AN APPLICATION OF THE MULTIVARIATE
LAGRANGE-BÜRMANN EXPANSION
IN MATHEMATICAL GEODESY *

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Abstract. In the simplified model of geodesy where the earth is conceived as a rotational ellipsoid, if the eccentricity of the ellipsoid is to be determined from gravity measurements, an equation of the form \( y = x - zh(x) \) is to be solved for \( x \), where \( y \) and \( z \) are small parameters whose values can be measured and \( h \) is a known function. We obtain the expansion of \( x \) in powers of \( y \) and \( z \) by means of the general Lagrange–Bürmann formula.

1. The problem. Using the standard notations of physical geodesy,

\( a = \) major axis of the earth ellipsoid,
\( GM = \) product of the earth’s mass and the gravitational constant,
\( J_2 = \) a constant in the expansion of the normal gravity field in spherical harmonics, and
\( \omega = \) angular velocity of the earth,

the equation satisfied by the eccentricity \( e \) of the ellipsoid may be stated as follows [1, 4]:

\[
3J_2 = e^2 - \frac{4}{15} \frac{\omega^2 a^3}{GM} \frac{e^3}{2q_0}.
\] (1)

Here \( 2q_0 \) is a known function of \( e \),

\[
2q_0 = \left(1 + \frac{3}{e'^2}\right) \arctan e' - \frac{3}{e'},
\] (2)

where

\[
e' = \frac{e}{\sqrt{1 - e^2}}.
\] (3)

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is the “second eccentricity.” The constants $a$, $GM$, $J_2$, $\omega$ are either known or can be obtained accurately from gravity measurements. Equation (1) thus serves to obtain accurate values of $e$ from gravity measurements. Our concern is with solving the equation and with exhibiting the dependence of the solution on the parameters.

The equation has the form

$$y = x - zh(x), \quad (4)$$

where

$$y = 3J_2, \quad z = \frac{\omega^2 a^3}{GM},$$

are known and $x = e^2$ is to be determined. The function

$$h(x) = \frac{4}{15} \frac{x^{3/2}}{2q_0(\sqrt{x})} \quad (5)$$

is known. In the physical problem on hand, the numerical values of $y$ and $z$ are both of the order of $3 \times 10^{-3}$.

2. Numerical solution of the equation. This is discussed very thoroughly in [1], and values of $e$ are obtained that are more accurate than those given in the literature. It follows from Eq. (4) of [1] that

$$\frac{1}{h(x)} = F\left(\frac{3}{2}, \frac{3}{2}; \frac{7}{2}; x\right), \quad (6)$$

where $F$ is the hypergeometric function. Thus $h$ is analytic not only for $0 < x < 1$ but also at $x = 0$. Moreover, since all coefficients in the series (6) are positive, as $x$ increases from 0 to 1, $h(x)$ decreases from $h(0) = 1$ to $h(1) = 4/15\pi$. By writing (4) as a fixed point equation,

$$x = y + zh(x), \quad (7)$$

we see that for positive $y$ and $z$ such that $y + z < 1$ the equation has precisely one solution, which, if $z$ satisfies the additional condition

$$|zh'(y + z)| < 1,$$

can be found as the limit of the iteration sequence defined by $x_0 = 0$,

$$x_{n+1} = y + zh(x_n), \quad n = 0, 1, 2, \ldots.$$  

The only numerical problem that arises is a considerable loss of accuracy, due to subtracting large numbers that are nearly equal, if $h$ is evaluated by means of the defining relations (5) and (2). It is much preferable to compute $h$ from the series expansion (6), which converges rapidly if $x$ is small.

3. Analytical solution. Iteration does furnish a numerical solution of (4) for given $y$ and $z$, but it does not show how this solution depends on the parameters. We therefore endeavor to find a series solution for (4). Our tool is the multidimensional Lagrange–Bürmann formula as discussed in [3]. We summarize these results briefly for convenience.
Let $P = (P_1, P_2, \ldots, P_n)$ be an admissible system of $n$ power series in $n$ indeterminates $x = (x_1, x_2, \ldots, x_n)$. ["Admissible" means that $P_j = c_j x_j + \text{higher-order terms}$, where $c_j \neq 0$.] Let $Q$ denote the inverse system of $P$. ["Inverse" means that $Q$ substituted into $P$ yields $x$.] Let $R$ be an arbitrary (single) Laurent series in $x$. Then the series obtained by substituting $Q$ into $R$ is given by

$$R \circ Q = \sum_k \text{Res}(RP^{-k}eP')x^k,$$

where the summation is with respect to all index vectors $k = (k_1, k_2, \ldots, k_n)$, and where

$$x^k = x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n},$$

$$e = (1, 1, \ldots, 1),$$

$P'$ is the Jacobian determinant of the system $P$, and $\text{Res}$ denotes the residue, that is, the coefficient of $x^{-e}$, in a Laurent series. The result (8) holds formally, that is, regardless of whether or not the series involved are convergent.

We require an application of (8), also given in [3]. Here we consider two systems of complex variables,

$$x = (x_1, \ldots, x_p), \quad y = (y_1, \ldots, y_q),$$

and a system of $p$ functions

$$f_i(x, y), \quad i = 1, 2, \ldots, p,$$

analytic near $(0, 0)$. We write $f = (f_1, \ldots, f_p)$, and we denote by $f'$ the Jacobian determinant of this system with respect to the $x_i$, regarding the $y_j$ as parameters. Assuming

$$f(0, 0) = 0, \quad f'(0, 0) \neq 0,$$

the system of equations

$$f(x, y) = 0$$

for sufficiently small $|y|$ has precisely one solution $x(y)$ which is analytic in $y$ and which satisfies $x(0) = 0$. We wish to find the coefficients of the power series $x(y)$ or, more generally, of $r(x(y), y)$, where $r$ is a given analytic function.

For a solution by means of the Lagrange-Bürmann formula we assume, without loss of generality, that the matrix

$$\left( \frac{\partial f_i}{\partial x_j} (0, 0) \right), \quad i, j = 1, \ldots, p,$$

is the identity. (This can be achieved by forming suitable linear combinations of the functions $f_i$ and of the variables $x_j$.) In the power series expansion of $f(x, y)$, let $By$ denote the terms that are linear in the $y_j$, that is,

$$f(x, y) = x + By + \text{terms of degree } \geq 2.$$  

($B$ is a matrix with $p$ rows and $q$ columns; we think of $y$ as a column vector.) Consider the map of a $(p + q)$-dimensional neighborhood of $(0, 0)$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(x, y) - By \\ y \end{pmatrix}.$$  

(10)
The system of \( p + q \) power series representing this map near \((0, 0)\) is admissible; in fact, its Jacobian matrix at \((0, 0)\) is the identity. Hence the inverse system

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
x(u, v) \\
y(u, v)
\end{pmatrix}
\] (11)

exists and can be represented by the Lagrange–Bürmann series. Letting

\[ P = f(x, y) - By, \]

and noting that the Jacobian determinant of the whole system (10) is just \( P' \), the Jacobian with respect to \( x \), one obtains in view of \( y = v \) for an arbitrary function \( r \)

\[
r(x(u, v), v) = \sum_{k \in \mathbb{Z}^p, m \in \mathbb{Z}^q} \text{Res}\{r(x, y)P^{-k}e^{y^{-m}}P'(x, y)\}u^ky^m. \] (12)

Now evidently \( f(x, y) = 0 \) if and only if \( u = -Bv \). Since \( v = y \), the solution of (8) thus is

\[ x(y) = x(-By, y), \]

and from (12) we find the explicit series expansion

\[
r(x(y), y) = \sum_{k \in \mathbb{Z}^p, m \in \mathbb{Z}^q} \text{Res}\{ \cdots \}(-By)^ky^m, \] (13)

where the residues are the same as in (12).

4. Application to the geodesic equation. To apply (13) to the solution of (4), we let \( p = 1, q = 2 \),

\[ x = (x), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}. \]

The equation to be solved is \( f(x, y) = 0 \), where

\[ f(x, y) = x - y - zh(x), \]

which in order to isolate first-order terms we write in the form

\[ f(x, y) = x - y - z - zxg(x), \]

where

\[ g(x) = \frac{1}{x}(h(x) - 1) = O(1). \]

We see that

\[ By = -y - z. \]

The map (10) in our case is thus

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}
P \\
y \\
z
\end{pmatrix}, \quad P = x(1 - zg(x)).
\]
If $r(x, y) = x$, (13) now yields

$$x(y, z) = \sum_{k > 0} \sum_{(m, n) \in \mathbb{Z}^2} \text{Res}_x \left\{ x P^{-k-1} y^{m-1} z^{n-1} \right\} (y + z)^k y^m z^n,$$

(14)

and it only remains to evaluate the residues.

Since $P$ does not depend on $y$, we need $m = 0$ to obtain a residue in $y$. Using

$$p^{-k-1}p' = -\frac{1}{k} \left( p^{-k} \right)' ,$$

we thus get

$$x(y, z) = -\sum_{k > 0} \frac{1}{k} \text{Res}_{x,z} \left\{ x (P^{-k})' z^{n-1} \right\} (y + z)^k z^n,$$

where the residue now is taken only with respect to the variables $x$ and $z$. In view of

$$P = x(1 - zg(x)),$$

we may use the binomial series to obtain

$$p^{-k} = x^{-k} (1 - zg(x))^{-k}$$

$$= x^{-k} \sum_{l=0}^{\infty} (-1)^l \binom{-k}{l} z^l g^l ,$$

where $\binom{-k}{l}$ is a binomial coefficient. Now for given $k > 0$ and $n \geq 0$,

$$\text{Res}_{x,z} \left\{ x (P^{-k})' z^{n-1} \right\} = \text{Res}_x \text{ of coefficient of } z^n \text{ in } x (P^{-k})'$$

$$= -\text{Res}_x \text{ of coefficient of } z^n \text{ in } P^{-k}$$

$$= -\text{coefficient of } x^{k-1} \text{ in } (-1)^n \binom{-k}{n} g^n$$

$$= (-1)^{n+1} \binom{-k}{n} g^{(n)}_{k-1} ,$$

where $\text{Res}_x$ denotes the residue with respect to the single variable $x$, and where the coefficients $g^{(n)}_k$ are defined by

$$[g(x)]^n = \sum_{k=0}^{\infty} g^{(n)}_k x^k .$$

We thus finally let

$$x(y, z) = \sum_{k > 0} \sum_{n \geq 0} \frac{(-1)^n}{k} \binom{-k}{n} g^{(n)}_{k-1} (y + z)^k z^n$$

$$= y + z + \sum_{k > 0} \sum_{n > 0} \frac{(-1)^n}{k} \binom{-k}{n} g^{(n)}_{k-1} (y + z)^k z^n .$$

(15)
5. **Truncation error.** In numerical computation, the series (15) will have to be truncated, for instance, by neglecting the terms where \( k + n \geq p \) for some positive integer \( p \). We therefore estimate the truncation error

\[
    t_p(y, z) = \sum_{k,n=0}^{k+n=p} \frac{(-1)^n}{k} \binom{-k}{n} s_{k-1}^{(n)}(y + z) dz^n.
\]

From (6), the coefficients \( a_n \) in

\[
    [h(x)]^{-1} = \sum_{n=0}^{\infty} a_n x^n
\]

are easily seen to satisfy \( |a_n| \leq 1 \). In view of \( a_0 = 1 \) we therefore have for \( |x| \leq \rho, \rho < 1, \)

\[
    |[h(x)]^{-1}| \geq 1 - \rho - \rho^2 - \cdots = (1 - 2\rho)/(1 - \rho),
\]

and thus, if \( 0 \leq \rho < 1/2, \)

\[
    |h(x)| \leq (1 - \rho)/(1 - 2\rho).
\]

Using the principle of the maximum, there follows for \( |x| \leq \rho < 1/2, \)

\[
    |g(x)| \leq \frac{1}{\rho} \left| \frac{1 - \rho}{1 - 2\rho} - 1 \right| = \frac{1}{1 - 2\rho}.
\]

Cauchy's estimate now yields

\[
    |g^{(n)}(x)| \leq \frac{1}{1 - 2\rho} \frac{1}{\rho^n}, \quad 0 < \rho < 1/2.
\]

Now let \( |y + z| \leq \rho_1, |z| \leq \rho_2 \). In view of

\[
    \frac{(-1)^n}{k} \binom{-k}{n} = \frac{1}{k} \binom{k + n - 1}{n} = \frac{1}{k + n} \binom{k + n}{n},
\]

there follows

\[
    \left| \sum_{k+n=q}^{k+n=q} \frac{(-1)^n}{k} \binom{-k}{n} g_{k-1}^{(n)}(y + z) dz^n \right| \leq \frac{1}{q} \sum_{k+n=q}^{q} \binom{q}{n} (1 - 2\rho)^{-n} \rho^{-k+1} \frac{\rho_1}{\rho} \rho_2^q
\]

\[
    = \frac{\rho}{q} \left( \frac{\rho_2}{1 - 2\rho} + \frac{\rho_1}{\rho} \right)^q.
\]

Therefore, if

\[
    \sigma = \frac{\rho_2}{1 - 2\rho} + \frac{\rho_1}{\rho} < 1, \quad (16)
\]

we find the truncation error estimate

\[
    |t_p(y, z)| \leq \frac{\rho}{\rho} \frac{\sigma^p}{1 - \sigma}. \quad (17)
\]
Choosing, for instance, \( \rho = \frac{1}{3} \), there results the simple formula

\[
\left| t_\rho(y, z) \right| \leq \frac{1}{3\rho} \left( \frac{3\rho_1 + 3\rho_2}{1 - (3\rho_1 + 3\rho_2)} \right)^\rho.
\]  

(18)

6. Numerical values. It remains to compute the coefficients \( g_k(n) \). This is a routine computation which is best performed with a symbolic manipulator. Using the MAPLE program of the University of Waterloo [2] we computed the \( g_k(n) \) as well as the coefficients of the series (15) in rational arithmetic for \( 1 \leq k \leq 10, 1 \leq n \leq 10 \). Complete tables of these values are available from the authors on request. Here we give only the values that are required to write the terms of the series for \( k + n < 5 \):

\[
h(x) = 1 - \frac{9}{14} x - \frac{13}{392} x^2 - \frac{4189}{181104} x^3 - \cdots.
\]

\[
g(x) = -\frac{9}{14} - \frac{13}{392} x - \frac{4189}{181104} x^2 - \cdots.
\]

\[
[g(x)]^2 = \frac{81}{196} + \frac{117}{2744} x + \cdots.
\]

\[
[g(x)]^3 = -\frac{729}{2744} - \cdots.
\]

This results in

\[
x(y, z) = (y + z) \left( 1 - \frac{9}{14} z - \frac{81}{196} z^2 - \frac{729}{2744} z^3 + \cdots \right)
\]

\[
+ (y + z) \left( -\frac{13}{392} z + \frac{351}{5488} z^2 + \cdots \right)
\]

\[
+ (y + z)^3 \left( -\frac{4189}{181104} z + \cdots \right)
\]

+ \cdots.  

(19)

From the values of the parameters given in [1] we have

\[
y = 3.247890 \times 10^{-3}, \quad z = 3.461391 \times 10^{-3}.
\]

Substituting these into (19) we get

\[
x = 6.694379 \times 10^{-3}
\]

with a truncation error \( t_5(y, z) \), which by (18) is less than

\[
\frac{1}{15} \left( \frac{3 \times 6.709281 \times 10^{-3}}{0.979872} \right)^5 = 2.25 \times 10^{-10},
\]

and which thus is less than the error in \( x \) due to rounding or measuring errors in \( y \) and \( z \).
REFERENCES