WEIGHTED LIAPUNOV FUNCTIONS
FOR A CLASS OF THIRD-ORDER
AUTONOMOUS DIFFERENTIAL EQUATIONS*

BY

LARRY R. ANDERSON

Whitman College, Walla Walla, Washington

1. Introduction. In 1963, Walter Leighton published a paper [7] in which he provided Liapunov functions for general classes of second- and third-order differential equations. In a subsequent paper [3], the current author and Walter Leighton considered Liapunov functions for second-order systems that were more general than those given in Leighton’s 1963 paper. Further, this latter paper gave a class of weighted Liapunov functions for certain second-order systems. Further work utilizing weighted Liapunov functions for second-order equations was published by the current author in [1] and by S. Duchich and the current author in [2]. A separate paper by A. Skidmore [8] extended Leighton’s 1963 work on third-order equations to fourth-order equations, but no “weighting” was considered in either paper.

The purpose of this paper is to extend Leighton’s work (in [7]) on third-order systems by providing a class of weighted Liapunov functions for an equation

\[ \ddot{x} + \phi(x, \dot{x}, \ddot{x}) = 0 \]

and the associated system

\[ \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -\phi(x, y, z). \] (1.1)

This system is a bit more general than third-order systems considered by Leighton in [7]. Moreover, for third-order systems considered by Leighton in [7], our weight functions \( V_w \) (\( w = \text{weight} \)) reduce to those given by Leighton when \( w = 1 \). In turn, as pointed out by Leighton in [7], his Liapunov functions are “sharp” in that they provide stability criteria in linear cases when the Routh-Hurwitz condition holds. Thus our class of functions provides breadth and sensitivity.

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2. A class of Liapunov functions. We assume here that \( \phi \) is of class \( C'' \) in an open set containing the origin and that the origin is an isolated critical point of (1.1).

We will consider weight functions \( w(x, y, z) \) which are of class \( C'' \) neighboring the origin and are positive in a deleted neighborhood of the origin.

For such a function \( w \) and a constant \( a \), we define

\[
V_w(x, y, z) = \int_0^x w(x, 0, v) dv + \int_0^x w(x, u, 0) \phi(x, u, 0) du \\
+ a \left[ zyw(x, y, 0) + \int_0^x w(t, 0, 0) \phi(t, 0, 0) dt \\
+ \int_0^x u \left[ w(x, u, 0) \phi_x(x, u, 0) + w_x(x, u, 0) \phi(x, u, 0) \right] du \right].
\]

If \( w \equiv 1 \) and if \( \phi(x, y, z) = z\psi(x, y) + \theta(x, y) \), then this reduces to the function considered by Leighton in [7].

In the sequel it will be convenient to define

\[
A(x, y) = w(x, y, 0) \phi(x, y, 0)
\]
and

\[
B(x, y) = yw(x, y, 0) \phi_x(x, y, 0) + yw_x(x, y, 0) \phi(x, y, 0).
\]

To compute \( \dot{V}_w \), we first compute the various first partial derivatives. We obtain

\[
\frac{\partial V_w}{\partial x} = \frac{\partial}{\partial x} \left( \int_0^x w(x, 0, v) dv + \int_0^x (A_x(x, u) + aB_x(x, u)) du \right) \\
+ aw(x, 0, 0) \phi(x, 0, 0) + a\phi_x(x, 0, 0), \\
\frac{\partial V_w}{\partial y} = A(x, y) + aw(x, y, 0) z + a\phi_x(x, y, 0) + aB(x, y), \\
\frac{\partial V_w}{\partial z} = zw(x, 0, z) + ayw(x, y, 0).
\]

We then have

\[
\dot{V}_w = y \int_0^x w(x, 0, v) dv + y \int_0^x (A_x(x, u) + aB_x(x, u)) du \\
+ ayw(x, 0, 0) \phi(x, 0, 0) + a\phi_x(x, 0, 0) \\
+ z \left[ w(x, y, 0) \phi(x, y, 0) + aw(x, y, 0) z + a\phi_x(x, y, 0) + aB(x, y) \right] \\
- \phi(x, y, z) \left[ zw(x, 0, z) + ayw(x, y, 0) \right] \\
= y^2 \left[ \frac{1}{y} \int_0^x (A_x(x, u) + aB_x(x, u)) du \right] + z^2 \left[ aw(x, y, 0) + ayw_x(x, y, 0) \right] \\
+ zy \left( \frac{aB(x, y)}{y} + \frac{1}{z} \int_0^x w(x, 0, v) dv + ayw_x(x, y, 0) \right) \\
+ zw(x, y, 0) \phi(x, y, 0) + ayw(x, 0, 0) \phi(x, 0, 0) \\
- z\phi(x, y, z) w(x, 0, z) - ayw(x, y, 0) \phi(x, y, z) \\
= Cy^2 + Dzy + Ez^2,
\]

(2.1)
where $C$, $D$, and $E$ are given by

$$
C = \frac{1}{y} \int_0^y \left( A_x(x, u) + \alpha B_x(x, u) - A_y(x, u) \right) \, du,
$$

$$
D = \frac{\alpha B(x, y)}{y} + \frac{1}{z} \int_0^z \left( zw_x(x, 0, v) - \alpha w(x, y, 0) \phi_x(x, y, v) \right) \, dv
$$

$$
+ \alpha y w_x(x, y, 0) + \phi(x, y, z) \frac{1}{y} \int_0^y w_x(x, u, v) \, du,
$$

$$
E = \alpha w(x, y, 0) + \alpha y w_x(x, y, 0) - \frac{1}{z} \int_0^z \frac{\partial (\phi w)}{\partial z}(x, y, v) \, dv.
$$

We obtain the following result.

**Theorem 1.** If there exists a weight function $w$ and a constant $\alpha$ such that $V_w$ is positive definite neighboring the origin and such that $E < 0$ and $D^2 - 4EC < 0$, then the origin is an asymptotically stable critical point of (1.1) and $V_w$ is a Liapunov function for (1.1).

Here we assume that $\phi$ and $w$ satisfy the regularity conditions given at the beginning of Sec. 2. Further, the theorem allows us to conclude asymptotic stability since the set $V_w = 0$ is the $x$ axis, which is an invariant set (see [4]).

**Example 1.** We let $w = 1$ and find sufficient conditions for the asymptotic stability of the origin. If $\phi(x, y, z) = z \psi(x, y) + \theta(x, y)$, these conditions agree with those given in [7].

We first find conditions which insure that $V_w (w = 1)$ will be locally positive definite. Consider the matrix

$$
(a_{ij}) = \left( \frac{\partial^2 V_w}{\partial x_i \partial x_j} \right), \quad 1 \leq i \leq 3, \quad 1 \leq j \leq 3.
$$

It is easily shown that

$$
\det(a_{ij})^0 = \begin{vmatrix} \alpha \phi_x^0 & \phi_x^0 & 0 \\ \phi_x^0 & \phi_y^0 + \alpha \phi_z^0 & \alpha \\ 0 & \alpha & 1 \end{vmatrix},
$$

where the superscript denotes evaluation at the origin. If we assume that

$$
\phi_z^0 > \alpha, \quad \phi_x^0 > 0, \quad \phi_y^0 > 0 \quad (2.2)
$$

and let $\alpha = (\phi_x^0)(\phi_y^0)^{-1}$, then the principal minors of the above determinant will be positive and hence $V$ will be locally positive definite.

The discriminant $D^2 - 4EC$ of (2.1) evaluated at the origin is given by

$$
(D^2 - 4EC)^0 = (-4EC)^0 = -4(\alpha - \phi_z^0)(\phi_x^0 - \phi_y^0),
$$

which will be negative neighboring the origin if $\phi_x^0 < \phi_y^0$. Since $E^0 = \alpha - \phi_z^0$, it follows that $V_w (w = 1)$ is a Liapunov function for (1.1) if (2.2) holds and if $\phi_x^0 < \phi_y^0$ (here $\alpha = (\phi_x^0)(\phi_y^0)^{-1}$).
If the above conditions hold and $\phi$ contains higher-order terms in $z$, $V_w (w = 1)$ is a Liapunov function. For instance, $V_w$ is a Liapunov function when $\phi(x, y, z) = 4z + z^3 + x + 2y$. Such cases weren’t considered by Leighton in [7].

Now assume that $w$ is any weight function satisfying the regularity conditions given at the beginning of Sec. 2. Further, assume that $w^0 > 0$ and that (2.2) holds and that $\phi_x^0 < \phi_y^0$. It can be shown that

$$\det \left( \frac{\partial^2 V_w^0}{\partial x_i \partial x_j} \right) = \begin{vmatrix} \alpha w^0 \phi_x^0 & w^0 \phi_y^0 & 0 \\ w^0 \phi_x^0 & w^0 \phi_y^0 + \alpha w^0 \phi_z^0 & \alpha w^0 \\ 0 & \alpha w^0 & w^0 \end{vmatrix}.$$ 

Also,

$$(D^2 - 4EC)^0 = -4(\alpha w^0 - w^0 \phi_x^0)(w^0 \phi_y^0 - w^0 \phi_z^0).$$

We obtain the following result, which in certain cases provides a large class of Liapunov functions for (1.1).

**Theorem 2.** Let $w^0 > 0$ and assume that $\phi$ satisfies (2.2) and that $\phi_x^0 < \phi_y^0$. Then $V_w$ is a Liapunov function for the system (1.1), where $\alpha = (\phi_y^0)^{-1}.$

In some earlier papers (see [1], [2], [3]) concerning second-order systems, particular attention was given to considering estimates of regions of asymptotic stability obtained from $V_w$ by varying the weight function $w$. For certain weight functions and certain second-order systems, one obtains better estimates than with the single estimate provided by $V_w$ with $w = 1$. In certain cases (see [1], [2]) optimal estimates over certain subclasses of weight functions may be found.

The following example illustrates that some of these methods carry over to the third-order case.

**Example 2.** In the following example we consider a system (see [5, p. 228])

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -y - bz - f(x).$$

Under appropriate conditions, the equation $V = V(P_0)$ will be an estimate of the region of asymptotic stability, where $V$ is the function above with $w = 1$ for the system (2.3) and $V(P_0)$ is the smallest positive critical value of $V$. We will show how to choose a weight function $w$ such that the manifold $V_w = V_w(P_0)$ bounds a subregion of the region of asymptotic stability of the origin and wholly contains the region $V = V(P_0)$ (see, in particular, [2]).

We assume that $f \in C'(-\infty, \infty)$, and that $f$ vanishes only at $x = 0$ and at some $x_0 > 0$. We assume further that $0 < f'(x) < m < b$ for all $x$, that $xf(x) \geq 0$ and $f'(0) > 0$, and that

$$\lim_{x \to -\infty} f(x) = L_1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = L_2,$$

where $L_1$ and $L_2$ are finite.
With \( w = 1 \) and \( \alpha = 1 \) we have
\[
V = \frac{z^2}{2} + \frac{y^2}{2} + f(x)y + zy + \int_0^x f(u) \, du + \frac{by^2}{2}.
\]
and with \( w = 1 + Bx^2 \) we have
\[
V_w = \frac{z^2}{2} (1 + Bx^2) + \frac{y^2}{2} (1 + Bx^2) + (1 + Bx^2) f(x)y
+ zy(1 + Bx^2) + \int_0^x (1 + Bu^2) f(u) \, du + \frac{by^2}{2} (1 + Bx^2).
\]

We summarize several facts concerning \( V \) and \( V_w \) and omit the computational details.

(i) \( V \) and \( V_w \) are locally positive definite.

(ii) For \( B \) sufficiently small and positive and for \( B = 0 \), the points \((x_0, 0, 0)\) and \((0, 0, 0)\) are the only finite critical points of \( V_w \).

(iii) For all \( B \geq 0 \), \( V_w \) has no infinite critical points, which follows from the fact that there is no sequence \( \{X_n\} \) in \( \mathbb{R}^3 \) such that
\[
|X_n| \to \infty \quad \text{and} \quad \frac{\partial V_w}{\partial x_i} (X_n) \to 0 \quad \text{as} \ n \to \infty \quad (1 \leq i \leq 3).
\]

(iv) From the above and [6], it follows that for \( B \) small and positive
\[
V(x, y, z) = V(x_0, 0, 0)
\]
and
\[
V_w(x, y, z) = V_w(x_0, 0, 0)
\]
form closed 2 manifolds containing the origin, both of which lie in a subset of the region of asymptotic stability of the origin.

We now show that the region (2.5) wholly contains (2.4) and that for \( B \) sufficiently small and positive, \( \dot{V}_w \leq 0 \) inside (2.5).

We first verify the latter. It may be shown that
\[
\dot{V}_w = z^2(xyB + (1 + Bx^2)(1 - b))
+ y^2(xyB + 2xf(x)B + 2xzB + xybB + [f'(x) - 1][1 + Bx^2]).
\]
Since the region (2.5) is compact, it follows that for \( B \) sufficiently small, \( \dot{V}_w \leq 0 \) inside this region.

To verify that (2.5) contains (2.4), we solve for \( z \) in terms of \( x \) and \( y \). For the surface (2.5), we have that
\[
z = -y \pm 2^{1/2} \left[ \frac{\int_0^x (1 + Bu^2) f(u) \, du}{1 + Bx^2} - \frac{y^2b}{2} - f(x)y - \frac{\int_0^x (1 + Bu^2) f(u) \, du}{1 + Bx^2} \right]^{1/2}.
\]
(2.6)
and for (2.4) we have that

$$z = -y \pm 2^{1/2} \left[ \int_0^{x_0} f(u) \, du - \int_{x_0}^x f(u) \, du - \frac{by^2}{2} - f(x)y \right]^{1/2}$$

$$= -y \pm 2^{1/2}b^{-1/2} \left[ \int_0^{x_0} f(u)[b - f'(u)] \, du - \frac{b^2}{2} \left( y + \frac{f(x)}{b} \right)^2 - \int_0^x f(u)[b - f'(u)] \, du \right]^{1/2}. \quad (2.7)$$

If we set the quantity in square brackets in (2.7) equal to zero, it may be seen that this is a closed curve bounding the domain of $z$ in (2.7). The projection of this curve on the $x$ axis is the interval $[x_1, x_0]$, where $x_1$ is the unique negative value such that

$$\int_0^{x_0} f(u)(b - f'(u)) \, du = \int_0^{x_1} f(u)(b - f'(u)) \, du.$$

To prove our assertion, it suffices to prove that the quantity in square brackets in (2.6) exceeds the quantity in square brackets in (2.7). This reduces to proving that

$$\int_x^{x_0} u^2 f(u) \, du \geq x^2 \int_x^{x_0} f(u) \, du, \quad x_1 \leq x \leq x_0.$$

If we put

$$G(x) = \int_x^{x_0} u^2 f(u) \, du - x^2 \int_x^{x_0} f(u) \, du$$

and note that

$$G'(x) = -2x \int_x^{x_0} f(u) \, du, \quad x_1 \leq x \leq x_0,$$

the inequality (which is strict in $(x_1, x_0)$) follows at once.

**References**