QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIV, NUMBER 3
OCTOBER 1986, PAGES 425-430

ON THE ROOTS OF $f_n(z) = a + K_{n+1}(z)/zK_n(z)$*

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Abstract. The function appearing in the title is examined to determine the number of roots and their general location. It is found that the roots appear in complex conjugate pairs, that the number of roots depends on n, and that, with the exception of a positive real root arising for $a < 0$, all roots lie in the left half-plane.

1. Introduction. In the solution of certain wave propagation problems concerning exterior regions in cylindrical coordinates, transcendental functions of the form

$$f_n(z) = a + K_{n+1}(z)/zK_n(z) \tag{1}$$

arise, where n is a nonnegative integer, a is a real, nonzero constant, and it is desired to determine the number of roots of $f_n(z)$ and the quadrant of the complex plane in which they lie. Examples of such problems may be found in the work of Selberg [1], Miklowitz [2], Biot [3], and Conant and Mussulman [4].

The nature of the roots of Eq. (1) appears to have been first studied by Selberg [1]. He restricts his interest to the upper half-plane and concludes that Eq. (1) has exactly one root, located on the interior of the first quadrant for $a < 0$, or on the interior of the second quadrant for $a > 0$. Miklowitz [2] shows that, for $n = 0$, Eq. (1) has no roots on the imaginary axis. Erdélyi and Kermack [5] examine a more general form of Eq. (1) and show conditions under which no right half-plane roots exist.

In the present paper, we will seek roots of Eq. (1) in the entire complex plane and show that, in contrast to Selberg's results, the number of roots is not independent of n, and the single right half-plane root lies on the real line.

2. Preliminaries. The number of roots of Eq. (1) can be determined by employing the principle of the argument, which can be stated as follows: If $f(z)$ is analytic everywhere on and within a simple closed curve Γ, except for a finite number of poles inside Γ, and if

* Received November 30, 1984.

$f(z)$ has no zeros on Γ, then

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}\,dz = Z - P, \tag{2}$$

where Z and P are the number of zeros and poles, respectively, inside Γ. Each zero or pole is counted according to its multiplicity. Since the integrand in Eq. (2) can be written as $d/dz[\ln f(z)]$, $Z - P$ can be shown to be equal to $1/2\pi$ times the change in argument of $f(z)$ as Γ is traversed once in the positive sense.

The modified Bessel function of the second kind, $K_n(z)$, has no poles except at the origin of the complex plane. Consequently, the poles of Eq. (1) are at the origin and at the zeros of $K_n(z)$. According to Watson [6], $K_n(z)$ has no zeros in the half-plane $\operatorname{Re} z \geqslant 0$, and since n is an integer, the number of zeros is given by $n + [(-1)^n - 1]/2$. Now $K_n(z)$ satisfies the reflection principle, so that its zeros are situated symmetrically about the negative real line. Further, $K_n(z)$ is a branch function with a branch cut usually taken along the negative real line.

We note also that $f_n(z)$ satisfies the reflection principle, in consequence of which its roots occur in complex conjugate pairs.

3. Right half-plane roots. Let us now examine the behavior of $f_n(z)$ as the contour shown in Fig. 1, consisting of small and large semicircles of radius r and R, respectively, joined by the line segments AB and CD lying along the imaginary axis, is traversed. On BC, $z = Re^{i\theta}$. Thus, considering the asymptotic expansion for $K_n(z)$ when z is large,

$$\lim_{R\to\infty} f_n(z) = a \tag{3}$$

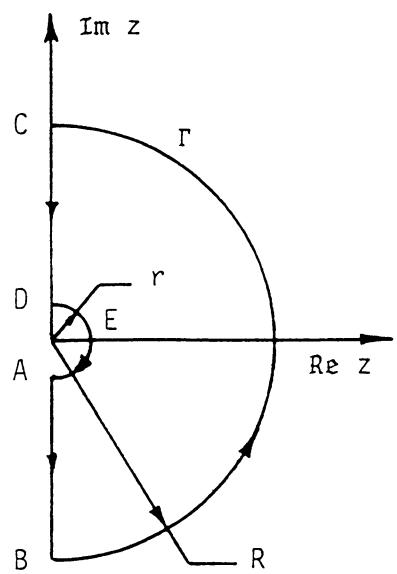

FIG. 1. Integration contour for right half-plane roots.

on BC. For small z,

$$K_n(z) = \begin{cases} \frac{1}{2}(n-1)!(z/2)^{-n}, & n \neq 0, \\ -\ln z, & n = 0, \end{cases} \tag{4}$$

so that

$$K_{n+1}(z)/K_n(z) = \begin{cases} 2n/z, & n \neq 0, \\ -1/z \ln z, & n = 0. \end{cases} \tag{5}$$

Thus on DA, where $z = re^{i\theta}$, Eq. (1) reduces to

$$f_n(z) = \begin{cases} 2ne^{-2i\theta}/r^2, & n \neq 0, \\ e^{-2i\theta}/r^2|\ln r|, & n = 0. \end{cases} \tag{6}$$

Now for positive imaginary arguments, $z = i\xi$, $\xi > 0$. Hence, with the aid of the relations

$$K_n(iz) = -\pi i e^{-in\pi/2}[J_n(z) - iY_n(z)]/2, \tag{7}$$

$$J_n(z)Y_{n+1}(z) - J_{n+1}(z)Y_n(z) = -2/\pi z, \tag{8}$$

it can be shown that

$$\frac{K_{n+1}(i\xi)}{K_n(i\xi)} = \frac{2}{\pi}\frac{1}{\xi[J_n^2(\xi) + Y_n^2(\xi)]} - i\frac{J_n(\xi)J_{n+1}(\xi) + Y_n(\xi)Y_{n+1}(\xi)}{[J_n^2(\xi) + Y_n^2(\xi)]}. \tag{9}$$

Therefore on CD, $\operatorname{Im} f_n(z) < 0$. If we now take the complex conjugate of Eq. (9) and recall that

$$\overline{K_n(z)} = K_n(\bar{z}), \tag{10}$$

we find that on AB, $\operatorname{Im} f_n(z) > 0$. In light of these results and Eqs. (3) and (6), $f_n(z)$ is nonvanishing on Γ.

To find the number of zeros in the right half-plane, we map the contour Γ into the w-plane through the mapping

$$w = f_n(z) \tag{11}$$

and note the number of times the contour Γ_w encircles the origin of the w-plane. The mapping is shown in Fig. 2. Note from Eq. (6) that on DA as θ varies from $\pi/2$ to $-\pi/2$, the argument of $f_n(z)$ varies from $-\pi$ to π and from Eq. (3) that on BC as θ varies from $-\pi/2$ to $\pi/2$ the argument of $f_n(z)$ remains unchanged. We see, then, that in the right half-plane the number of zeros of $f_n(z)$ is zero or one, according to whether a is positive or negative. It follows that the single root must lie on the real line. The existence of a positive real root for $a < 0$ can also be seen by recalling the large- and small-argument approximations for the modified Bessel functions and noting from Eq. (1) that for large x, $f_n(x)$ is negative while for small x, it is positive.

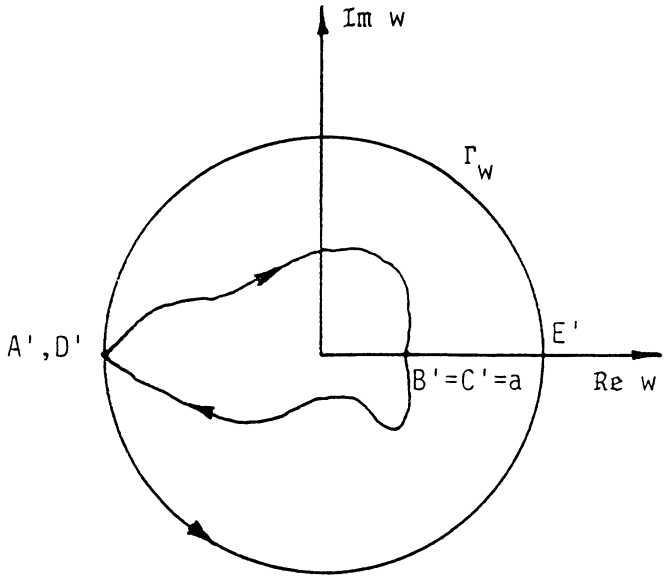

FIG. 2. Image contour for right half-plane roots.

4. Left half-plane roots. Because of the branch cut associated with $K_n(z)$, we make use of a result obtained previously, namely, that the roots occur in complex conjugate pairs, and examine only the second quadrant using the contour shown in Fig. 3.

We note that Eqs. (3) and (6) apply to segments LM and JK, respectively, of Fig. 3, and that from Eq. (9), $\operatorname{Im} f_n(z) < 0$ on KL.

On MJ, $z = -\xi$, $\xi > 0$. With the aid of the relations

$$K_n(-z) = e^{-n\pi i}K_n(z) - \pi i I_n(z), \tag{12}$$

$$K_n(z)I_{n+1}(z) + K_{n+1}(z)I_n(z) = 1/z, \tag{13}$$

we obtain, after some manipulation,

$$\frac{K_{n+1}(-\xi)}{K_n(-\xi)} = \frac{-K_{n+1}(\xi)K_n(\xi) + \pi^2 I_{n+1}(\xi)I_n(\xi) - i\pi(-1)^n/\xi}{K_n^2(\xi) + \pi^2 I_n^2(\xi)}. \tag{14}$$

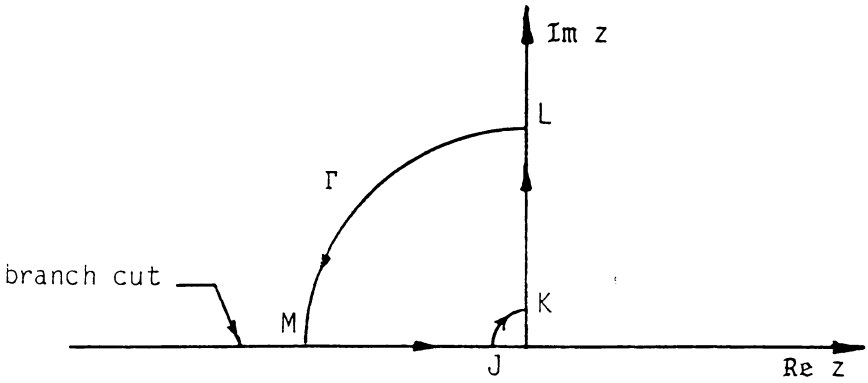

FIG. 3. Integration contour for second quadrant roots.

Thus on MJ, $\operatorname{Im} f_n(z)$ is greater or less than zero according to whether n is even or odd. Consequently, $f_n(z)$ does not vanish anywhere on Γ.

The mapping of Γ into the w-plane is accomplished as described previously, and it is shown in Fig. 4 for n even and in Fig. 5 for n odd. From these figures we see that for even values of n,

$$Z - P = \begin{cases} 1, & a > 0, \\ 0, & a < 0, \end{cases} \tag{15}$$

while for odd values of n,

$$Z - P = 1, \qquad a \neq 0. \tag{16}$$

It follows that in the half-plane $\operatorname{Re} z < 0$, the number of zeros of Eq. 1 is given by

$$Z = \begin{cases} n + [(-1)^n + 3]/2, & a > 0, \\ n - [(-1)^n - 1]/2, & a < 0. \end{cases} \tag{17}$$

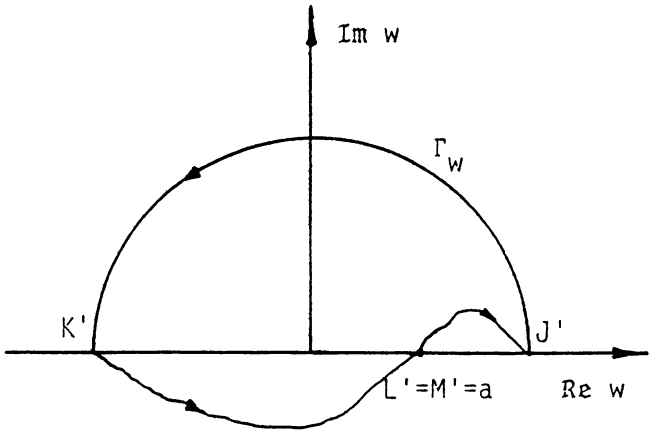

FIG. 4. Image contour for second quadrant roots $-n$ even.

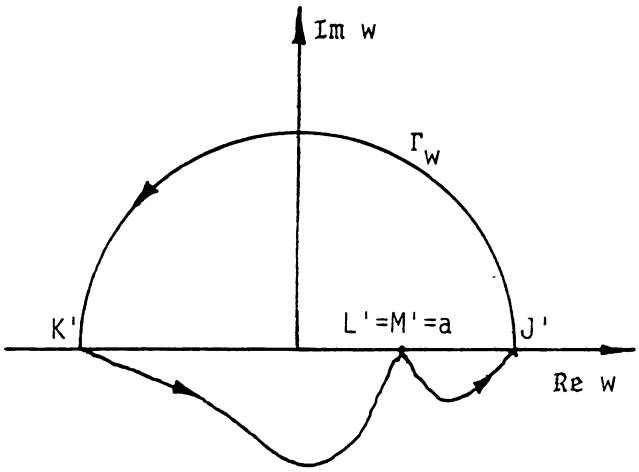

FIG. 5. Image contour for second quadrant roots $-n$ odd.

5. Summary. We can now summarize the results for the right and left half-plane as follows: The number of zeros of Eq. (1) is given by

$$Z = n + \left[(-1)^n + 3\right]/2 \quad \text{for } a > 0, \tag{18}$$

all of which are in the interior of the left half-plane, and

$$Z = n - \left[(-1)^n - 3\right]/2 \quad \text{for } a < 0, \tag{19}$$

one of which is on the positive real line, the remainder being in the interior of the left half-plane.

Complex zeros appear in conjugate pairs.

6. Discussion. The number of roots of Eq. (1) found in this paper differs markedly from the number predicted by Selberg. In his work, no account is taken of the poles of Eq. (1), and he incorrectly interprets the behavior of $K_{n+1}(z)/K_n(z)$ for negative real values of z. In examining the zeros in the first quadrant, he apparently chooses a path that includes the real line, resulting in a contour on which $f_n(z)$ vanishes. It should be noted, however, that Selberg's specific interest is in upper half-plane roots for the case $n = 0$, $a > 0$. For this case, his conclusions are in agreement with those obtained from Eq. (18).

Acknowledgment. This problem arose during the course of an investigation sponsored by the Montana State University Engineering Experiment Station. Their support is acknowledged. The author is grateful to Professor R. L. Mussulman for several beneficial discussions concerning the subject of this paper.

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