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# A NOTE ON CERTAIN IDENTITIES INVOLVING SERIES OF LEGENDRE POLYNOMIALS*

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**1. Introduction.** Legendre polynomials $P_n(x)$ and Legendre functions occur frequently in the solution of a number of boundary value problems in spherical domains (see [6] and [8]). Motivated by their need in the analysis of a class of mixed boundary value problems involving Laplace's equation in polar coordinates, Johnson [7] has recently proved several identities for sums of Legendre polynomials. We first recall here Johnson's main results (which are of special interest to us) in the following equivalent forms:

$$\sum_{n=0}^{\infty} P_n(\cos\theta)P_n(\cos\phi) = \frac{1}{\pi}\int_0^{\min(\theta,\phi)} \frac{ds}{(\cos s - \cos\theta)^{1/2}(\cos s - \cos\phi)^{1/2}}, \tag{1}$$

$$(0 < \theta \leqslant \pi, 0 < \phi \leqslant \pi, \theta + \phi < 2\pi);$$

$$\sum_{n=0}^{\infty} \frac{P_n(\cos\theta)P_n(\cos\phi)}{n+\frac{1}{2}} = \frac{2}{\pi}\boldsymbol{K}\left(\sin\frac{\theta}{2}\right)\boldsymbol{K}\left(\cos\frac{\phi}{2}\right), \qquad (0 \leqslant \theta < \phi \leqslant \pi), \tag{2}$$

where $\boldsymbol{K}(k)$ denotes the complete elliptic integral of the first kind defined by (see, e.g., [11, p. 35, Eq. (32)])

$$\boldsymbol{K}(k) = \int_0^{\pi/2} \frac{ds}{(1-k^2\sin^2 s)^{1/2}} = \tfrac{1}{2}\pi F\left[\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right], \tag{3}$$

in terms of the Gaussian hypergeometric series

$$F[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{4}$$

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For $\theta = 0$, (2) evidently yields

$$\sum_{n=0}^{\infty} \frac{P_n(\cos\phi)}{n+\frac{1}{2}} = \boldsymbol{K}\left(\cos\frac{\phi}{2}\right), \qquad 0 < \phi \leqslant \pi, \tag{5}$$

or, equivalently,

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n+\frac{1}{2}} = \tfrac{1}{2}\pi F\left[\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{1+x}{2}\right] = \tfrac{1}{2}\pi P_{-1/2}(-x), \qquad (-1 \leqslant x < 1), \tag{6}$$

in terms of the Legendre function of order $-\frac{1}{2}$.

For $\phi = \pi$, (2) reduces to the form:

$$\sum_{n=0}^{\infty} (-1)^n \frac{P_n(\cos\theta)}{n+\frac{1}{2}} = \boldsymbol{K}\left(\sin\frac{\theta}{2}\right), \qquad 0 \leqslant \theta < \pi, \tag{7}$$

or, equivalently,

$$\sum_{n=0}^{\infty} (-1)^n \frac{P_n(x)}{n+\frac{1}{2}} = \tfrac{1}{2}\pi F\left[\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{1-x}{2}\right] = \tfrac{1}{2}\pi P_{-1/2}(x), \qquad (-1 < x \leqslant 1). \tag{8}$$

Since

$$P_n(-x) = (-1)^n P_n(x), \qquad n = 0, 1, 2, \ldots, \tag{9}$$

the result (7) follows from (5) upon setting $\phi = \pi - \theta$, and the result (6) with $x$ replaced by $-x$ immediately yields (8).

Setting $\theta = \pi$ in (1) yields

$$\sum_{n=0}^{\infty} (-1)^n P_n(\cos\phi) = \frac{1}{\pi}\sec(\phi/2)\int_0^{\phi} \frac{\frac{1}{2}\sec^2(s/2)\,ds}{\left[\tan^2(\phi/2) - \tan^2(s/2)\right]^{1/2}}, \tag{10}$$

and we readily have

$$\sum_{n=0}^{\infty} (-1)^n P_n(\cos\phi) = \tfrac{1}{2}\sec(\phi/2), \qquad 0 < \phi < \pi, \tag{11}$$

which, for $\phi = \pi - \theta$, becomes

$$\sum_{n=0}^{\infty} P_n(\cos\theta) = \tfrac{1}{2}\csc(\theta/2), \qquad 0 < \theta < \pi, \tag{12}$$

in view of the identity (9).

Formulas (5) to (8), (11), and (12) are well known in the theory of Legendre polynomials (see, e.g., [3, vol. 2], [4, 5, 9, 10]). The object of the present note is to show how readily their parent formulas (1) and (2), and indeed also several analogous results of possible interest in the analysis of mixed boundary value problems, can be derived in *closed* forms as useful consequences of substantially more general results available in the literature.

**2. Generalizations of the series (1).** The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F\left[-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right], \tag{13}$$

reduce (for special values of the parameters $\alpha$ and $\beta$) not only to the Legendre polynomials $P_n(x)$, but also to the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$ and

the Tchebycheff polynomials $T_n(x)$ and $U_n(x)$. In fact, we have

$$P_n^{(0,0)}(x) = P_n(x), \tag{14}$$

$$P_n^{(\nu-1/2,\nu-1/2)}(x) = \frac{(\nu+\frac{1}{2})_n}{(2\nu)_n} C_n^{\nu}(x), \tag{15}$$

$$P_n^{(-1/2,-1/2)}(x) = \frac{(\frac{1}{2})_n}{n!} T_n(x), \qquad P_n^{(1/2,1/2)}(x) = \frac{(\frac{3}{2})_n}{(n+1)!} U_n(x), \tag{16}$$

and there are various limit relationships that connect Jacobi polynomials with (for instance) the Bessel function $J_\nu(x)$, Hermite polynomials $H_n(x)$, and Laguerre polynomials $L_n^{(\alpha)}(x)$. In terms of the general Jacobi polynomials, it is known that ([12, p. 224, Eq. (4.1)], [5, p. 297, Eq. (45.5.5)])

$$\sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+\frac{1}{2})}{\{\Gamma(\alpha+n+1)\}^2\Gamma(\beta+n+\frac{3}{2})}(\alpha+\beta+2n+1)P_n^{(\alpha,\beta)}(\cos\theta)P_n^{(\alpha,\beta)}(\cos\phi)$$
$$= \frac{2}{\pi}\{\csc(\theta/2)\csc(\phi/2)\}^{2\alpha}\int_0^{\min(\theta,\phi)} \frac{\{\sin(s/2)\}^{2\alpha}\{\sec(s/2)\}^{2\beta}\,ds}{(\cos s-\cos\theta)^{1/2}(\cos s-\cos\phi)^{1/2}}, \tag{17}$$
$$(0<\theta\leqslant\pi,\ 0<\phi\leqslant\pi,\ \theta+\phi<2\pi),$$

which, in view of (14), immediately yields (in the special case $\alpha=\beta=0$) formula (1).

For the Jacobi polynomials, the following bilinear generating function is due to Bailey [2] (see also [1, p. 102, Ex. 19], [5, p. 298, Eq. (45.5.7)], and [11, p. 116, Eq. (47)]):

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)t^n$$
$$= (1+t)^{-\alpha-\beta-1}F_4\left[\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha+\beta+2);\ \alpha+1,\beta+1;\ X,Y\right]$$
$$= (1+t)^{-\alpha-\beta-1}\sum_{l,m=0}^{\infty}\frac{(\alpha+\beta+1)_{2l+2m}}{(\alpha+1)_l(\beta+1)_m}\frac{(X/4)^l}{l!}\frac{(Y/4)^m}{m!}, \tag{18}$$

where, for convenience,

$$X = \frac{(1-x)(1-y)t}{(1+t)^2}, \qquad Y = \frac{(1+x)(1+y)t}{(1+t)^2}. \tag{19}$$

Setting $\alpha=\beta=0$ and $t=1$ in (18), we obtain

$$\sum_{n=0}^{\infty} P_n(x)P_n(y) = \tfrac{1}{2}F_4\left[\tfrac{1}{2},1;1,1;\tfrac{1}{4}(1-x)(1-y),\tfrac{1}{4}(1+x)(1+y)\right], \tag{20}$$

and since [1, p. 102, Ex. 20 (iv)]

$$F_4\left[\alpha,\beta;\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right]$$
$$= (1-x)^{\alpha}(1-y)^{\alpha}F[\alpha,\alpha-\beta+1;\beta;xy], \tag{21}$$

formula (20) assumes the elegant form:

$$\sum_{n=0}^{\infty} P_n(x)P_n(y) = (1+x)^{-1/2}(1-y)^{-1/2}F\left[\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{(1-x)(1+y)}{(1+x)(1-y)}\right], \tag{22}$$

or, alternatively,

$$\sum_{n=0}^{\infty} P_n(x)P_n(y) = \frac{2}{\pi}(1+x)^{-1/2}(1-y)^{-1/2}\boldsymbol{K}\left(\left[\frac{(1-x)(1+y)}{(1+x)(1-y)}\right]^{1/2}\right), \tag{23}$$

in terms of the complete elliptic integral $\boldsymbol{K}(k)$ defined by (3), it being understood in (22) and (23) that

$$-1 < x \leqslant 1, -1 \leqslant y < 1, \quad \text{and} \quad |(1-x)(1+y)| < |(1+x)(1-y)|, \tag{24}$$

provided that exceptional values of $x$ and $y$ (which would make the series diverge) are tacitly excluded.

For $x = \cos\theta$ and $y = \cos\phi$, (23) readily yields the following closed-form expression for the series in (1):

$$\sum_{n=0}^{\infty} P_n(\cos\theta)P_n(\cos\phi) = \frac{1}{\pi}\sec(\theta/2)\csc(\phi/2)\boldsymbol{K}\left(\tan\frac{\theta}{2}\cot\frac{\phi}{2}\right), \tag{25}$$

$$(0 \leqslant \theta < \phi \leqslant \pi).$$

Since $\boldsymbol{K}(0) = \pi/2$, both (11) and (12) are obvious consequences of (25).

Formula (25) and its interesting variations are stated in [5, p. 305, Eq. (46.8.1)]. More generally, in terms of the Gegenbauer polynomials $C_n^{\nu}(x)$, the bilinear generating function (18) can be reduced to the form ([5, p. 311, Eq. (47.6.11)], [11, p. 189, Problem 54]):

$$\sum_{n=0}^{\infty} \frac{n!}{(2\nu)_n} C_n^{\nu}(\cos\theta)C_n^{\nu}(\cos\phi)t^n = 2^{2\nu}(\xi+\eta)^{-2\nu}F\left[\nu, \tfrac{1}{2}; \nu+\tfrac{1}{2}; \left(\frac{\xi-\eta}{\xi+\eta}\right)^2\right], \tag{26}$$

which, for $\nu = \frac{1}{2}$, provides the following (known) generalization of (25) [5, p. 306, Eq. (46.8.11)]:

$$\sum_{n=0}^{\infty} P_n(\cos\theta)P_n(\cos\phi)t^n = \frac{4}{\pi(\xi+\eta)}\boldsymbol{K}\left(\frac{\xi-\eta}{\xi+\eta}\right), \tag{27}$$

where

$$\xi = \left[1 - 2t\cos(\theta+\phi) + t^2\right]^{1/2}, \qquad \eta = \left[1 - 2t\cos(\theta-\phi) + t^2\right]^{1/2}. \tag{28}$$

**3. Generalizations of the series (2).** From much more general results involving associated Legendre functions (see, e.g., [3, vol. 1, p. 167, Eq. (8); p. 168, Eq. (9)], [4, p. 1013, Eq. (8.792)], [5, p. 373, Eqs. (56.6.7) and (56.6.8)], and [9, p. 183]), it follows readily that

$$\sum_{n=0}^{\infty} \frac{2n+1}{(\mu-n)(\mu+n+1)} P_n(\cos\theta)P_n(\cos\phi) = \pi\csc(\pi\mu)P_\mu(\cos\theta)P_\mu(-\cos\phi), \tag{29}$$

$$(\mu \neq 0, \pm 1, \pm 2, \ldots; 0 < \phi \pm \theta < 2\pi),$$

or, equivalently,

$$\sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(\mu-n)(\mu+n+1)} P_n(\cos\theta) P_n(\cos\phi) = \pi \csc(\pi\mu) P_\mu(\cos\theta) P_\mu(\cos\phi), \tag{30}$$

$$(\mu \neq 0, \pm 1, \pm 2, \ldots; -\pi < \phi \pm \theta < \pi).$$

In the special case when $\mu = -\frac{1}{2}$, (29) and (30) become

$$\sum_{n=0}^{\infty} \frac{P_n(\cos\theta) P_n(\cos\phi)}{n+\frac{1}{2}} = \tfrac{1}{2} P_{-1/2}(\cos\theta) P_{-1/2}(-\cos\phi) \tag{31}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \frac{P_n(\cos\theta) P_n(\cos\phi)}{n+\frac{1}{2}} = \tfrac{1}{2} P_{-1/2}(\cos\theta) P_{-1/2}(\cos\phi), \tag{32}$$

which, in view of the definition (3), are essentially the same as formula (2).

Formula (29) with $\theta = 0$ (or $\phi = \pi$) yields

$$\sum_{n=0}^{\infty} \frac{2n+1}{(\mu-n)(\mu+n+1)} P_n(\cos\phi) = \pi \csc(\pi\mu) P_\mu(-\cos\phi), \tag{33}$$

while (30) similarly reduces to

$$\sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(\mu-n)(\mu+n+1)} P_n(\cos\phi) = \pi \csc(\pi\mu) P_\mu(\cos\phi). \tag{34}$$

By suitably specializing the noninteger parameter $\mu$, one can obtain a large number of closed-form sums for series of Legendre polynomials from each of the formulas (29), (30), (33), and (34).

Finally, we recall the following result involving Jacobi polynomials (see, e.g., [3, vol. 2, p. 212, Eq. (3)], [5, p. 294, Eq. (45.1.13)], and [9, p. 217]):

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)\Gamma(1-\lambda+n)}{\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+\lambda+n+1)} P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(1-\lambda)}{\Gamma(\alpha+\lambda)} \left(\frac{1-x}{2}\right)^{\lambda-1}, \tag{35}$$

$$-1 < x < 1, -\lambda < \min\{\alpha, (2\alpha-1)/4\},$$

which, for $\alpha = \beta = 0$, yields

$$\sum_{n=0}^{\infty} \frac{(1-\lambda)_n}{(1+\lambda)_n} (2n+1) P_n(x) = \lambda \left(\frac{1-x}{2}\right)^{\lambda-1}, \qquad -1 < x < 1, \lambda > \tfrac{1}{4}. \tag{36}$$

Formula (36) provides an interesting generalization of such well-known results as (11) and (12).

## References


[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, London and New York (1935)

[2] W. N. Bailey, *The generating function of Jacobi polynomials*, J. London Math. Soc. **13**, 8–12 (1938)

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, vols. 1, 2, McGraw-Hill, New York, Toronto and London (1953)

[4] I. S. Gradshteyn and I. M. Ryžhik, *Tables of integrals, series, and products*, Academic Press, New York and London (1965)

[5] E. R. Hansen, *A table of series and products*, Prentice-Hall, Englewood Cliffs, New Jersey (1975)

[6] E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Chelsea Publishing Co., New York and London (1955)

[7] H. L. Johnson, *Some new identities for Legendre polynomials*, Quart. Appl. Math. **41**, 353–355 (1983)

[8] T. M. MacRobert, *Spherical harmonics*, Third edition, Pergamon Press, Oxford and New York (1967)

[9] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Third edition, Springer-Verlag, Berlin, New York and Heidelberg (1966)

[10] V. Mangulis, *Handbook of series for scientists and engineers*, Academic Press, New York and London (1965)

[11] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto (1984)

[12] K. N. Srivastava, *On triple series equations involving series of Jacobi polynomials*, Proc. Edinburgh Math. Soc. (2) **15**, 221–231 (1967)