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NONISOTHERMAL DYNAMIC PHASE TRANSITIONS*

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Abstract. In this work we prove existence of travelling wave solutions to the Korteweg theory of capillarity regularization of conservation laws governing the motion of a van der Waals fluid. These solutions connect states in different phases and thus can be interpreted as dynamic phase transitions.

I. Introduction. In this paper we prove the existence, under suitable conditions, of travelling wave solutions to the Korteweg capillarity theory regularization of conservation laws governing van der Waals fluids. Thus we consider the following mass and momentum conservation laws:

$$w_t = u_X, \tag{1.1a}$$

$$u_t = \tau_x, \tag{1.1b}$$

where Lagrangian coordinates are used, w is the specific volume, u is the velocity, and τ is the stress. In what follows, the constitutive relation among w, u, τ takes the form

$$\tau(w) = \tau_1 + \tau_2 = (-p(w) + \mu u_x) - c w_{xx}. \tag{1.2}$$

Here μ and c are, respectively, the viscosity and capillarity coefficients. $p(w)$ is the van der Waals equation of state

$$p(w, \theta) = \frac{R\theta}{w - b} - \frac{a}{w^2}, \tag{1.3}$$

where a, b are constants, R is the gas constant, and θ is the absolute temperature. Note that the stress allows decomposition into two parts, the bracketed one being the intertial and viscous contribution, and the other part coming from the Korteweg capillarity theory. For more details on the reasons for employing this theory, as well as on the derivation of (1.1a)–(1.1b) and (1.2), the reader is referred to the work of Slemrod [7, 8, 9]. We now complement (1.1a)–(1.1b) with an energy balance equation, in which the so-called Felderhof postulate [9] has been used:

$$E_t = h_x + (\tau_1 u)_x.$$

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Here E is the specific total energy and h is the heat flux. In terms of the full stress τ this equation takes the form

$$E_t = (\tau u)_x + h_x + (cu_x w_x)_x. \tag{1.1c}$$

We also follow Slemrod in introducing the following scaling:

$$c = \mu^2 A, \tag{1.4}$$

with A a positive constant, as well as using the specific total energy in the form

$$E = \frac{u^2}{2} + \frac{cw_x^2}{2} + \varepsilon, \tag{1.5}$$

where the specific internal energy ε is given (from thermodynamic reasons) by

$$\varepsilon(w, \theta) = -\frac{a}{w} + c_v\theta + \text{const}. \tag{1.6}$$

Here c_v is the specific heat at constant volume and a is as in (1.2). For the reader's convenience and future use, we note that the following thermodyanmic relations hold:

$$\psi(w, \theta) = \varepsilon(w, \theta) - \theta\eta(w, \theta),$$
$$\eta = -\frac{\partial\psi}{\partial\theta}, \qquad p = -\frac{\partial\psi}{\partial w}.$$

Here ψ is the specific Helmholtz free energy; η is the specific entropy.

Finally, we assume throughout that the heat flux obeys the Fourier law,

$$h = \alpha\theta_x, \tag{1.7}$$

with α a positive constant, the coefficient of thermal conductivity. With assumptions (1.2)–(1.7) incorporated into (1.1), the resulting system of equations can be regarded as a realistic model of a van der Waals fluid. Of special interest is the case of subcritical temperature, when $p(w, \theta)$ is nonmonotone in w. Then travelling wave solutions of (1.1) connecting states in different phases (on different isotherms) can be considered as "nonisothermal phase transitions." Such phenomena have been experimentally observed [4]. Existence results for the case of small viscosity and high specific heat at constant volume have been obtained by Slemrod [8]. In fact, this paper is an extension (using topological methods of Conley [2], [3]) of Slemrod's results. See also [6] for related results on nonuniqueness.

In the subcritical regime, the van der Waals hydrostatic pressure is given in Fig. 1.

The region $w \in (b, w_\alpha)$ is called the α-(liquid) phase, $w \in (w_\beta, \infty)$ is the β-(vapor) phase. The region $[\bar{w}_-, w_\alpha) \cup (w_\beta, \bar{w}_+]$ is called the metastable region.

Note, that $\lim_{w\to\infty} p'(w) = 0$. If now we put in (1.1) $\xi = (X - Ut)/\mu$, employ all the assumptions (1.2)–(1.7), integrate from $-\infty$ to ξ, imposing the conditions

$$w(-\infty) = w_-, \qquad u(-\infty) = u_-, \qquad E(-\infty) = E_-, \qquad \theta(-\infty) = \theta_-,$$

express E in terms of w, u, θ and eliminate all terms involving u, we obtain the following system of ordinary differential equations:

$$w' = v, \tag{1.8a}$$

$$Av' = -Uv - U^2(w - w_-) + p_- - p(w, \theta), \tag{1.8b}$$

$$\theta' = \frac{\mu}{\alpha} U \left\{ -(\varepsilon(w, \theta) - \varepsilon) - p_-(w - w_-) + \frac{U^2}{2}(w - w_-)^2 + \frac{Av^2}{2} \right\}, \tag{1.8c}$$

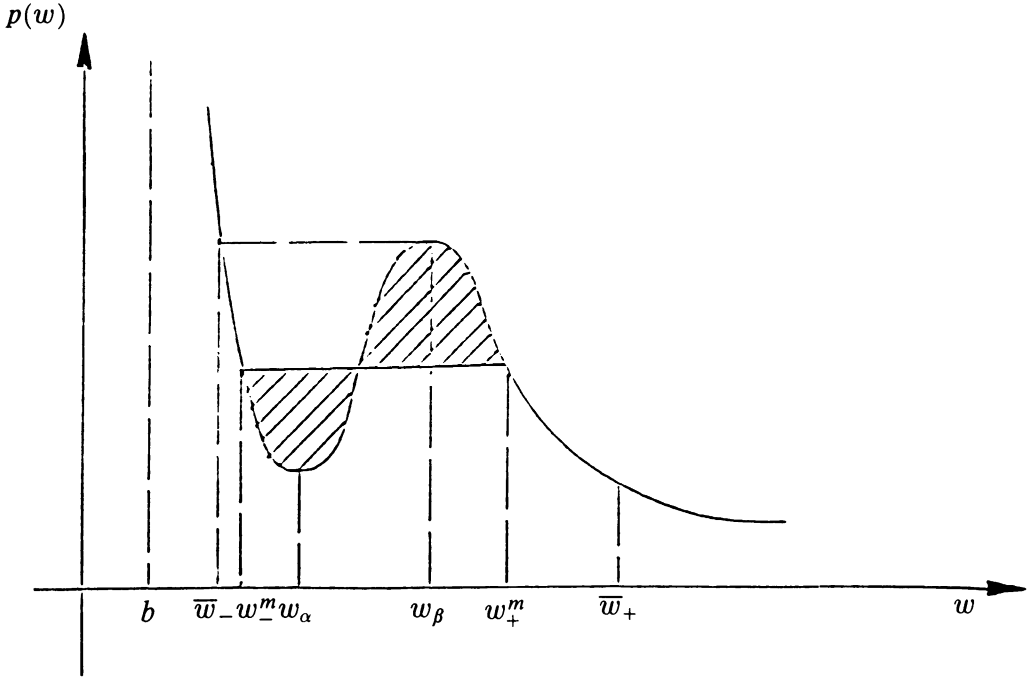

FIG. 1.

where $' = d/d\xi$ and we introduced for convenience $p_- = p(w_-, \theta_-)$ and $\varepsilon_- = \varepsilon(w_-, \theta_-)$.

Following the motivational remarks in the beginning of the Introduction, we now pose the following problem:

> Find a solution of (1.8) connecting the equilibrium points $(w_-, 0, \theta_-)$ with w_- in the α-phase of the θ_- isotherm and $(w_+, 0, \theta_+)$ with w_+ in the β-phase of the θ_+ isotherm. (P)
>
> This is what we mean by a nonisothermal dynamic phase transition.

We also remark that for the sake of brevity we have not included a discussion of "connection triples," i.e., of the relativized index formalism of Conley index theory. This aspect of the index is lucidly exposed in [2], and the reader is well advised to refer to that paper, as it is the main tool used in the present work.

Another point worth making is that (see below) both saddle–saddle and saddle–node connections are possible. In what follows, we restrict ourselves to saddle–saddle connections, which are nongeneric.

II. Preliminaries. In this section we organize the material necessary for stating and proving our results concerning (1.1). To this end, we start by considering the "isothermal system," i.e., the problem equivalent to (P) when θ, the temperature, is kept constant. In other words, only (1.8a) and (1.8b) are taken into account and $\theta(-\infty) = \theta(+\infty) = \theta_-$, a subcritical temperature. Physically, it corresponds either to having α or c_v infinitely high. This problem has been treated in [7] and [5], and we just summarize the salient points.

1. There are at most 4 critical points for the system (1.8a)–(1.8b). All are of the form $(w, 0)$ and their w-coordinate can be arranged in the order $w_- < w_1(U) < w_+(U) < w_2(U)$; for U small enough, $(w_-, 0)$ is the only critical point in the α-phase, $(w_1, 0)$ is in the unstable region, $(w_+, 0)$ and $(w_2, 0)$ are in the β-phase. When the relevant points are defined and nondegenerate, $(w_-, 0)$, $(w_+, 0)$ are saddles and $(w_1, 0)$, $(w_2, 0)$ are sinks. From now on we restrict ourselves to the (nongeneric) saddle–saddle connections.

2. Define $U_0 = \min\{U \geq 0 | w_+(U) \text{ exists}\}$, $U_1 = \max\{U \geq 0 | w_+(U) \text{ exists}\}$. Then, if $w_- < \bar{w}$ (refer to Fig. 1), $U_0 \neq 0$.

3. The system (1.8a)–(1.8b) can be considered as a dissipative perturbation of a Hamiltonian system with the Hamiltonian

$$U(w, v, U) = \frac{1}{2}v^2 + \frac{1}{A}\int_{w_-}^{w} [p(s) - p(w_-, \theta_-) + U^2(s - w_-)]\, ds. \tag{2.1}$$

Define a value of U, $\overline{U}$, implicitly by $H(w_+(\overline{U}), 0, \overline{U}) = 0$. Note that $\overline{U}$ does not always exist. If it does, $\overline{U} \leq U_1$. A necessary condition for the problem (P) reduced to (1.8a)–(1.8b) to have a solution is that, given w_-, $H(w_+(U_0), 0, U_0) < 0$. This stems from the dissipative nature of the process. This condition is satisfied iff $w_- < w_-^m$. In fact, this condition is also sometimes sufficient, as shown in Theorem 2.1 below. The same result has been proven by Slemrod [7], and the main reason for including it is that the proof involves an index computation that will be used in Theorem 3.4. The computation is very much in the spirit of Example B of [2, §2] and is therefore only sketched.

THEOREM 2.1. *If $H(w_+(0), 0, 0) < 0$, $U_0 = 0$ and $\overline{U}$ exists, the problem (P1) restricted to (1.8a)–(1.8b) with $\theta_+ = \theta_- = \theta$ subcritical, admits a saddle–saddle connection (i.e., an isothermal dynamic phase transition). Moreover, the solution is monotone in w and its speed is $U \in (0, \overline{U})$.*

Proof.

Step 1. (a) For $U = 0$ the system (1.8a)–(1.8b) is Hamiltonian. If $\bar{w} < w_- < w_-^m$, the portion of the unstable manifold of $(w_-, 0)$ in the positive quadrant of the (w, v)-plane will cross the line $w = w_+(0)$ with a positive value of v.

(b) If now $U = \overline{U}$, $H(w_+(\overline{U}), 0, \overline{U}) = 0$ and therefore by dissipativity, the corresponding portion of the unstable manifold will have to hit the line $v = 0$ at some point $(w, 0)$, $w < w_+(\overline{U})$. Of course, by continuity with respect to the parameter, the results of (a) are still true for $U \in (0, \varepsilon)$ and the results of (b) are still true for $U \in (\overline{U} - \varepsilon, \overline{U})$ for some $\varepsilon > 0$.

Step 2. Next, we augment (1.8a)–(1.8b) with the equation

$$U' = A\psi(w, v, U)\left(U - \frac{\overline{U}}{2}\right) \tag{2.2}$$

with $\psi(w, v, U)$ continuous and such that it is positive on a neighborhood X_1 of $(w_-, 0, \varepsilon/2) \cup (w_-, 0, \overline{U} - \varepsilon/2)$ in $\mathbb{R}^3$, negative on a neighborhood X_2 of $(w + (\varepsilon/2), 0, \varepsilon/2) \cup (w_+(\overline{U} - \varepsilon/2), 0, \overline{U} - \varepsilon/2_-)$, and zero otherwise. Now let D_1 be a compact neighborhood in $\mathbb{R}^3$ of the set $(w_-, 0) \times [\varepsilon/2, \overline{U} - \varepsilon/2]$ and let D_2

be a compact neighborhood of $\prod_{v\in[\varepsilon/2,\overline{U}-\varepsilon/2]}(w_+(U),0)$. Now denote by $I(D_i)$ the isolated invariant of D_i and set

$$S' = I(D_1), \qquad S'' = I(D_2).$$

Of course, these neighborhoods can be chosen to be disjoint. It is clear that S', S'' are isolated invariant sets for the product system on the cylinder $\mathbf{R}^2 \times [\varepsilon/2, \overline{U} - \varepsilon/2]$.

Next, we calculate the Conley indices of these isolated invariant sets, which we denote by $h(\cdot)$. Since both critical points are saddles for each U, one can construct isolating blocks for them in the form of (possibly twisted) rectangular cylinders. Clearly, for S'', exactly two (opposing) sides of the lateral four will comprise the exit set. In the case of S', the exist set will consist of 2 lateral sides together with the "top" and "bottom" sides (due to our specifications of $\psi(w,v,U)$). Therefore, by taking strong deformation retracts we have

$$\begin{aligned} h(S') &= \Sigma^2, && \text{a pointed 2-sphere} \\ h(S'') &= \Sigma^1, && \text{a pointed 1-sphere and} \\ h(S') \vee h(S'') &= \Sigma^1 \vee \Sigma^2 \neq \overline{0}. \end{aligned} \tag{2.3}$$

(The latter inequality is proved in Smoller [10, Chap. 22]; the reader is referred to this work for general information on the Conley index.)

Step 3. Next we construct a compact isolating neighborhood N in the cylinder $\mathbf{R}^2 \times [\varepsilon/2, \overline{U} - \varepsilon/2]$ such that for the product flow, (S', S'', S), where $S = I(N)$, form a connection triple in the parlance of [2], and their existence results can therefore be applied.

For each U define $N_1(U)$ to be the set in the (w,v)-plane bounded by the straight lines $w = w_-$, $w = w_+(U)$, the straight line $v = 0$ and the portion of the curve $H(w,v,U) = K(U)$ which is contained between the two vertical lines in the positive quadrant for some large positive $K(U)$.

Next, form the set $N_0(U) = N_1(U) \cup B_1(U) \cup B_2(U) - D(U)$ where $B_1(U)$, $B_2(U)$ are "standard" blocks for $(w_-,0)$, $(w_+(U),0)$, respectively, of small enough diameter r, and $D(U)$ is the set

$$D(U) = N_1(U) \cap \{w, v \in \mathbf{R}^2 | H(w,v,U) \leq \delta(U)\},$$

with $\delta(U)$ negative and so small that $D(U) \cap B_i(V) = \varnothing$ for $i = 1, 2$.

Clearly, given $K(U)$, r can be chosen so small that neither branch of the curve $H(w,v,U) = K(U)$ intersects any of $B_i(U)$, $i = 1, 2$. This fact will be used in Theorem 3.4.

The set $N_0(U)$ is drawn in Fig. 2. The exit set of $N_0(U)$, which we denote $N^-(U)$, is indicated by darker lines. Importantly, it is disconnected and has three components, A_1, A_2, A_3.

Denoting $[\varepsilon/2, \overline{U} - \varepsilon/2]$ by I, we remark that $\delta(U)$, $K(U)$ in the above construction can be chosen to be smooth in U for U in I. Now we define the sets

$$N = N_1 = \prod_{U\in I} N_0(U) \qquad \text{and} \qquad N^- = \prod_{U\in I} N^-(U).$$

Finally, set $S = I(N)$. Then, obviously, (S', S'', S) is a connection triple for U in I.

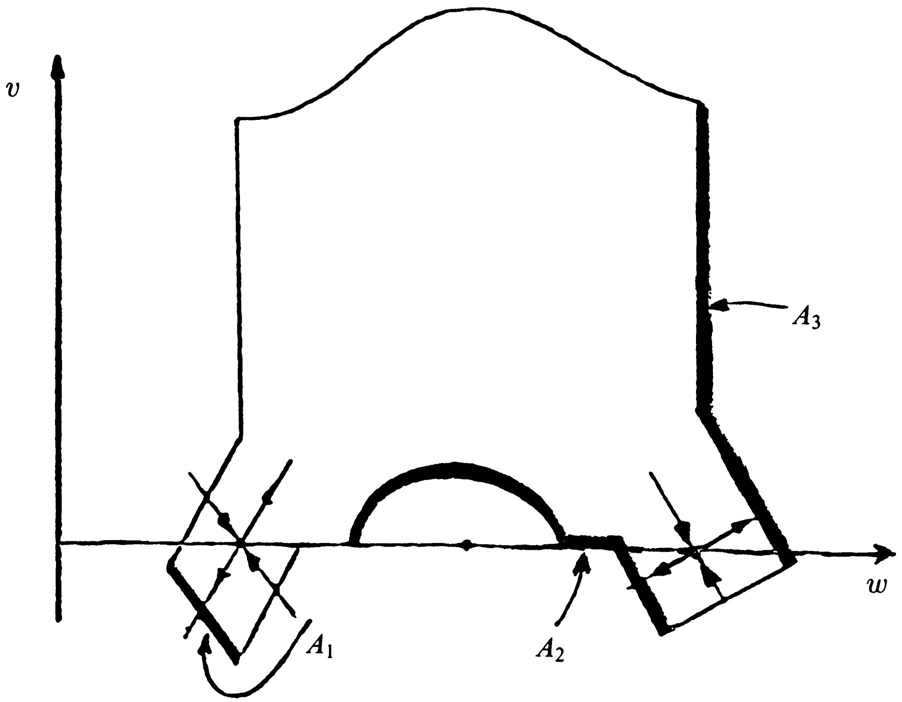

FIG. 2.

Step 4. To decide whether or not $S = S' \cup S''$, we must compute $h(S)$. To this end we construct an index pair for S, (N_1, N_2) in N, taking N_1 as above.

To satisfy the requirement that all points leaving N hit N_2 before doing so, we must have that $N_2 \supset N^-$. The inclusion, furthermore, has to be proper. To see the reason, consider the slice $N_0(\varepsilon/2)$. Trajectories starting there and leaving N have to leave through N_2. Thus N_2 has to contain the set $X_1 \cap \{\varepsilon/2\}$, which is an open set in $N_0(\varepsilon/2)$. To ensure compactness of N_2, $\mathrm{cl}(X_1 \cap \{\varepsilon/2\})$ must be in N_2. But then by positive invariance of N_2 we must have that $\mathrm{cl}(X_1 \cap \{\varepsilon/2\}) \cdot \mathbf{R}_+ \cap N \subset N_z$. Therefore, in the construction of $N_2 \cap \{\varepsilon/2\}$, we must use a closed neighborhood of the segment of the unstable manifold through $(w_-, 0)$ contained in $N_0(\varepsilon/2)$. If the set X_1 and this neighborhood have been chosen small enough, all the points leaving N through N^- will leave either through A_1 or A_3. In a similar construction for $N_2 \cap \{\overline{U} - \varepsilon/2\}$, the trajectories will leave through A_1 and A_2. (See Step 1.) By the theory of [2], [3], $h(S', S'', S) = h(S)$ is the homotopy type of the pointed space $(N_1/N_2, [N_2])$ and well defined, i.e., independent of λ for λ small enough and independent of $\psi(w, v, U)$.

But $h(S) = \overline{0}$. The reason for that is that N_1 is homeomorphic to a solid cylinder and N_2 is contractible to a point. (Refer to the remarks above.)

Thus, $h(S) \neq h(S') \vee h(S'')$. Therefore $S \neq S' \cup S''$. Therefore, by passing to the limit $\lambda \to 0$ (this is justified in [2], or, alternatively, see [1]), we conclude that there exists a $U \in I$ such that the isothermal problem (P) has a solution. Its properties are direct consequences of the construction.

In the next section we show that under certain conditions on μ and c_v the non-isothermal system, in some region of the phase space, can be factored out into a (smash) product of the system treated above and an attracting fixed point. Thus the Conley index of the resulting isolated invariant set is $\overline{0}$ and the reasoning above can be used again.

III. Existence results. First of all, we collect all the information necessary for stating the assumptions under which our results are valid.

The locus of critical points of (1.8a)–(1.8c) is the set in the $v = 0$ plane in $\mathbf{R}^3$ of the intersection points of the two curves

$$f_1(w) = \frac{w-b}{R}\left[\frac{a}{w^2} + p_- - U^2(w - w_-)\right], \tag{3.1a}$$

$$f_2(w) = \frac{1}{c_v}\left[\frac{a}{w} + \varepsilon_- - p_-(w - w_-) + \frac{U^2}{2}(w - w_-)^2\right]. \tag{3.1b}$$

A useful relation between these functions is

$$c_v f_2'(w) = \frac{R}{w-b} f_1(w), \tag{3.2}$$

from which it is clear that $f_2(w)$ decreases for $w > b$ as long as $f_1(w)$ is positive. Now, since

$$\left[\frac{f_1(w)}{(w-b)}\right]' = -\frac{2a}{w^3} - U^2 < 0,$$

for positive w, we see that $f_1(w)$ has only one zero apart from $w = b$. Therefore it is true that if $(w, 0, \theta)$ is a critical point of (1.8) with $w > w_-$, then $\theta < \theta_-$.

Next consider possible intersections of the curves in (3.1) with $U = 0$. It is not hard to convince oneself that there exists a value of c_v, say c_0, such that for $c_v < c_0$ there is only one point of intersection, and for $c_v > c_0$, there are three. c_0 depends on w_-; our approach is to fix w_- and treat c_v and U as parameters. Now, if $U \neq 0$, the graph of $f_1(w)$ will acquire a downward kink for large enough w, while the opposite will be true for $f_2(w)$. This will result (depending on c_v) in either two or four intersection points. If we now change U, say from U_1 to U_2, $U_2 > U_1$, we see that for $f_1(w)$, the curve corresponding to U_1 will lie entirely above the one corresponding to U_2, while for $f_2(w)$ the situation is exactly the opposite. A typical situation is depicted (for c_v large enough) in Fig. 3.

From all these considerations, we have the following.

LEMMA 3.1. Given any c_v, w_- in the α-phase of the θ_--isotherm, there exists a minimal value of U, $U_0^* \geq 0$, for which the system (1.8) has four critical points. Moreover, there exists another value of U, U_1^*, such that for $U > U_1^*$ only two critical points remain.

When the four critical points exist and are nondegenerate we have the following result by linearizing and employing the global Liapunov function (see below).

LEMMA 3.2.

(i) $(w_-, 0, \theta_-)$, $(w_+, 0, \theta_+)$ are saddle points.

(ii) $(w_1, 0, \theta_1)$, $(w_2, 0, \theta_2)$ are asymptotically stable.

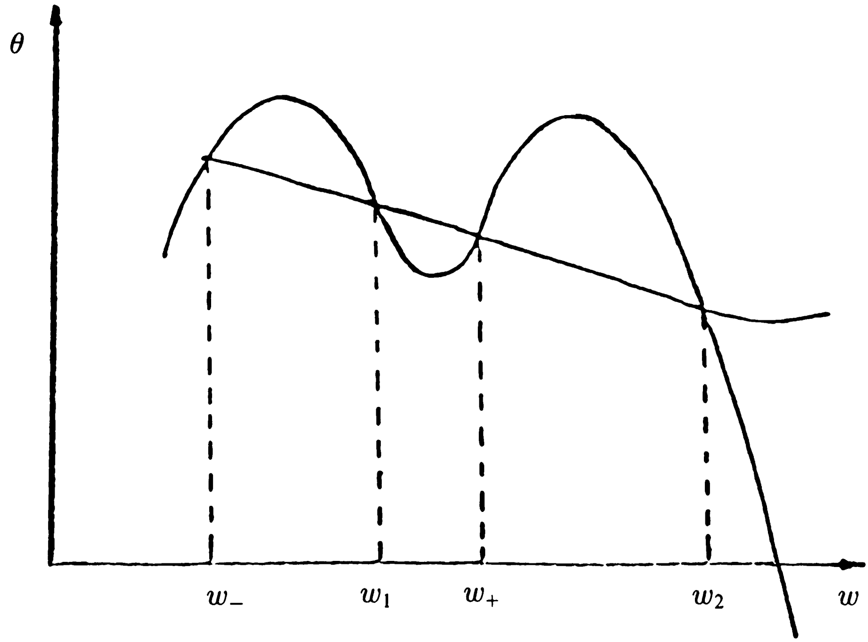

FIG. 3.

Proof. The details of the proof are in [6]. The proof of (ii) uses the Liapunov function (see [7])

$$V(v, w, \theta) = \eta(w, \theta) + \frac{Av^2}{2\theta} + \frac{1}{\theta}\left\{-(\varepsilon(w, \theta) - \varepsilon_-) - p_-(w - w_-) + \frac{U^2}{2}(w - w_-)^2\right\}. \tag{3.3}$$

Finally, we derive a necessary condition for (P) to have a solution and define an equivalent of $\overline{U}$. Since $V(w, v, \theta)$ decreases along trajectories, a necessary condition for having a saddle–saddle connection with speed U is that $V(w_-, 0, \theta_-) > V(w_+(U), 0, \theta_+(U))$. In other words, we want $\eta(w_-, \theta_-) > \eta(w_+(U), \theta_+(U))$. This, of course, is consistent with the expectation of entropy increase across the phase boundary. Thus we have:

LEMMA 3.3. A necessary condition for the existence of a saddle–saddle connection in (P) is that

$$\int_{w_-}^{w_+(U_0^*)} \frac{R}{s-b}\left(1 - \frac{f_1(s)}{f_2(s)}\right) ds < 0. \tag{3.4}$$

Proof. Using the explicit form of $\eta(w, \theta)$ and omitting the U-dependence, the remarks before the statement of the lemma mean that a necessary condition is

$$R \log \frac{w_+ - b}{w_- - b} + c_v \log \frac{\theta_+}{\theta_-} < 0.$$

The left-hand side can be rewritten as

$$\int_{w_-}^{w_+} \left[\frac{R}{s-b} + c_v \frac{d(\log f_2(s))}{ds}\right] ds < 0.$$

But this is the same as requiring

$$\int_{w_-}^{w_+} \left[\frac{R}{s-b} + c_v \frac{f_2'(s)}{f_2(s)} \right] ds < 0.$$

Using (3.2) we can rewrite this as

$$\int_{w_-}^{w_+} \frac{R}{s-b} \left(1 - \frac{f_1(s)}{f_2(s)} \right) ds < 0. \tag{3.5}$$

If now one differentiates under the integral sign with respect to U, one sees that the integral is a monotone function of U. Therefore the conclusion of the lemma follows.

Note that if $f_2(w) \equiv \theta_-$, as in the isothermal case, the necessary condition for that case is recovered from (3.5).

Consider the left-hand side of (3.5) as a function of U, say $G(U)$. If $G(U_0^*) < 0$ and $G(U_1^*) > 0$, then there exists a unique value of U, $\overline{U}^*$, such that $G(\overline{U}^*) = 0$. This is the equivalent of $\overline{U}$ for the problem (P).

There are three possible cases to be considered:

(i) $U_0^* > 0$,

(ii) $U_0^* = 0$, $\overline{U}^*$ exists,

(iii) $U_0^* = 0$, $\overline{U}^*$ does not exist.

In case (iii) it is quite possible that (P) has no solution (by which we reiterate that a saddle–saddle connection is meant). Geometrical reasoning for this statement in the isothermal case is given in [6]. In what follows we treat case (ii), under certain restrictions on c_v (but no restrictions on μ). Case (i), which arises, e.g., when w_- is chosen far enough from the Maxwell line of the θ_--isotherm, is treated in [5]. As the proof of existence in this case follows quite closely the proof of Theorem 2.2 in [6], we shall only give the precise statement of the theorem in this case in Sec. IV and comment on the restrictions under which it is valid. Note that in this case the necessary condition of Lemma 3.3 is automatically satisfied.

The main result of this section is the following Theorem.

THEOREM 3.4. If $U_0^* = 0$, $\overline{U}^*$ exists, for any μ and c_v large enough (see Assumptions I, II below) (P) admits a saddle–saddle connection which is monotone in θ. Before we prove this theorem, we note that the lower bound that is effectively introduced on c_v for the theorem to hold is artificial in the sense that it results from our choice of isolating neighborhoods in the index argument.

The idea of the proof of Theorem 3.4 is as follows: we use the construction of the proof of Theorem 2.1 to construct an isolating neighborhood that "traps" the saddle–saddle connection for a range of parameters μ and c_v. Next, a different isolating neighborhood is used which allows continuation from the case above to the case of arbitrary viscosity and an enlarged range of admissible values of c_v. Therefore we start with the following theorem (proven also in [8], but without explicitly stating the requirements on c_v; the proof there employs a singular perturbation approach and cannot be used to obtain an existence result for all μ; here the continuation properties of the index become invaluable).

Theorem 3.5. If $U_0^* = 0$, $\overline{U}^*$ exists, then for c_v high enough and μ sufficiently small, (P) has a solution.

Proof of Theorem 3.5. From the condition $U_0^* = 0$ we immediately infer that if only (1.8a)–(1.8b) are considered (i.e., in the limit $\mu \to 0$), $U_0 = 0$.

Furthermore, $\overline{U}^* = 0$ means that

$$\int_{w_-}^{w_+(\overline{U}^*)} \left[\frac{R}{s-b}\left(1 - \frac{f_1(s)}{f_2(s)}\right)\right] ds = 0,$$

and thus

$$\int_{w_-}^{w_+(\overline{U}^*)} \left[\frac{R}{s-b}\left(1 - \frac{f_1(s)}{\theta_-}\right)\right] ds > 0.$$

But this means that for the limit $\mu \to 0$ of (P), the critical point $(w_+(\overline{U}^*), 0)$ has $H(w_+(\overline{U}^*), 0, \overline{U}^*) > H(w_-, 0, \overline{U}^*)$. Of course, it can happen that $\overline{U}^* > U_1$. This will not happen if c_v is large enough. This prompts the introduction of our first assumption.

Assumption I. Let c_v be large enough so that $\overline{U}^* < U_1$. Remarks above also yield immediately that $\overline{U}$ can be defined and that, moreover, $\overline{U} < \overline{U}^*$.

Next consider the set $N_0(U)$ of Sec. II. We remind the reader that this set is

$$N_0(U) = \{(v, w) \in \mathbb{R}^2 | v \geq 0, w \in [w_-, w_+(U)], H(w, v, U) \in [\varepsilon(U), K(U)]\} \\ \cup B_r(w_-, 0) \cup B_r(w_+(U), 0),$$

with $\varepsilon(U)$, $K(U)$ smooth in U and subject to the restrictions

(i) $\varepsilon(U) < 0 < K(U)$,

(ii) $\{H(w, vU) = \varepsilon(U)\} \cap \{B_r(w_-, 0) \cup B_r(w_+(U), 0)\} = \varnothing$.

Let I be any compact subinterval of $(0, U_1)$. Then, by a standard theorem on isolating blocks, which utilizes the fact that transversality is an open condition, we get the following proposition.

Proposition 3.6. Let $\mu = 0$ and consider θ as a parameter. Then for all $U \in I$, there exists a $\gamma(U) > 0$ such that $N_0(U)$ is an isolating block for the resulting equations for all $\theta \in [\theta_- - \gamma(U), \theta_- + \gamma(U)]$.

Pick $I = [\delta, U_1 - \delta]$ with δ small enough such that

$$U_1 - \delta > \overline{U}^*, \qquad \text{and let} \quad \gamma = \min\{\gamma(U) | U \in I\}.$$

Finally, we have the following proposition for $M(U) = N_0(U) \times [\theta_- - \gamma, \theta_- + \gamma]$.

Proposition 3.7. For μ positive and small enough and c_v large enough,

(a) $M(U)$ is an isolating block for (1.8a)–(1.8c), $U \in I$,

(b)(i) if in (1.8a)–(1.8c) $U = \delta$, then the unstable manifold through $(w_-, 0, \theta_-)$ leaves $M(\delta)$ through $A_1(\delta) \times [\theta_- - \gamma, \theta_- + \gamma]$

(ii) if in (1.8a)–(1.8c) $U = U_1 - \delta$, the unstable manifold through $(w_-, 0, \theta_-)$ leaves $M(U_1 - \delta)$ through $A_2(U_1 - \delta) \times [\theta_- 0 - \gamma, \theta_- + \gamma]$.

The proof of the proposition above is by examining the behavior of the vector field on different subsets of the boundary of $M(U)$ and by appealing to the standard

ODE theorems on dependence of flow on parameter. Moreover, the following fact about the behavior of the vector field on the boundary of $M(U)$ is found to be true: the exit set of$M(U)$, $M^-(U)$, is just $\{A_1(U) \cup A_2(U) \cup A_3(U)\} \times [\theta_- - \gamma, \theta_- + \gamma]$. A detailed proof of Proposition 3.7 and of the remarks that follow it is given in [5].

To conclude the proof of Theorem 3.5, append to (1.8a)–(1.8c) the equation

$$U' = \lambda \tilde{\psi}(w, v, \xi, U)(U - U_1/2), \tag{3.6}$$

where, as in the proof of Theorem 2.1, $\tilde{\psi}$ is continuous and positive in a neighborhood of $(w_-, 0, \theta_- - \delta) \cup (w_-, 0, \theta_-, U_1 - \delta)$, negative in a neighborhood of $(w_+(\delta), 0, \theta_+(\delta), \delta) \cup (w_+(U_1 - \delta), 0, \theta_+(U_1 - \delta), U_1 - \delta)$ in $\mathbb{R}^4$ and zero otherwise. Set $M = \prod_{U \in I} M(U)$, an isolating neighborhood of (1.8a)–(1.8c) with (3.6) appended. Let D_1 be a neighborhood in $\mathbb{R}^4$ of $(w_-, 0, \theta_-) \times I$ and D_2 a neighborhood in $\mathbb{R}^4$ of $\prod_{U \in I}(w_+(U), 0, \theta_+(U))$ such that $I(D_i) \cap \{U\}$ is at most a singleton for $i = 1, 2$, $U \in I$. Denote $R' = I(D_1)$, $R'' = I(D_2)$, $R = I(M)$. Then, by using the strong deformation retraction $r : M \times [0, 1] \to M$, $r(w, v, \theta, U, t) = (w, v, t\theta_- + (1 - t)\theta, U)$, we see that as before $h(R') = \Sigma^2$ and $h(R'') = \Sigma^1$. Moreover, the information of Proposition 3.6 means that $h(R) = h(S) \wedge \overline{1}$, i.e.,

$$h(R) = \overline{0} \neq h(R') \vee h(R'') = \Sigma^2 \vee \Sigma^1.$$

Since (R', R'', R) form a connection triple, it means that $R \neq R' \cup R''$ so that there exists a value of $U \in I$ such that the problem (P) admits a solution. By the construction of $M(U)$ and by remarks following (3.2), we also conclude that the solution is monotone increasing in w. □

The result obtained so far is a singular perturbation one. Clearly, M is an "unsuitable" isolating neighborhood for (1.8) and (3.5) with μ, c_v outside a very restricted range. Below, we define a different isolating neighborhood for $I(M)$, P, such that continuation is made possible to any value of μ and a large range of c_v's.

Lemma 3.7. Let the condition be such that Theorem 3.5 is valid. Then any saddle–saddle connection that solves (P) is monotone decreasing in θ.

Proof. Suppose that the saddle–saddle connection has speed $\widetilde{U}$. Linearization shows that the relevant eigenvectors of the two critical points have the correct direction (i.e., the unstable eigenvector of $(w_-, 0, \theta_-)$ points into the "funnel" $\theta' \leq 0$, etc.). Invoking the Hartmann–Grobman theorem, we conclude that all that has to be shown is that if the saddle–saddle connection leaves the portion of the funnel $\theta' \leq 0$ lying outside of $B_r(w_-, 0, \theta_-) \times [\theta_- - \gamma, \theta_- + \gamma]$ and $B_r(w_+(\widetilde{U}), 0, \theta_+(\widetilde{U})) \times [\theta_- - \gamma, \theta_- + \gamma]$ for r small enough, it has to leave $M(\widetilde{U})$.

Suppose the solution leaves the funnel at $\xi = \xi_0$. Then $\theta'(\xi_0) = 0$ and $v(\xi_0)$ is nonnegative (by definition of $M(\widetilde{U})$). But then

$$\begin{aligned}\theta''(\xi_0) &= \frac{\mu \widetilde{U}}{\alpha}\left[\left(-\frac{a}{w^2} - p_- + \widetilde{U}^2(w_- w_-)\right) v + Avv'\right]\Big|_{\xi=\xi_0} \\ &= -\left(\frac{\mu \widetilde{U}}{\alpha}(\widetilde{U} v^2(\xi_0)) + \frac{R\theta(\xi_0) v(\xi_0)}{w(\xi_0) - b}\right),\end{aligned}$$

which is negative for positive $v(\xi_0)$. This means that in the case of positive $v(\xi_0)$, $\theta(\xi_0)$ is a maximum point for $\theta(\xi)$, the θ-component of the solution. This in turn means that the point $(w(\xi_0), v(\xi_0), \theta(\xi_0))$ cannot be a point of exit from the funnel. If $v(\xi_0) = 0$, and $v'(\xi_0) = 0$, we are at a critical point; if $v'(\xi_0) \neq 0$, then the solution through $(w(\xi_0), v(\xi_0), \xi(\xi_0))$ leaves $M(\widetilde{U})$ immediately either in forward or backward time direction and so cannot be the saddle–saddle connection isolated by $M(\widetilde{U})$. This concludes the proof.

Next, consider the set

$$P_1(U) = \{w, v, \theta | \theta \leq \theta_-,\ \theta' \leq 0,\ w \in [w_-, w_+(U)],\ v \geq 0\}.$$

By the preceding lemma, $P_1(\widetilde{U})$ contains the closure of the saddle–saddle connection. To make $P_1(\widetilde{U})$ into an isolating neighborhood, we must enlarge it around the critical points we need (the two saddles) and cut out a piece which includes the point we do not need, $(w_1(\widetilde{U}), 0, \theta_1(\widetilde{U}))$. In fact, we have to cut out more, since one cannot rule out that $I(P_1(\widetilde{U}))$ does not touch the boundary of $P_1(\widetilde{U})$ along the curve $\theta = f_1(w)$, $w \in [f_1^{-1}(\theta_-), f_1^{-1}(\theta_1(\widetilde{U}))]$. To get rid of all that curve, we introduce our second assumption on c_v.

Assumption II. Let c_v be large enough so that

$$V(f_1^{-1}(\theta_-), 0, \theta_-) < V(w_+(0), 0, \theta_+(0)).$$

Inspection shows that this inequality will then hold for all positive U.

Finally, choose numbers ρ, ν small enough and define

$$P(U) = P_1(U) \cup W_\rho(w_-, 0, \theta_-) \cup W_\rho(w_+(U), 0, \theta_+(U)) - Z(U),$$

where $W_\rho(p)$ is a closed ball of radius ρ around the point p and

$$Z(U) = \{w, v, \theta \in \mathbb{R}^3 | V(w, v, \theta) \leq V(f_1^{-1}(\theta_-), 0, \theta_-) + \nu\}.$$

Let also $P = \prod_{U \in I} P(U)$. Note that $P(\widetilde{U})$ is an isolating neighborhood for $I(M(\widetilde{U}))$. In fact, it is obvious that the proof of Theorem 3.5 can go through using $P(U)$, P instead of $M(U)$, M. By well definedness of the index,

$$h(I(P)) = h(I(M)) = \overline{0}; \qquad (R', R'', R) \text{ is a connection triple in } P.$$

Now only a simple continuation argument is required to conclude the proof of Theorem 3.4. Theorem 3.5 provided us with a range $\mu \in [0, \mu_0]$ and $c_v \in [c_v^0, \infty)$ of parameters for which (P) admits a solution. Let $\tilde{\mu}$, $\tilde{c}_v$ be in that range. Denote the dependence of isolating neighborhoods and invariant sets on viscosity by $P(U, \mu)$, $P(\mu)$, $R'(\mu)$, etc. Note that $P(U, \mu) \equiv P(U)$, $P(\mu) \equiv P$. Fix $c_v = \tilde{c}_v$. We now claim that the connection triplets $(R'(\tilde{\mu}), R''(\tilde{\mu}), R(\widetilde{M}))$ and $(R'(\bar{\mu}), R''(\bar{\mu}), R(\bar{\mu}))$ for *any* value of $\bar{\mu}$ are related by continuation for $U \in I$. To substantiate this claim we must first show that for any μ, $(R'(\bar{\mu}), R''(\bar{\mu}), R(\bar{\mu}))$, where $R'(\bar{\mu})$, $R''(\bar{\mu})$ are just the curves of saddle points and $R(\bar{\mu}) = I(R(\bar{\mu})) = I(P)$ under the flow of (1.8) and (3.5) with $\mu = \bar{\mu}$, are a connection triple. But this is obvious since $\overline{U}^*$ is an upper bound on the wave speed irrespective of μ. We also must exhibit an isolating neighborhood for the continuation. But this is exactly the reason for introducing P, which will serve as such an isolating neighborhood. Thus,

$$h(R'(\tilde{\mu}), R''(\tilde{\mu}), R(\tilde{\mu})) = h(R'(\bar{\mu}), R''(\bar{\mu}), R''(\bar{\mu}), R(\bar{\mu})) = \overline{0},$$

and the reasoning of Theorem 2.1, and the arbitrariness of $\bar{\mu}$ gives us that (P) has a solution for any value of μ.

Finally, one wants to see to which values of c_v can one continue $(R'(\tilde{c}_v), R''(\tilde{c}_v), R(c_v))$ as a connection triple. We do not give a complete answer to this question, but note, that using $P(c_v)$ as an isolating neighborhood, continuation will work to any value of c_v as long as Assumptions I and II hold (i.e., this will ensure both that $I(P(c_v)) \cap \partial P(c_v)$ is empty and that $R(c_v) \cap \{\delta\} \supset [R'(c_v) \cap \{\delta\}] \cup [R''(c_v) \cap \{\delta\}]$, and that strict inclusion also holds at the other endpoint of the interval). The properties of the solution follow directly from the construction of P.

IV. Remarks.

1. The case treated above is the most interesting one as it is the one that has to be considered if the initial data is chosen close enough to the Maxwell line of the θ_--isotherm. In case $U_0^* > 0$, we also have an existence result. Again, there is a lower bound on c_v involved in its formulation and again the lower bound is technical in the sense that it comes from having chosen a certain construction in the proof. Note that $f_1(w)$ is monotone decreasing as long as $f_2(w)$ is positive. This makes it possible for the point $(w_2(U), 0, \theta_2(U))$ to have (nonphysical) negative θ-coordinate. Let the minimum of $f_1(w)$ be $\theta_{\min}(U)$ (for *any* c_v and U small enough, $\theta_{\min}(U) < 0$). The proof of the theorem below, Theorem 3.8, uses Wazewski's principle and employs crucially the level sets of the Liapunov function (3.3). Therefore, since this function is only defined for $\theta > 0$, it is sensible to introduce the following assumption.

Assumption Ia. Let c_v be high enough so that $\theta_{\min}(U_0^*) > 0$. Then it is possible to prove:

THEOREM 3.8. If $U_0^* > 0$, c_v is such that Assumption Ia holds, (P) has a solution with speed $U \in [U_0^*, U_1^*]$. We do not give the proof here since it is lengthy and quite similar to the proof of Theorem 2.2 in [6]. See also [5].

So the remaining questions as far as existence is concerned, are

(a) Is it possible to prove/disprove existence in case $U_0^* = 0$, $\overline{U}^*$ does not exist?

(b) Is it possible to prove/disprove existence if c_v is outside the range given by Assumptions I, II or Assumption Ia?

2. As far as uniqueness is concerned, the coefficient A of scaling ($c = \mu^2 A$) is of utmost importance. For the two-dimensional case various nonuniqueness results are proved in [6].

3. Finally, as μ only enters in the quotient μ/α, assuming that thermal conductivity is proportional to μ, one can pass to the limit $\mu = 0$ (i.e., $\xi \to \pm\infty$) and obtain a solution of the (nonisothermal) Riemann problem which is admissible with respect to the viscosity-capillarity criterion (for much more information on relevance of this regularization for obtaining solutions to the Riemann problem, the reader is referred to [7], [8]).

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