RESPONSE BOUNDS FOR HYSTERETIC SECOND ORDER SYSTEMS*

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The behavior of many engineering systems is governed by the second order differential equation

\[ \ddot{U}(t) + F[U(t)] = \ddot{G}(t), \]  

where \( \ddot{G}(t) \) is a specified oscillatory function of time \( t \), a dot denotes differentiation with respect to \( t \), \( F(U) \) is a nonlinear restoring function representing the system hysteresis, as shown in Fig. 1, and \( U(0) = \dot{U}(0) = 0 \). In a recent paper [1], it has been shown that

\[ f = F(u) \leq \frac{1}{\alpha} \ddot{g}, \]  

where \( f = \sup|F(U)|, u = \sup|U(t)|, \ddot{g} = \sup|\ddot{G}(t)|, \) and \( 0 < \alpha \leq 1 \) is given by

\[ \alpha = \frac{A}{4fu}, \]  

where \( A \) is the area of the hysteresis loop.

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The quality of the upper bound on $U(t)$, as given by inequality (2), deteriorates with the reduction in the initial slope of $F(U)$. That is, the estimate provided by inequality (2) is good for stiff systems. For soft systems (i.e., systems for which the initial slope of $F(U)$ is small), this estimate may be much higher than the actual values of $U(t)$.

To find a bound on $U(t)$ suitable for soft systems, both sides of Eq. (1) are multiplied by $\dot{U}(t)$, and the resulting expression is integrated over the interval $[t_i, t_{i+1}]$ to yield

$$\int_{t_i}^{t_{i+1}} F(U) \dot{U}(t) \, dt = \int_{t_i}^{t_{i+1}} G(t) \dot{U}(t) \, dt,$$  

where $t_i$ and $t_{i+1}$ are two consecutive times of zero crossing of $\dot{U}(t)$. If both sides of Eq. (1) are multiplied by $G(t)$ and the resulting expression is integrated over the same interval $[t_i, t_{i+1}]$, it yields

$$\int_{t_i}^{t_{i+1}} \dot{U}(t) G(t) \, dt + \int_{t_i}^{t_{i+1}} F(U) G(t) \, dt = \frac{3}{2} \dot{G}^2(t),$$

where $\dot{G}$ is the derivative of $G(t)$.

Comparing Eqs. (4) and (6), one obtains

$$\int_{t_i}^{t_{i+1}} \dot{U}(t) G(t) \, dt = -\int_{t_i}^{t_{i+1}} F(U) \dot{U}(t) \, dt.$$  

Substitution from Eq. (7) into Eq. (5) results in

$$\int_{t_i}^{t_{i+1}} F(U) \dot{U}(t) \, dt + \frac{3}{2} \dot{G}^2(t_{i+1}) = \int_{t_i}^{t_{i+1}} F(U) G(t) \, dt + \frac{3}{2} \dot{G}^2(t_i).$$

Since $F(U)$ has at most one zero crossing in the time interval $[t_i, t_{i+1}]$, then

$$\int_{t_i}^{t_{i+1}} F(U) \dot{U}(t) \, dt = \int_{G(t_{i+1})}^{G(t_i)} F(U) \, dG \leq 2fg,$$

where $g = \sup |G(t)|$. Also,

$$\int_{t_i}^{t_{i+1}} F(U) \dot{U}(t) \, dt = \int_{U(t_i)}^{U(t_{i+1})} F(U) \, dU = \alpha(2fu),$$

where $\alpha$ is the reduction factor which makes the equality satisfied, and its value is given by relation (3). Comparisons of equalities (8) and (10) and inequality (9) yield

$$\alpha fu \leq fg + \left(\frac{1}{2}\right) \dot{g}^2,$$

where $\dot{g} = \sup |\dot{G}(t)|$. The simultaneous occurrence of $f$ and $u$ is an implicit assumption in both inequalities (2) and (11). Equivalently, inequality (11) can be written as

$$auF(u) \leq gF(u) + \left(\frac{1}{4}\right) \dot{g}^2.$$
This inequality yields an upper bound on $u$ with $\alpha$ as a parameter suitable for soft systems.

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**References**