D}_2^{(2)}\cos 2qx\right)\mathbf{U}, \tag{44}$$

where

$$\mathbf{A} = \begin{pmatrix} -2\rho & \varepsilon^2\Omega_2 \\ 2 - \varepsilon^2\Omega_2 & 0 \end{pmatrix} + \varepsilon^2 \mathbf{D}_2^{(0)}. \tag{45}$$

To solve (44) approximately we use a Galerkin type of approach. We introduce the projection operator P_N,

$$P_N\mathbf{U} = \sum_{n=0}^{N} \mathbf{U}^{(n)}\cos(nqx), \tag{46}$$

and consider the system generated by

$$P_N(\lambda - L)P_N\mathbf{U} = 0. \tag{47}$$

This in turn leads to a finite matrix system,

$$\lambda \mathbf{Q}_N = \mathbf{R}_N \mathbf{Q}_N \tag{48}$$

where

$$\mathbf{Q}_N = \begin{pmatrix} \mathbf{U}^{(0)} \\ \mathbf{U}^{(1)} \\ \vdots \\ \mathbf{U}^{(N)} \end{pmatrix}, \tag{49}$$

$$\mathbf{R}_1 = \begin{pmatrix} \mathbf{A} & \frac{1}{2}\varepsilon \mathbf{D}_1 \\ \varepsilon \mathbf{D}_1 & \mathbf{A} - q^2\mathbf{C} \end{pmatrix}, \tag{50}$$

$$\mathbf{R}_2 = \begin{pmatrix} \mathbf{A} & \frac{1}{2}\varepsilon \mathbf{D}_1 & \frac{1}{2}\varepsilon^2 \mathbf{D}_2^{(2)} \\ \varepsilon \mathbf{D}_1 & \mathbf{A} - q^2\mathbf{C} + \frac{\varepsilon^2}{2}\mathbf{D}_2^{(2)} & \frac{\varepsilon}{2}\mathbf{D}_1 \\ \varepsilon^2 \mathbf{D}_2^{(2)} & \frac{\varepsilon}{2}\mathbf{D}_1 & \mathbf{A} - 4q^2\mathbf{C} \end{pmatrix}. \tag{51}$$

The stability criterion is now simply obtained from (48). For any particular truncation N, we find the value of ε for which $\mathrm{Re}(\lambda) = 0$. For $N = 1$, $N = 2$ the eigenvalues have been computed, and the secondary instability curve in the c_0 vs. q^2 plane is shown in Figure 1 for the value $c_1 = 1$ and the simplest truncation $N = 1$. In particular, for $c_0 = .25$ the secondary transition point $q^2 \approx 1.22$ agrees very closely with that given in [6]. The secondary instability curve has also been computed using the exact (numerical) limit cycle solution [9] and we find that the $N = 1$ truncation gives a good approximation (in particular, for $c_0 = .25$, $c_1 = 1$, the transition point is $q^2 \approx 1.12$). For the $N = 2$ truncation, we find that the secondary transition curve is not as accurate as for the $N = 1$ truncation for reasons which are not entirely clear (in particular, for $c_0 = .25$, $c_1 = 1$, the transition point is $q \approx 1.33$).

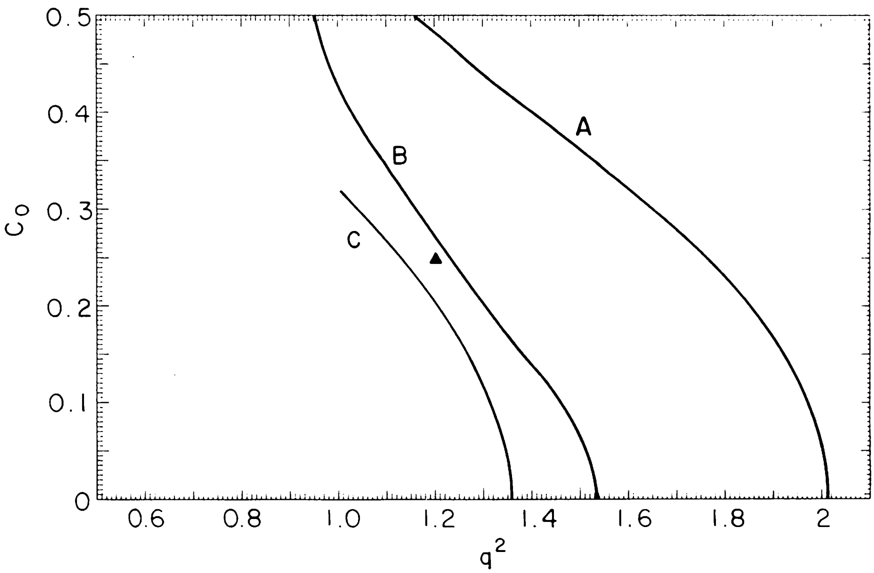

FIG. 1. Primary and secondary transition curves in the c_0 vs. q^2 plane for $c_1 = 1$, $N = 1$ truncation.
A: Primary transition
B: Secondary transition

In view of the closeness of the approximate to the exact limit cycle it is puzzling that (42) furnishes a relatively poor approximation to the exact value of secondary bifurcation. A partial explanation is gotten by analyzing what happens with the projection approximations treated above. Each of these ($N = 1, 2, \ldots$) under the perturbation analysis given in Section 2 leads exactly to (42) at $O(\varepsilon^2)$. A detailed examination then shows that this eigenvalue at some value of ε^2 meets an eigenvalue coming from the origin and bifurcates from the real axis. This is illustrated in Fig. 2. As ε^2 is further increased, the pair of

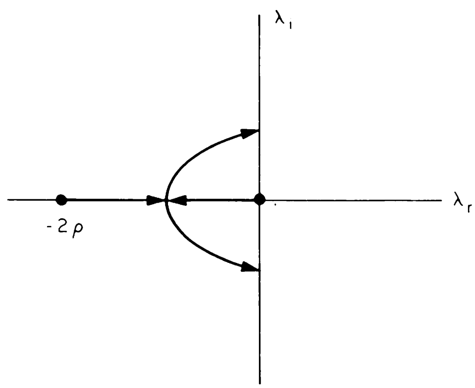

FIG. 2. Eigenvalue trajectory showing crossing imaginary axis, ε^2 (or q^2) is the parameter along curves. Bifurcation point tends toward origin as $N \uparrow \infty$.

eigenvalues eventually cross the imaginary axis, and at the corresponding value of ε^2 (or q^2) instability occurs. Thus we see that although (42) provides the initial rate of movement toward the imaginary axis it is qualitatively and quantitatively wrong in describing the crossing of the imaginary axis.

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