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AN ASYMPTOTIC STABILITY CONDITION FOR INHOMOGENEOUS SIMPLE SHEAR*

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Abstract. Analytic steady solutions of inhomogeneous simple shear with isothermal and stress boundary conditions are found. The material is assumed to be thermoviscous and inertia and heat conduction effects are included. The basic inhomogeneous solution is spatially dependent, but time independent. Bifurcation of this solution, as the parameters vary, is analyzed. It is shown that there is a critical value of the parameter, corresponding to thermal softening, below which two steady state solutions exist for specified values of other parameters. A linear perturbation method, which gives rise to a linear set of partial differential equations (with spatially dependent coefficients), is used to distinguish the stable branch of the bifurcation diagram. After separation of variables, the existence of eigenvalues and eigenfunctions of the resulting fourth-order system is demonstrated. An asymptotic solution to the eigenvalue problem, for the case when an appropriate parameter is set equal to zero, is obtained explicitly. An integral method is then used in the general case to obtain a sufficient condition for stability.

1. Introduction. Shear bands, which are narrow regions of intense plastic shearing, have been extensively studied since the early work by Zener and Holloman [24] in 1944. These bands play an important role in processes such as high-speed machining, forming and ballistic penetration. Because the thermo-plastic instability is widely regarded as the mechanism for the formation of these shear bands (see, for example, Rogers [19] and Hutchinson [14]), a number of authors have studied the critical condition for the onset of this instability. Bai [3] and Burns and Trucano [5] derived instability conditions using a linear perturbation method for quasi-static simple shear of an infinite domain and a finite plate respectively. Douglas and Chen [9] examined adiabatic anti-plane shear under quasi-static conditions. The stability of simple homogeneous shear with mixed thermal boundary conditions was examined by Douglas, Malek-Madani, and Chen [10]. Burns [4] gave an approximate linear stability analysis of the formation of adiabatic shear bands under dynamic loading conditions

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by the two-timing perturbation method. Fressengeas and Molinari [11] studied a similar dynamic problem of simple shear using a relative perturbation method.

Nonlinear stability analyses of both solids and fluids have been carried out by a number of investigators who consider the behavior of solutions in the complete half-infinite time interval $0 \leq t < \infty$. Dafermos [7] examined the adiabatic flow of a Newtonian liquid which is caused by steady shearing of the boundary, and showed the existence of the global smooth solution to the initial-boundary value problem. Dafermos and Hsiao [8] obtained global smooth solutions in a problem arising in one-dimensional nonlinear thermoviscoelasticity with stress-free and thermally insulated boundary conditions. Tzavaras [21] studied the adiabatic shearing of a non-Newtonian liquid with temperature dependent viscosity and proved that a uniform shearing solution is asymptotically stable as $t \to \infty$ for the material obeying a certain constitutive law in which the coefficient of the thermal softening is assumed to be less than the coefficient of the strain rate sensitivity. Tzavaras [22] also considered the adiabatic plastic shearing of infinite slab of unit thickness subjected to prescribed traction at the boundaries. By assuming that the thermal softening prevails over strain hardening, he established a bound on the physical conditions for which the solution will blow up in finite time.

All the stability analyses cited above assume that the basic solution is homogeneous in strain and temperature. Wright [23] has studied inhomogeneous simple shear, in which he describes two stages of the deformation process after shear bands are formed from imperfections caused by thermal softening overcoming the combined effects of strain hardening and strain rate hardening. In the first stage, the region outside the band (the band width is defined as the distance from the center of the band to the position at which the plastic strain rate falls to one tenth of its maximum value) is at a low temperature and in an elastic state. In the region inside the band the temperature gradient steepens while the band narrows because the mechanism causing shear localization is still operative. However, if the process lasts long enough the heat conduction from the boundary—which depends upon the temperature gradient and the band width—will become so significant that no more energy is accumulated inside the band. Therefore the process of localization slows down and is finally stopped. The shear band thus stabilizes and the final stage of deformation is a quasi-steady motion. Wright gave a general form of the steady solution, which is illustrated by numerical examples for four commonly used viscoplastic flow laws. He also gave limitations on the constitutive law which allow such a steady solution. These different visco-plastic flow laws were all calibrated against the same hypothetical data set and a rough agreement between his numerical calculations and the physical measurements [17] of adiabatic shear banding was found. Wright concluded that the steady inhomogeneous deformation can be considered as the "core structure" of a shear band.

In the work presented here we examine the stability of an inhomogeneous strain and temperature field in a plate that leads to a shear-band-like solution. In other words, we are seeking a stability condition under which the shear band will stabilize as Wright [23] described it. In Sec. 2, the governing equations of the inhomogeneous simple shear are given and the steady state solutions are obtained. It will be shown

that once all parameters except for the one corresponding to thermal softening are specified, there is a critical value of this parameter above which no steady state solution exists—while for values less than this critical value two solutions exist. A detailed description of the bifurcation diagrams is presented. Section 3 contains the linear stability analysis of these solutions. After separating the time and space variables in the linearized equations, the existence of eigenvalues and eigenfunctions to the resulting fourth-order boundary value problem with nonconstant coefficients is demonstrated. In Sec. 4 an approximate solution to the linearized problem is obtained when an appropriate parameter is set equal to zero. The reduced system of ordinary differential equations is second order and the analysis of this eigenvalue problem leads to the determination of a critical value of $\sqrt{A}\beta/2$, where A is the amplitude of the steady state solution and β is the coefficient of thermal softening, in terms of which the stable branch of the bifurcation curve obtained in Sec. 2 is distinguished. It is shown that when this critical value is exceeded there is a solution to the linearized equations that grows exponentially in time. In Sec. 5 we reconsider the fourth-order system and use an energy method and obtain a bound on $\sqrt{A}\beta/2$ that guarantees the linear stability of the steady-state solution. This bound is less than the above mentioned critical value and it remains an open problem whether one could improve upon it. Section 6 contains the summary and a discussion of the results.

2. Governing equations. We consider the one-dimensional shearing of an infinite slab bounded by the planes $\bar{y} = \pm h$, such that

$$\bar{x} = \bar{X} + \bar{u}(\bar{Y}, \bar{t}), \qquad \bar{y} = \bar{Y}, \qquad \bar{z} = \bar{Z}, \tag{2.1}$$

where $\bar{x}, \bar{y}, \bar{z}$ and $\bar{X}, \bar{Y}, \bar{Z}$ are the Eulerian and the Lagrangian coordinates of the material respectively, $\bar{u}$ is the only nonzero component of displacement (which is in the $\bar{X}$ direction) and $\bar{t}$ is time. We assume that the displacement in the directions $\bar{y}$ and $\bar{z}$ together with all partial derivatives with respect to $\bar{x}$ and $\bar{z}$ are zero for all motions. It is also assumed that the plastic strains dominate the elastic strains. Therefore the velocity gradient $\bar{v}_{\bar{y}}$ is approximately equal to the plastic strain rate.

The governing equations, which are derived from the balance of the linear momentum, conservation of energy, Fourier's law of heat conduction, and the constitutive law of the material, are:

$$\bar{\rho}\bar{v}_{\bar{t}} = \bar{\tau}_{\bar{y}}, \tag{2.2}$$

$$\bar{k}\bar{\tau}\bar{v}_{\bar{y}} = \bar{\rho}\bar{C}\bar{\theta}_{\bar{t}} - \bar{\lambda}\bar{\theta}_{\bar{y}\bar{y}}, \tag{2.3}$$

$$\bar{\tau} = \bar{\mu}\exp(-\bar{\beta}\bar{\theta})\bar{v}_{\bar{y}}. \tag{2.4}$$

Here, $\bar{\rho}$ is the density of the material, $\bar{v}$ is the particle velocity (in the $\bar{x}$ direction), $\bar{\tau}$ is the only nonzero Cauchy stress component, $\bar{k}(\approx 0.9)$ is the proportion of plastic work converted into heat (Taylor and Quinney, [20]), $\bar{C}$ is the specific heat at constant volume, $\bar{\lambda}$ is the thermal conductivity of the material, $\bar{\theta}$ is the temperature, $\bar{\mu}$ is the strain rate sensitivity and $\bar{\beta}$ the thermal softening coefficient. Subscripts denote partial differentiation with respect to that variable. The parameters $\bar{\rho}, \bar{k}, \bar{C}, \bar{\lambda}, \bar{\mu}, \bar{\beta}$ are all assumed to be constant in time and space.

The slab is subjected to shearing stress (in the $\bar{x}$ direction) of $\bar{\tau}_b$ on both upper and lower faces, while the temperature is kept constant at $\bar{\theta}_b$ on both faces, as shown in Figure 2.1. Equations (2.2)–(2.4) are therefore complimented by the boundary conditions

$$\bar{\tau}(\pm h) = \bar{\tau}_b \quad \text{and} \quad \bar{\theta}(\pm h) = \bar{\theta}_b. \tag{2.5}$$

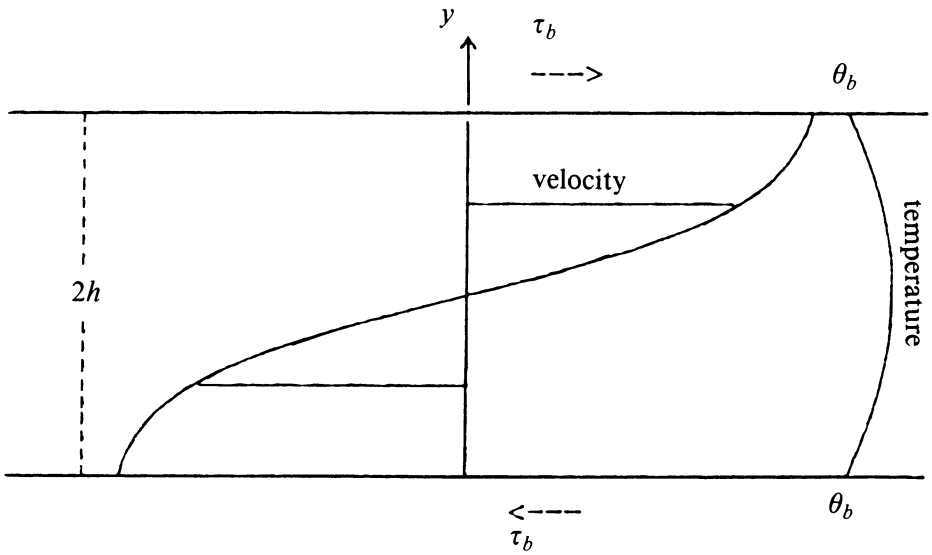

FIG. 2.1. The slab is subjected to uniform stress and temperature on both upper and lower boundaries.

We begin the analysis of the stability of the physical system described by equations (2.2)–(2.5) by introducing the nondimensional variables:

$$y = \frac{\bar{y}}{h}, \quad \theta = \frac{\bar{\theta}}{\bar{\theta}_b}, \quad \tau = \frac{\bar{\tau}}{\bar{\tau}_b}, \quad t = \frac{\bar{t}\bar{\lambda}}{\bar{\rho}h^2\bar{C}}, \quad \text{and} \quad v = \frac{\bar{v}\bar{\rho}h\bar{C}}{\bar{\lambda}}. \tag{2.6}$$

A simple change of variables allows the governing equations to be written

$$\rho v_t = \tau_y, \tag{2.7}$$

$$k\tau v_y = \theta_t - \theta_{yy}, \tag{2.8}$$

$$\tau = \mu \exp(-\beta\theta) v_y, \tag{2.9}$$

with boundary conditions,

$$\theta(-1,t) = \theta(1,t) = 1, \tag{2.10}$$

$$\tau(-1,t) = \tau(1,t) = 1. \tag{2.11}$$

The nondimensional parameters in the nondimensional governing equations (2.7)–(2.9) are given by

$$\rho = \frac{\bar{\lambda}^2}{\bar{\tau}_b\bar{\rho}h^2\bar{C}^2}, \quad k = \frac{\bar{k}\bar{\tau}_b}{\bar{\rho}\bar{C}\bar{\theta}_b}, \quad \mu = \frac{\bar{\mu}\bar{\lambda}}{\bar{\tau}_b\bar{\rho}h^2\bar{C}}, \quad \beta = \bar{\beta}\bar{\theta}_b. \tag{2.12}$$

Steady solutions to these governing equations are obtained by setting terms containing time derivatives in Eqs. (2.7)–(2.9) equal to zero. Equations (2.7) and (2.11)

lead to a uniform stress field $\tau_0 = 1$. To obtain the steady inhomogeneous temperature field, we eliminate v_y from (2.8) and (2.9) to get

$$\theta_{yy} + k\mu^{-1}\exp(\beta\theta) = 0, \tag{2.13}$$

with θ satisfying (2.10). If we now multiply this equation by θ_y and integrate twice with respect to y, the basic solution for the temperature is found to be

$$\theta_0 = \frac{1}{\beta}\ln\left[\frac{A\beta\mu}{2k}\operatorname{sech}^2\left(\frac{\sqrt{A}\beta y}{2}\right)\right], \tag{2.14}$$

where the symmetry of the thermal boundary conditions is used. The velocity can then be found by substituting (2.14) into (2.8) and integrating with respect to y. Since rigid motions can be ignored, the constant of integration is set to zero, and the steady basic solution for the inhomogeneous velocity is derived:

$$v_0 = \frac{\sqrt{A}}{k}\tanh\left(\frac{\sqrt{A}\beta y}{2}\right). \tag{2.15}$$

We note that (2.15) describes the velocity of a steady state solution to (2.7)–(2.11) whose slope at $y = 0$ can be made as steep as one wishes by calibrating the parameters μ, k, and β. In that sense we think of this solution as shear-band-like.

The constant of integration A is now computed from the boundary condition (2.10), namely $\theta(1, t) = 1$, which leads to the following algebraic equation for A:

$$\exp(\beta) = \frac{A\beta\mu}{2k}\operatorname{sech}^2\left(\frac{\sqrt{A}\beta}{2}\right). \tag{2.16}$$

The above equation defines the bifurcation diagram for the steady-state solutions of (2.7)–(2.9). As can be seen by drawing the graphs of the two functions of β in (2.16), there are values of the parameters β, A for which (2.16) has two roots, one root, or no roots, as shown in Figure 2.2.

As is typical in most bifurcation problems, the points (β, A) at which the implicit function theorem fails to describe A as a function of β determine the critical points in the bifurcation diagram. To compute these critical points, we define f by

$$f\left(\beta, A, \frac{\mu}{k}\right) = \exp(\beta) - \frac{A\beta\mu}{2k}\operatorname{sech}^2\left(\frac{\sqrt{A}\beta}{2}\right), \tag{2.17}$$

so that Eq. (2.16) is now expressed as $f(\beta, A, \mu/k) = 0$. The critical point (β, A) that starts the bifurcation diagram is the solution of $f_A(\beta, A, \mu/k) = 0$. A simple differentiation of f with respect to A shows that this point is the solution to the algebraic equation

$$a\tanh(a) = 1 \tag{2.18}$$

where

$$a = \sqrt{A}\beta/2. \tag{2.19}$$

The solution to (2.18) is $a_* = 1.20$. When $a = a_* = 1.20$ there is a unique steady-state solution to (2.7)–(2.9). Using the critical value of a $(= a_*)$ in (2.17) we eliminate A

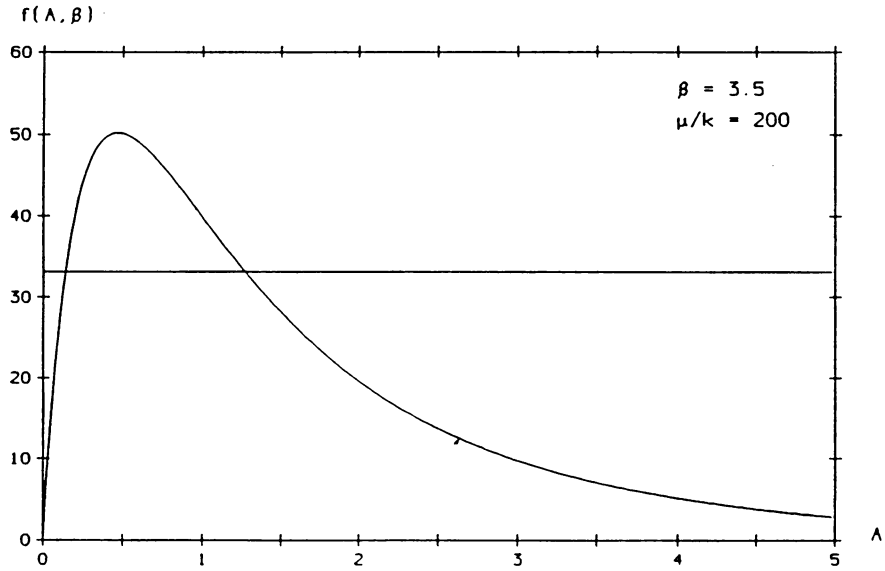

FIG. 2.2. The graph of the two functions in (2.16) for typical values of the parameters β and μ/k.

between (2.19) and $f(\beta, A, \mu/k) = 0$. This leads to the following bifurcation equation involving β, μ, and k:

$$\beta e^{\beta} = 2\mu(a_*^2 - 1)/k. \tag{2.20}$$

This equation can be interpreted as an equation in β, the parameter that measures thermal softening, as a function of the parameters μ and k. Once μ and k are given, the critical value of β is determined from (2.20) so that for β's larger than this critical value there are no steady-state solutions to our problem, whereas for β's less than this value there are two steady-state solutions. We also note that since the function $(0, \infty) \ni \beta \mapsto \exp(\beta)$ is globally invertible, we can alternatively write (2.20) as

$$\beta = g(2\mu(a_*^2 - 1)/k), \tag{2.21}$$

where g is the inverse of the above function. We have thus proved the following theorem.

THEOREM 2.1. For each value of $\mu > 0$ and $k > 0$ there is a critical value of β, denoted by β_* and determined from (2.21), such that there is no steady-state solution to (2.7)–(2.9) when $\beta > \beta_*$ and that there are two steady-state solutions when $\beta < \beta_*$.

Figure 2.3 contains the graphs of $f(\beta, A, \mu/k) = 0$ for several choices of μ/k. The points at which each curve has an extremum (when viewed, locally, as a function of A or β) are obtained by solving $f_A(\beta, A, \mu/k) = 0$ or $f_\beta(\beta, A, \mu/k) = 0$ simultaneously with $f(\beta, A, \mu/k) = 0$. These systems are solved numerically and the results are

depicted in Fig. 2.3. The main object of study in this paper is to show that for a fixed triple (β, μ, k), the steady-state solution whose amplitude A satisfies $f_A(\beta, A, \mu/k) < 0$ is stable while the one with $f_A(\beta, A, \mu/k) > 0$ is not. Thus the roots of f_A determine the stable and unstable branch on each curve.

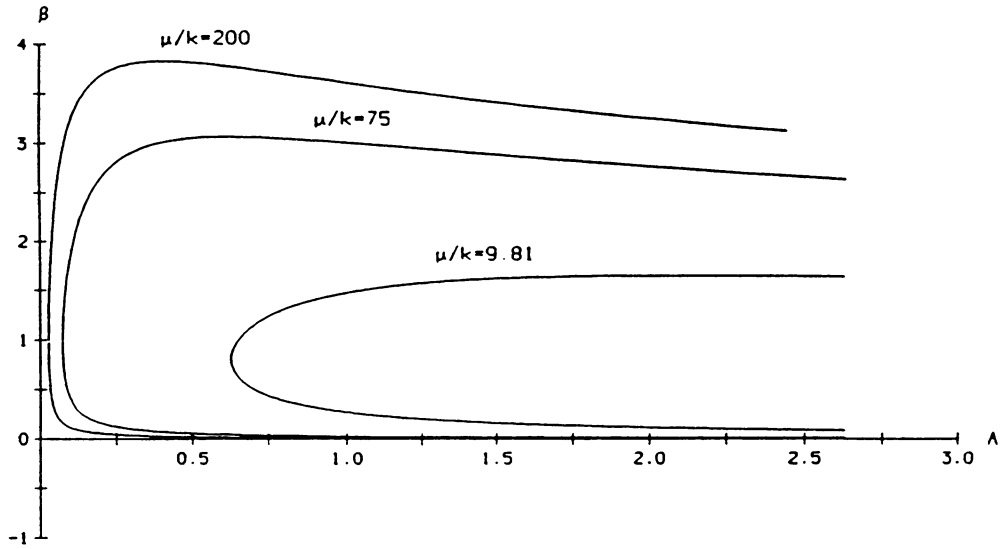

FIG. 2.3. Graphs of $f(\beta, A, \mu/k) = 0$ for several choices of μ/k.

We point out, in passing, that f_β is not zero at this critical value of β_*, so that, in the language of standard bifurcation theory (cf. Joseph and Iooss [16]) β_* is called a regular turning point and its stability, when this point is viewed as an equilibrium point of an ordinary differential equation, is well known. In the following sections we will study the stability of these steady-state solutions as solutions of the system of partial differential equations (2.7)–(2.9).

3. Linear perturbation analysis. The notion of asymptotic stability in the sense of Liapunov [13] requires that small perturbations of a basic solution decay to zero exponentially and that this decay rate be a function of the initial (in time) deviation from the basic solution and not depend on any specific perturbation. The proper study of such a notion requires an existence and uniqueness theorem for (2.7)–(2.11) which defines the natural norms in which the size of the initial perturbations should be measured. Theorems of this kind have been proved in [7], [8], [21], and [22] for systems that are very closely related to (2.7)–(2.9). The main difference between (2.7)–(2.9) and the systems considered in the aforementioned papers is in the choice of our boundary conditions. Although an existence theorem similar to the ones obtained in [7], [8], [21], and [22] is desirable, and its proof uses very similar techniques to the ones outlined in the above papers, this extension is not simple and is rather tedious. We will give an account of a proper existence theorem elsewhere and will not pursue this point in this paper any further. Therefore, we will confine our attention to a

more classical notion of stability, namely, that of linear stability. We will demonstrate that under reasonable physical conditions, the equations (2.7)–(2.9), linearized about the basic steady-state solution, have normal modes that decay to zero exponentially. Further work in showing that this notion is equivalent to the asymptotic stability of the fully nonlinear system is in progress.

The stability of the steady inhomogeneous solution (τ_0, v_0, θ_0) is investigated by introducing the linear perturbations:

$$v(y,t) = v_0(y) + \delta v(y,t), \tag{3.1}$$

$$\tau(y,t) = \tau_0(y) + \delta\tau(y,t), \tag{3.2}$$

$$\theta(y,t) = \theta_0(y) + \delta\theta(y,t), \tag{3.3}$$

where $\delta v, \delta\tau$, and $\delta\theta$ denote the perturbations on the velocity, stress, and temperature, respectively, all of which are assumed to be small compared with the respective basic solutions v_0, τ_0, and θ_0. Substitution of equations (3.1)–(3.3) into the governing equations (2.7)–(2.9), leads to the following system of linear partial differential equations, the, so-called, "perturbation equations",

$$\rho\delta v_t = \delta\tau_y, \tag{3.4}$$

$$k(v_{0y}\delta\tau + \delta v_y) = \delta\theta_t - \delta\theta_{yy}, \tag{3.5}$$

$$\delta\tau = \frac{\delta v_y}{v_{0y}} - \beta\delta\theta. \tag{3.6}$$

In order to satisfy the boundary conditions (2.10) and (2.11) the perturbations must meet the requirements

$$\delta\theta(-1,t) = \delta\theta(1,t) = \delta\tau(-1,t) = \delta\tau(1,t) = 0. \tag{3.7}$$

Since Eqs. (3.4)–(3.6) are linear partial differential equations with time independent coefficients, the standard separation of variables technique is used to write the perturbation functions as

$$\delta\tau(y,t) = \tau^*(y)e^{\alpha t}, \tag{3.8}$$

$$\delta\theta(y,t) = \theta^*(y)e^{\alpha t}, \tag{3.9}$$

$$\delta v(y,t) = v^*(y)e^{\alpha t}, \tag{3.10}$$

where (τ^*, θ^*, v^*) is the eigenfunction, and represents a perturbation mode, with associated eigenvalue α. The general perturbation of the basic solution will be a linear combination of these perturbation modes. Substituting (3.8)–(3.10) into the linear equations (3.4)–(3.6), we obtain a system of ordinary differential equations:

$$\rho\alpha v^* = \tau^*_y, \tag{3.11}$$

$$k(v_{0y}\tau^* + v^*_y) = \alpha\theta^* - \theta^*_{yy}, \tag{3.12}$$

$$\tau^* = \frac{v^*_y}{v_{0y}} - \beta\theta^*, \tag{3.13}$$

with boundary conditions,

$$\theta^*(-1) = \theta^*(1) = \tau^*(-1) = \tau^*(1) = 0. \tag{3.14}$$

These perturbation equations (3.11)–(3.13) can be further reduced to

$$k(2\tau^* v_{0y} + \beta\theta^* v_{0y}) = \alpha\theta^* - \theta^*_{yy}, \tag{3.15}$$

$$\tau^*_{yy} = \rho\alpha(\tau^* + \beta\theta^*)v_{0y}. \tag{3.16}$$

The eigenvalue α can be interpreted as the growth rate of the perturbation mode (v^*, τ^*, θ^*). If α is negative, this perturbation mode decays exponentially. If all α's are negative, we call the system (2.14) and (2.15) linearly asymptotically stable.

We close this section by proving an existence theorem to (3.14)–(3.16). We note that this system is a fourth-order boundary value problem with nonconstant coefficients. We will use an existence theorem of Cheng [6] that is tailored for systems like ours.

Let F denote the set of all 2×2 matrix functions $Q(\hat{y}) = (q_{ij}(\hat{y}))$ such that $q_{11} \geq 0$, $q_{12} > 0$, $q_{21} > 0$, and $q_{22} \geq 0$. Let I be an interval of R and let $M(I)$ be the set of all 2×2 continuous matrix-valued functions $P(\hat{y}, \hat{\alpha}) \equiv P_{\hat{\alpha}}(\hat{y}) = (P_{ij}(\hat{y}, \hat{\alpha}))$ on $[0, \infty)\times I$, where $p_{11} > 0$, $p_{12} > 0$, $p_{21} > 0$, and $p_{22} \geq 0$. For each $P_{\hat{\alpha}} \in M(I)$, we should call

$$\mathbf{w}_{\hat{y}\hat{y}} + P_{\hat{\alpha}}(\hat{y})\mathbf{w} = \mathbf{0}, \tag{3.17}$$

$$\mathbf{w}(a) = \mathbf{0} = \mathbf{w}(b), \tag{3.18}$$

the first boundary problem of $[a, b]$ relative to $P_{\hat{\alpha}}$. Let the system-conjugate point of a relative to Q, denoted by $\eta(a, Q)$, be defined as the smallest $b > a$ such that Eq. (3.18) is satisfied by a nontrivial solution of

$$\mathbf{w}_{\hat{y}\hat{y}} + Q(\hat{y})\mathbf{w} = \mathbf{0}. \tag{3.19}$$

THEOREM 3.1. (Cheng [6]) Let $I = [\hat{\alpha}_1, \hat{\alpha}_2]$ and let $P_{\hat{\alpha}} \in M(I)$. Assume $Q, R \in F$ such that $\eta(a, Q) \geq b$ and $\eta(a, R) \leq b$ where $0 \leq a \leq b$. If $P_{\hat{\alpha}} \in F$ for $\hat{\alpha} \in I$, $P_{\hat{\alpha}_1}(\hat{y}) \leq Q(\hat{y})$ on $[a, \eta(a, Q)]$ and $P_{\hat{\alpha}_2}(y) \geq R(\hat{y})$ on $[a, \eta(a, R)]$, then there exists an eigenvalue in I for the first boundary problem on $[a, b]$ relative to $P_{\hat{\alpha}}$.

Equations (3.15)–(3.16) can be written into the form of Eq. (3.17) if we let

$$\mathbf{w} = \frac{1}{4}\begin{bmatrix} \theta^* \\ \tau^* \end{bmatrix} \quad \text{and} \quad P_{\hat{\alpha}} = \begin{bmatrix} \hat{\alpha} + k\beta v_{0y}(\hat{y}) & 2kv_{0\hat{y}}(\hat{y}) \\ \rho\hat{\alpha}\beta v_{0y}(\hat{y}) & \rho\hat{\alpha}v_{0y}(\hat{y}) \end{bmatrix}. \tag{3.20}$$

If we define

$$Q = \begin{bmatrix} \hat{\alpha}_1 + k\beta\Gamma & 2k\Gamma \\ \rho\hat{\alpha}_1\beta\Gamma & \rho\hat{\alpha}_1\Gamma \end{bmatrix}, \qquad R = \begin{bmatrix} \hat{\alpha}_2 + k\beta\gamma & 2k\gamma \\ \rho\hat{\alpha}_2\beta\gamma & \rho\hat{\alpha}_2\gamma \end{bmatrix} \tag{3.21}$$

where

$$\Gamma = \mathrm{Max}[v_{0\hat{y}}(\hat{y})] \text{ and } \gamma = \mathrm{Min}[v_{0\hat{y}}(\hat{y})], \tag{3.22}$$

it can be easily observed that the inequalities associated with $P_{\hat{\alpha}_1}, Q$ and $P_{\hat{\alpha}_2}, R$ in Theorem 3.1 are satisfied. In order to satisfy the remaining inequalities in the above theorem we will confine our attention to a specific set of parameters:

$$\mu = 77,\ k = 0.375,\ A = 0.075,\ \beta = 2.92, \text{ and } \rho = 10^{-9}. \tag{3.23}$$

We emphasize that this set of parameters is typically observed in shear band formation for a high strength steel [23]. With these choices of the parameters a straightforward but tedious computation of the general solution to the fourth-order system

(3.19) with constant coefficient matrices (3.21) shows that the inequalities in Theorem 3.1, $\eta(0, Q) \geq 1$ and $\eta(0, R) \leq 1$, are satisfied for $\hat{\alpha}_1 = 1$ and $\hat{\alpha}_2 = 100$. We have thus proved the following theorem.

THEOREM 3.2. If the parameters μ, k, A, β, and ρ are set to specific values as in Eq. (3.23) then a nontrivial solution to Eq. (3.14)–(3.16) exists.

Remark. A close look at the inequalities used as hypotheses of Theorem 3.1 shows that the values of the parameters used in (3.23) can be varied considerably and still keep these inequalities valid. As the purpose of this paper is to obtain the stability of the steady-state solutions, we will not attempt to give the best possible existence theorem here.

In the next section we will obtain an asymptotic solution to this system when ρ is small.

4. Asymptotic solution of the perturbation equations. As pointed out in the previous section, there is a marked difference in the size of the parameters in the structural metals being considered here. Experimental results [18] have shown that physical conditions for the formation of shear bands exist under circumstances in which the nondimensional parameter ρ is small compared with the other physical parameters (λ, k, μ, β) in the governing equations (2.7)–(2.9). In this section we will take advantage of this fact and obtain information on the stability of the steady-state solutions where ρ is near zero. To this end we expand the eigenfunctions and the eigenvalue of (3.15) and (3.16) as Taylor series in ρ:

$$\tau^*(\rho, y) = \tau_0^*(y) + \rho\tau_1^*(y) + \cdots, \tag{4.1}$$

$$\theta^*(\rho, y) = \theta_0^*(y) + \rho\theta_1^*(y) + \cdots, \tag{4.2}$$

$$\alpha(\rho) = \alpha_0 + \rho\alpha_1 + \cdots. \tag{4.3}$$

Since the eigenfunctions τ^* and θ^* must meet the homogeneous boundary conditions (3.14), the first-order terms are subject to

$$\tau_0^*(-1) = \tau_0^*(1) = 0, \tag{4.4}$$

$$\theta_0^*(-1) = \theta_0^*(1) = 0. \tag{4.5}$$

Substituting Eqs. (4.1)–(4.3) into the perturbation equations (3.15) and (3.16), and collecting terms which do not contain ρ, we obtain two linear partial differential equations in the first-order terms τ_0^* and θ_0^*:

$$(2\tau_0^* + \beta\theta_0^*)\frac{A\beta^2}{2}\operatorname{sech}^2\left(\frac{\sqrt{A}\beta y}{2}\right) = \alpha_0\theta_0^* - \theta_{0yy}^*, \tag{4.6}$$

$$\tau_{0yy}^* = 0. \tag{4.7}$$

Equation (4.7) together with conditions (4.4) imply that $\tau_0^* = 0$ and Eq. (4.6) becomes a single ordinary differential equation in θ_0^*, namely

$$\theta_{0yy}^* + (2\alpha^2 \operatorname{sech}^2(ay) - \alpha_0)\theta_0^* = 0, \tag{4.8}$$

where

$$a = \sqrt{A}\beta/2. \tag{4.9}$$

We point out that Eq. (4.8) is an example of the Schrödinger equation with potential $2a^2 \operatorname{sech}^2(ay)$ and energy level $-\alpha_0$. In this context this equation has been studied extensively (cf. Gol'dman and Krivchenkov [12] pp. 53–55). However, these studies are not relevant to our problem since the boundary conditions in the quantum mechanical problem are often assigned at infinity and the parameters of interest do not enter into the potential the way a appears in (4.8). Therefore, we will give a direct account of the solutions to (4.8) and (4.5).

We begin by introducing the change of the independent variable from y to η where η is defined by

$$\eta = \tanh(ay). \tag{4.10}$$

Let η^* denote the value of the upper boundary in the new variable:

$$\eta^* = \tanh(a). \tag{4.11}$$

We denote $\hat{\theta}$ by the function that defines the new dependent variable:

$$\hat{\theta}(\eta) = \theta_0^*(y). \tag{4.12}$$

A simple calculation shows that $\hat{\theta}$ satisfies

$$\hat{\theta}_{\eta\eta} - \frac{2\eta}{1-\eta^2}\hat{\theta}_\eta + \left[\frac{2}{1-\eta^2} - \frac{\alpha_0}{a^2(1-\eta^2)^2}\right]\hat{\theta} = 0, \tag{4.13}$$

subject to the boundary conditions

$$\hat{\theta}(\eta^*) = \hat{\theta}(-\eta^*) = 0. \tag{4.14}$$

Equation (4.13) is a second-order ordinary differential equation with rational coefficients and regular singular points at $\eta = \pm 1$. We now look for power series solutions of (4.13). The classical theorem of Frobenius guarantees the uniform convergence of this power series in any compact subset of $(-1, 1)$. Let $\hat{\theta}$ be written as a power series in η:

$$\hat{\theta} = \sum_{m=0}^{\infty} c_m \eta^m. \tag{4.15}$$

Differentiation of (4.15) and substitution of the results into (4.13) give

$$(1-\eta^2)^2 \sum_{m=0}^{\infty} m(m-1)c_m \eta^{m-2} - \sum_{m=0}^{\infty} \left[2(m-1)(1-\eta^2) + \frac{\alpha_0}{a^2}\right] c_m \eta^m = 0, \tag{4.16}$$

with recurrence relation

$$c_{m+2} = \frac{(2m^2a^2 - 2a^2 + \alpha_0)c_m + ma^2(3-m)c_{m-2}}{(m+1)(m+2)a^2}, \quad m = 0, 1, 2, \ldots \tag{4.17}$$

where coefficients with negative indices are set to zero. Because of the symmetry of the homogeneous boundary conditions (4.14), the coefficients of the odd terms in the power series expansion of η must vanish, so that

$$c_{2m+1} = 0, \qquad m = 0, 1, 2, 3, \ldots. \tag{4.18}$$

Using the recurrence relation (4.17), the nontrivial perturbation of temperature can be expressed in terms of c_0 as

$$\hat{\theta}(\eta) = c_0 \left(1 - \frac{2a^2 - \alpha_0}{2a^2}\eta^2 + \cdots\right). \tag{4.19}$$

The perturbation of the velocity gradient can now be found from Eq. (3.13).

We note that the eigenvalue α_0 appears in (4.19). The final condition that determines this eigenvalue is the boundary condition

$$\hat{\theta}(\eta^*) = 0, \tag{4.20}$$

with η^* as defined in (4.11). Our goal is to obtain the range of the parameters (β, μ, k) for which α_0 is negative. To achieve this goal we will employ certain well-known results on Sturm-Liouville operators and the dependence of their eigenvalues on their coefficients and find the optimal bound on β for which α_0 is negative for all eigenvalues of (4.5) and (4.8). First we define the linear operator L by

$$L[\theta] = -\theta_{yy} - 2a^2 \operatorname{sech}^2(ay)\theta \tag{4.21}$$

subject to the Dirichlet boundary conditions (4.5). This operator is an example of the classical Sturm-Liouville operator and its potential is given by the hyperbolic secant term. We conclude that (cf. Ince [15]), among other properties, the spectrum of L is discrete and that all eigenvalues of L are simple. Next we note that the hyperbolic secant function is uniformly bounded in the interval in question for all values of a. An easy calculation demonstrates that when $a = 0$ the reduced second-order operator is positive definite. An application of the Poincaré inequality shows further that when a is a small positive number the operator L remains positive definite. This fact of course implies that all eigenvalues α_0 of (4.8) are negative, which gives us the linear stability of the reduced problem. We can find the critical value of a for which L loses its positive definiteness by analyzing the perturbation due to the hyperbolic secant a bit further. A theorem of Alexander and Antman ([1], p. 53) states that the eigenvalues of the Sturm–Liouville operator $T[u] = (pu')' + qu$, $u(-1) = u(1) = 0$, depend continuously on p and q as long as p and q are smooth functions of their arguments. Of course, the hyperbolic secant function depends analytically on a and y. Hence the smallest eigenvalue of L is a continuous function of a. Thus, the critical value of a for which this eigenvalue becomes zero determines the value of a for which positive definiteness fails. Our next task is to analyze this case of "marginal stability".

The following computations determine this critical value of a. First we note from (4.13) that $\hat{\theta}_0$, the eigenfunction corresponding to eigenvalue zero, must satisfy

$$\hat{\theta}_{0\eta\eta} - \frac{2\eta}{1-\eta^2}\hat{\theta}_{0\eta} + \frac{2}{1-\eta^2}\hat{\theta}_0 = 0, \tag{4.22}$$

and

$$\hat{\theta}_0(-\eta^*) = \hat{\theta}_0(\eta^*) = 0. \tag{4.23}$$

Equation (4.22) is the first-order Legendre equation with its general solution given by

$$\hat{\theta}_0 = c_1 P_1(\eta) + c_2 Q_1(\eta), \tag{4.24}$$

where P_1 and Q_1 are the first Legendre functions. Since $P_1(\eta)$ is an odd function and $Q_1(\eta)$ is an even function, c_1 must be zero from the symmetric homogeneous boundary conditions (4.23). Therefore

$$\hat{\theta}_0 = c_2\left(\frac{\eta}{2}\ln\frac{1+\eta}{1-\eta} - 1\right). \tag{4.25}$$

The boundary condition $\hat{\theta}_0(\eta^*) = 0$ leads to

$$\eta^* = 0.83, \tag{4.26}$$

which, in turn, gives a critical value for a:

$$a^* = \tanh^{-1}(\eta^*) = 1.20. \tag{4.27}$$

This condition also can be written down in terms of A, the amplitude of the steady-state solution, and β as

$$A\beta^2 = 5.76. \tag{4.28}$$

We note that this value of a is precisely the value we obtained by the bifurcation argument of Sec. 2. We have thus proved the following theorem.

THEOREM 4.1. The steady-state solutions of the reduced system ($\rho = 0$) are stable as long as $A\beta^2 < 5.76$.

The proof of the following corollary follows from the discussion at the end of Sec. 2.

COROLLARY 4.1. The steady-state solutions of the reduced system are stable if $f_A(\beta, A, \mu/k) < 0$.

This corollary states that the branch (cf. Fig. 2.3) of the curve defined by $f(\beta, A, \mu/k) = 0$ (with μ/k = const.) below and to the "left" of β_* is stable. In other words, of the two steady-state solutions, the one with the smaller amplitude velocity and temperature fields is stable.

5. A general criterion for stability. Since the general solutions to the perturbation equations (3.15) and (3.16) are not available and our main interest is only in the sign of the perturbation growth rate α, we now use an integral method to transform the ordinary differential equations (3.15) and (3.16) into an algebraic equation in α, in which the sign of every coefficient is known. Assuming that the parameters k, ρ, and β are positive, we will prove the following theorem.

THEOREM 5.1. Let θ_* and τ_* be nontrivial perturbations satisfying (3.14)–(3.16). Then the system prescribed by equations (2.7)–(2.11) is linearly asymptotically stable if the condition

$$A\beta^2 < \pi^2/2 \tag{5.1}$$

is satisfied. Here A is the constant of integration defined by (2.16).

Proof. We begin the proof by multiplying (3.16) by τ^*, integrating from -1 to 1. On integrating the left-hand side by parts, we obtain

$$-\int_{-1}^{1} |\tau_y^*|^2\,dy = \rho\alpha\int_{-1}^{1} |\tau^*|^2 v_{0y}\,dy + \rho\alpha\beta\int_{-1}^{1} \theta^*\tau^* v_{0y}\,dy. \tag{5.2}$$

Equation (3.15) is then multiplied by θ^* and similarly integrated by parts, to yield:

$$k\left[2\int_{-1}^{1}\tau^*\theta^* v_{0y}\,dy+\int_{-1}^{1}\beta|\theta^*|^2 v_{0y}\,dy\right]=\alpha\int_{-1}^{1}|\theta^*|^2\,dy+\int_{-1}^{1}|\theta_y^*|^2\,dy. \tag{5.3}$$

We next eliminate the term $\int_{-1}^{1}\tau^*\theta^* v_{0y}\,dy$ from the above two equations and employ Eq. (3.11) to arrive at

$$\alpha\left[\int_{-1}^{1}\beta|\theta^*|^2\,dy+2k\rho\int_{-1}^{1}|v^*|^2\,dy\right]+2k\int_{-1}^{1}|\tau^*|^2 v_{0y}\,dy$$
$$-k\beta^2\int_{-1}^{1}|\theta^*|^2 v_{0y}\,dy+\beta\int_{-1}^{1}|\theta_y^*|^2\,dy=0. \tag{5.4}$$

In deriving the above equation we have used the important fact that $\alpha=0$ is not an eigenvalue of (3.14)–(3.16). This fact was actually proved in the previous section if we notice that the case $\alpha=0$ for the present system (with ρ positive) and the reduced system ($\rho=0$) are the same. We have already shown in Sec. 4 that under condition (5.1) $\alpha=0$ is not an eigenvalue.

We note that the coefficient of α in (5.4) is always positive; hence the condition for $\alpha<0$ is equivalent to

$$-k\beta^2\int_{-1}^{1}|\theta^*|^2 v_{0y}\,dy+2k\int_{-1}^{1}|\tau^*|^2 v_{0y}\,dy+\beta\int_{-1}^{1}|\theta_y^*|^2\,dy>0. \tag{5.5}$$

Let Γ denote the maximum velocity gradient v_{0y} over the interval $[-1,1]$, which can be easily calculated from the basic solution (see equation (2.15)):

$$\Gamma=\frac{A\beta}{2k}. \tag{5.6}$$

This constant will allow us to bound the first term in (5.5). Next, we will use the Poincaré inequality to bound the L^2 norm of θ^* in terms of its gradient:

$$\frac{\pi^2}{4}\int_{-1}^{1}|\theta^*|^2\,dy\le\int_{-1}^{1}|\theta_y^*|^2\,dy. \tag{5.7}$$

The constant $\pi^2/4$ is, by the Dirichlet principle, the smallest eigenvalue of the operator $S[u]=u_{yy}$ subject to the boundary conditions $u(-1)=u(1)=0$. Substituting the value of Γ from (5.6) into (5.5) and using (5.7) shows that (5.5) is positive as long as (5.1) holds. This completes the proof.

The stability criterion (5.1) is only a sufficient condition for stability. Whether this criterion can be improved to the limit a_* obtained in Secs. 2 and 4 remains an open problem.

6. Summary and discussion. An analytic steady solution of inhomogeneous simple shear in a thermo-viscous material has been obtained for materials with sufficiently low coefficient of thermal softening. These velocity profiles exhibit the feature associated with shear bands, such as very steep gradients in velocity and displacement.

The materials considered are viscoplastic materials without work (strain) hardening. As pointed out by Wright [23], for a steady solution to exist, restrictions on the

constitutive law and the boundary conditions are required. (For example, no work hardening may be included in the constitutive law and no adiabatic boundary conditions are allowed.) As shown in Theorem 2.1, the steady solution exists only when the thermal softening coefficient is less than a critical value, which is determined by Eq. (2.20). Physically, this means that a material with sufficiently high thermal softening coefficient becomes so weak that it cannot support the applied traction without material acceleration, i.e., no steady solution is possible.

The stability of the steady solutions is studied by introducing linear perturbations, which result in perturbation equations with variable coefficients. The asymptotic solution of these perturbation equations for ρ small is obtained. The resulting stability criterion for this reduced system can be written as

$$A\beta^2 \leq 5.76. \tag{6.1}$$

Examination of the stability of the general system results in a fourth-order differential equation with variable coefficients which is obtained from the linear perturbation analysis. This equation can be transformed into an algebraic equation in the stability parameter α by an energy method using integrals of the field quantities. The stability criterion of the general system is given by

$$A\beta^2 \leq \frac{\pi^2}{2}. \tag{6.2}$$

This criterion is a sufficient condition for the stability of the general system. We have not been able to improve on this bound for the general problem.

Since it is assumed that the disturbance of the velocity has component in the x-direction only, the perturbation analysis performed in Sec. 3 is a one-dimensional analysis. However, a three-dimensional perturbation analysis, (see, for example, Anand, et. al. [2]), is necessary for a more complete understanding of stability in inhomogeneous fields.

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