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ROTATIONAL-TRANSLATIONAL ADDITION THEOREMS FOR SPHEROIDAL VECTOR WAVE FUNCTIONS*

By

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Abstract. Rotational-translational addition theorems for spherical and spheroidal vector wave functions are established. These theorems concern the vector wave functions $\mathbf{M}^a$ and $\mathbf{N}^a$ (with $a = r, x, y, z$) which can be obtained and used to treat various electromagnetic problems such as multiple scattering of a plane wave from prolate spheroids (with arbitrary spacings and orientations of their axes of symmetry) or radiation from thin-wire antennas. For sake of completeness, rotational-translational addition theorems for the vector wave function $\mathbf{L}$ are also established. This work is a natural extension of previous studies concerning simpler transformations of coordinate systems, such as rotation or translation. The two cases $r \geq d$ and $r \leq d$ are distinguished, where d is the distance between the centers of the spheroids.

1. Introduction. The establishment of rotational-translational addition theorems for spheroidal scalar wave functions [1, 2] requires the use of rotational and translational addition theorems for spherical scalar wave functions, as previously provided by Friedman and Russek [3], Stein [4], and Cruzan [5]. The reason is that spheroidal scalar wave functions can easily be converted into spherical ones and conversely, if both spheroidal and spherical coordinate systems are related to the same Cartesian system (Ref. [6], Eq. (5.3.9) and Ref. [7], Sec. 2.1, Eqs. (2) and (3)). In addition, starting from the last-mentioned equations, it is easy to deduce theorems converting spheroidal vector wave functions $\mathbf{M}^a$ and $\mathbf{N}^a$ into spherical ones ${}^{\cdot}\mathbf{M}^a$ and ${}^{\cdot}\mathbf{N}^a$ and conversely. Consequently, when spheroidal geometries are under consideration in studies on multiple scattering, it is necessary, first, to convert spheroidal vector wave functions into spherical ones in order to use rotational and translational addition theorems, which are given in the works by Stein [4] and by Cruzan [5]; the return to spheroidal vector wave functions can then be realized.

Starting from the expansions of an electromagnetic field in terms of vector wave functions $\mathbf{M}^a$, $\mathbf{N}^a$, and $\mathbf{L}$ expressed in the unprimed coordinate system (x, y, z) with

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origin at the center of one spheroid and axis of revolution along the z-axis:

1) rotational addition theorems allow us to represent each vector wave function $\mathbf{M}^a_{mn}$, $\mathbf{N}^a_{mn}$, and $\mathbf{L}_{mn}$ by a series expansion of spheroidal vector wave functions $\mathbf{M}^{a'}_{m'n}$, $\mathbf{N}^{a'}_{m'n}$, and $\mathbf{L}'_{m'n}$, associated with a second rotated primed system with its z'-axis arbitrarily oriented,

2) translational addition theorems allow us to represent each vector wave function $\mathbf{M}^a_{m'n}$, $\mathbf{N}^a_{m'n}$, and $\mathbf{L}_{m'n}$ by a series expansion of vector wave functions $\mathbf{M}^{a'}_{\mu\nu}$, $\mathbf{N}^{a'}_{\mu\nu}$, and $\mathbf{L}'_{\mu\nu}$, associated with a third translated primed system, centered at the origin of the other spheroid.

The combination gives rotational-translational addition theorems. In theory, the order of the two transformations is arbitrary but, in practice, its choice can provide convenience in computations.

The present theorems are established for the vector wave functions $\mathbf{M}^a$ and $\mathbf{N}^a$ in order to study the multiple scattering of electromagnetic waves from two (or more) spheroids with *arbitrary directions of their axes of revolution.* Such an event is frequently encountered in the description of scattering of radar signals by hydrometeors and visible light absorption by heterogeneous suspensions. For completeness, rotational-translational addition theorems concerning the vector wave function $\mathbf{L}$ are also established. They will be useful in studies of curl-free problems such as elastodynamic wave scattering problems, for example.

2. Expression of an electromagnetic field in terms of vector wave functions M and N. The basis of the integration of the vector wave equation by means of vector wave functions in a closed domain of a sourceless homogeneous isotropic medium was first mentioned by Hansen and then developed by Stratton. According to Stratton (Ref. [7], p. 393), the vector wave function $\mathbf{M}^a$ is obtained by $\mathbf{M}^a = \operatorname{curl}(\psi\mathbf{a})$, where ψ denotes the scalar function, which is a solution of the scalar wave equation $\Delta\psi + K^2\psi = 0$, where K is the wave number and $\mathbf{a}$ is such that $\operatorname{curl}\mathbf{a} = 0$ and $\operatorname{div}\mathbf{a} =$ constant. These latter conditions are solely fulfilled when $\mathbf{a}$ is the radius vector $\mathbf{r}$ or one of the three Cartesian unit vectors $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z$ (Appendix A). Hence, there are four vector wave functions $\mathbf{M}$, denoted $\mathbf{M}^a$, and four vector wave functions $\mathbf{N}^a = (1/K)\operatorname{curl}\mathbf{M}^a$ (with $a = r, x, y, z$) which correspond to $\mathbf{a} = \mathbf{r}$, $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$, respectively.

If we consider all the spheroidal ($\mathbf{M}^a$ and $\mathbf{N}^a$) and spherical (${}^{\cdot}\mathbf{M}^a$ and ${}^{\cdot}\mathbf{N}^a$) vector wave functions, with $a = r, x, y, z$, the following properties are to be pointed out:

1) each of them is a solution to the vector wave equation $(\operatorname{curlcurl} - K^2)\mathbf{V} = 0$;

2) only ${}^{\cdot}\mathbf{M}^r$ and ${}^{\cdot}\mathbf{N}^r$ are a set of orthogonal vector wave functions and ${}^{\cdot}\mathbf{M}^r$ is purely tangential (Ref. [7], Secs. 7–13 and 7–11);

3) ${}^{\cdot}\mathbf{M}^a$ and ${}^{\cdot}\mathbf{N}^a$ with $a = x, y, z$ are not orthogonal among themselves whatever the combination may be; however, each can be expressed in a series expansion of ${}^{\cdot}\mathbf{M}^r$ or ${}^{\cdot}\mathbf{N}^r$ (for ${}^{\cdot}\mathbf{M}^a$, see the work by Cruzan (Ref. [5], Eqs. (15), (16), (17)));

4) the spheroidal vector wave functions of a given set (with a equal to any one of the r, x, y, z) are, unfortunately, neither orthogonal themselves, nor orthogonal to those of the other sets (Ref. [6], p. 70);

5) each spheroidal vector wave function $\mathbf{M}^a$ or $\mathbf{N}^a$ is connected to the corresponding spherical vector wave function and conversely (see Eqs. (4)–(6)).

To conclude, only $\mathbf{M}^r$ and $\mathbf{N}^r$ are directly connected to a set of orthogonal spherical vector wave functions. The integration of the vector wave equation in terms of vector wave functions has found a direct and extensive application in studies on electromagnetic scattering and radiation by canonical objects such as spheres [7, 8] and cylinders [7] and by more complicated ones such as spheroids [9–18].

3. Theorems converting spheroidal vector wave functions into spherical ones and conversely. Let (x, y, z), (r, θ, ϕ), and (η, ξ, ϕ) be, respectively, Cartesian, spherical, and spheroidal coordinate systems having the same origin O. Let ${}^{\cdot}\psi_{mn}^{(i)}(r, \theta, \phi)$ and $\psi_{mn}^{(i)}(h; \eta, \xi, \phi)$ be the mnth spherical and spheroidal scalar wave functions where h denotes the semi-interfocal distance. For $i = 1, 2, 3, 4$, respectively, ${}^{\cdot}\psi^{(i)}(r, \theta, \phi)$ involves spherical, Bessel, Neumann, and Hankel functions of the first and second kinds; any $\psi^{(i)}(h; \eta, \xi, \phi)$ is directly related to the corresponding ${}^{\cdot}\psi^{(i)}(r, \theta, \phi)$. The choice of the index i depends on the time-dependence ($e^{-j\omega t}$ or $e^{j\omega t}$) and also on the behavior of the functions ${}^{\cdot}\psi^{(i)}$ and $\psi^{(i)}$ at infinity and at the origin.

We consider Eq. (5.3.9) in Ref. [6] and Eqs. (2) and (3) in Ref. [16], both valid for $i = 2, 3, 4$ and written as

$$\psi_{mn}^{(i)}(h; \eta, \xi, \phi) = \sum_{s=|m|,|m|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)\, {}^{\cdot}\psi_{ms}^{(i)}(r, \theta, \phi), \tag{1}$$

$${}^{\cdot}\psi_{mn}^{(i)}(r, \theta, \phi) = \sum_{l=|m|,|m|+1}^{\infty}{}' \Gamma_{ml}^{mn}(h) \psi_{ml}^{(i)}(h; \eta, \xi, \phi) \tag{2}$$

with

$$\begin{aligned} \Lambda_{ms}^{mn}(h) &= j^{s-n} d_{s-|m|}^{mn}(h), \\ \Gamma_{ml}^{mn}(h) &= j^{l-n} \frac{{}^{\cdot}N_{mn}}{N_{ml}(h)} d_{n-|m|}^{ml}(h) \end{aligned} \tag{3}$$

defined by Eqs. (8) and (14) in Ref. [2].

We recall that the prime over the summation sign in Eq. (1) (for example) indicates that the summation is over only even values of s when $n - |m|$ is even and over only odd values of s when $n - |m|$ is odd, from the initial value $|m|$ or $|m| + 1$ according to the parity of m. In addition, $d_s^{mn}(h)$ denotes the well-known coefficients of the expansion of the spheroidal angular function of the first kind $S_{mn}^{(1)}(h, \eta)$ (Ref. [6], p. 16) in terms of Legendre functions $P_m^{m+s}(\eta)$. ${}^{\cdot}\mathbf{N}_{mn}$ and $\mathbf{N}_{ml}(h)$ denote the normalization factors of the associated Legendre functions (Ref. [2], p. 748) and the spheroidal angle function $S_{ml}^{(1)}$, respectively (Ref. [6], p. 22).

Then, starting from Eqs. (1) and (2) and the definition $\mathbf{M}_{mn}^{(i)a} = \operatorname{curl}(\psi_{mn}^{(i)}\mathbf{a})$, since $\operatorname{curl}\mathbf{a} = 0$ and writing $\mathbf{M}_{mn}^{(i)a} = \operatorname{grad}\psi_{mn}^{(i)} \times \mathbf{a}$ and ${}^{\cdot}\mathbf{M}_{mn}^{(i)a} = \operatorname{grad}^{\cdot}\psi_{mn}^{(i)} \times \mathbf{a}$, we respectively obtain

$$\mathbf{M}_{mn}^{(i)a}(h;\eta,\xi,\phi) = \sum_{s=|m|,|m|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)\,{}^{\cdot}\mathbf{M}_{ms}^{(i)a}(r,\theta,\phi), \tag{4}$$

$${}^{\cdot}\mathbf{M}_{mn}^{(i)a}(r,\theta,\phi) = \sum_{l=|m|,|m|+1}^{\infty}{}' \Gamma_{ml}^{mn}(h)\mathbf{M}_{ml}^{(i)a}(h;\eta,\xi,\phi) \tag{5}$$

and, consequently,

$$\mathbf{N}_{mn}^{(i)a}(h;\eta,\xi,\phi) = \sum_{s=|m|,|m|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)\,{}^{\cdot}\mathbf{N}_{ms}^{(i)a}(r,\theta,\phi), \tag{6}$$

$${}^{\cdot}\mathbf{N}_{mn}^{(i)a}(r,\theta,\phi) = \sum_{l=|m|,|m|+1}^{\infty}{}' \Gamma_{ml}^{mn}(h)\mathbf{N}_{ml}^{(i)a}(h;\eta,\xi,\phi). \tag{7}$$

For completeness, theorems converting spheroidal vector wave functions $\mathbf{L}$ into spherical ones, and conversely, are now established. Starting from Eqs. (1) and (2), since $\mathbf{L}_{mn}^{(i)} = \operatorname{grad}\psi_{mn}^{(i)}$ and because the gradient operator is invariant to a transformation of the coordinate system, we immediately obtain

$$\mathbf{L}_{mn}^{(i)}(h;\eta,\xi,\phi) = \sum_{s=|m|,|m|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)\,{}^{\cdot}\mathbf{L}_{ms}^{(i)}(r,\theta,\phi), \tag{8}$$

$${}^{\cdot}\mathbf{L}_{mn}^{(i)}(r,\theta,\phi) = \sum_{l=|m|,|m|+1}^{\infty}{}' \Gamma_{ml}^{mn}(h)\mathbf{L}_{ml}^{(i)}(h;\eta,\xi,\phi). \tag{9}$$

4. Rotational-translational addition theorems for spheroidal vector wave functions.

4.1. *Definitions and notations.* Let (x, y, z) and (x', y', z') be two Cartesian coordinate systems whose origins are O and O', respectively (Fig. 1). In each of them, we consider two other spherical and spheroidal coordinate systems respectively denoted $(r,\theta,\phi), (\eta,\xi,\phi)$ and $(r',\theta',\phi'), (\eta',\xi',\phi')$. The unprimed systems are transformable into the primed systems either by a rotation (with O equal O') through the Euler angles (α,β,γ) as described by Edmonds (Ref. [19], third printing, p. 6) or by a translation defined by $\overrightarrow{OO'} = \mathbf{d}$. In the case of a rotation followed by a translation (rotational-translational addition theorems), it is understood that the first operation (rotation) transforms the unprimed systems into double primed systems and the second operation (translation) transforms the double primed systems into single primed systems. Let ${}^{\cdot}\psi_{mn}^{(i)}$ and $\psi_{mn}^{(i)}$, ${}^{\cdot}\psi'^{(i)}_{mn}$ and $\psi'^{(i)}_{mn}$, ${}^{\cdot}\psi''^{(i)}_{mn}$ and $\psi''^{(i)}_{mn}$ be the mnth spherical and spheroidal scalar wave functions expressed in the unprimed, primed, and double primed spherical and spheroidal coordinate systems, respectively; a similar notation is used for the corresponding vector wave functions $\mathbf{M}^a, \mathbf{N}^a$, and $\mathbf{L}$.

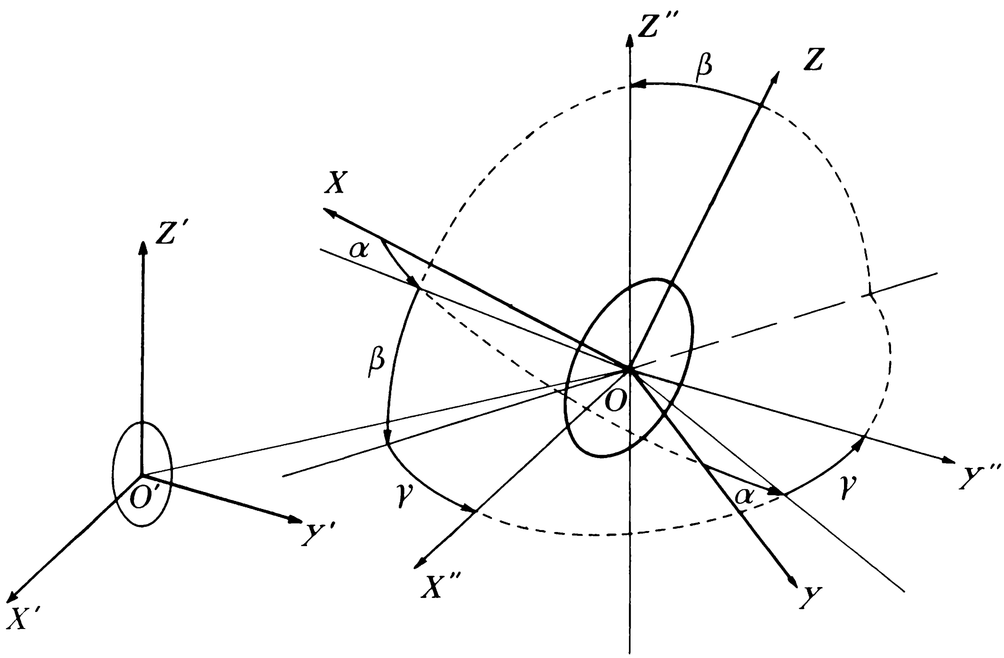

FIG. 1. Geometry of the problem. With the help of the Euler angles (α, β, γ), the system $OXYZ$ is transformed into the system $OX''Y''Z''$, which is parallel to the system $OX'Y'Z'$. The systems are separated by the vector $\mathbf{d} = \overrightarrow{OO'}$.

4.2. *Rotational addition theorems for spheroidal vector wave functions.* According to the notations given in the Appendix of Ref. [2], we have

$$^{\cdot}\psi_{mn}^{(i)}(r,\theta,\phi) = \sum_{m'=-n}^{n} R_{m'n}^{mn}(\alpha,\beta,\gamma)^{\cdot}\psi'^{(i)}_{m'n}(r',\theta',\phi'), \tag{10}$$

where the $R_{m'n}^{mn}(\alpha,\beta,\gamma)$ are related to the matrix elements $d_{m',m}^{(l)}(\beta)$ [19] by

$$R_{m'n}^{mn}(\alpha,\beta,\gamma) = (-1)^{m'-m}\sqrt{^{\cdot}\mathbf{N}_{mn}/^{\cdot}\mathbf{N}_{m'n}}\,e^{jm'\gamma}d_{m',m}^{(l)}(\beta)e^{jm\alpha} \tag{11}$$

(Eq. (A-1) in Ref. [2] and Eq. (A1–3) in Ref. [4] are incorrect; the variables α and γ on the right side of each equation must be interchanged).

Because $r' = r$, it is easy [4] to deduce from Eq. (10) the following vector addition theorem for $^{\cdot}\mathbf{M}$:

$$^{\cdot}\mathbf{M}_{mn}^{(i)r}(r,\theta,\phi) = \sum_{m'=-n}^{n} R_{m'n}^{mn}(\alpha,\beta,\gamma)^{\cdot}\mathbf{M}_{m'n}^{(i)r'}(r',\theta',\phi'). \tag{12}$$

Because the curl is invariant with respect to a rotation, we get

$$^{\cdot}\mathbf{N}_{mn}^{(i)r}(r,\theta,\phi) = \sum_{m'=-n}^{n} R_{m'n}^{mn}(\alpha,\beta,\gamma)^{\cdot}\mathbf{N}_{m'n}^{(i)r'}(r',\theta',\phi'). \tag{13}$$

Now, starting from Eq. (4) in which $\mathbf{a} = \mathbf{r}$ (expressing the mnth spheroidal unprimed vector wave function as a series expansion of msth spherical ones) and replacing each msth spherical vector wave function by means of Eq. (12), we obtain

$$\mathbf{M}_{mn}^{(i)r}(h;\eta,\xi,\phi) = \sum_{s=|m|,|m|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h) \sum_{\mu=-s}^{s} R_{\mu s}^{ms}(\alpha,\beta,\gamma)\,{}^{\cdot}\mathbf{M}_{\mu s}^{(i)r}(r',\theta',\phi'). \tag{14}$$

Then, in view of Eq. (5), the μsth spherical vector wave function is converted into a series expansion of spheroidal vector wave functions $\mathbf{M}_{\mu\nu}^{(i)r'}(h';\eta',\xi',\phi')$. The rearrangement of the order of the summations and their limits allows us to obtain the rotational addition theorem for the spheroidal vector wave function $\mathbf{M}^{(i)r}$:

$$\mathbf{M}_{mn}^{(i)r}(h;\eta,\zeta,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} \overline{R}_{\mu\nu}^{mn}(h,h';\alpha,\beta,\gamma)\mathbf{M}_{\mu\nu}^{(i)r'}(h';\eta',\zeta',\phi') \tag{15}$$

with

$$\overline{R}_{\mu\nu}^{mn}(h,h';\alpha,\beta,\gamma) = \sum_{s=s_0,\,s_0+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)R_{\mu s}^{ms}(\alpha,\beta,\gamma)\Gamma_{\mu\nu}^{\mu s}(h'). \tag{16}$$

The presence of the $d_{s-|m|}^{mn}$ and $d_{l-|\mu|}^{\mu\nu}$ coefficients in $\Lambda_{ms}^{mn}(h)$ and $\Gamma_{l-|\mu|}^{\mu\nu}(h')$, respectively, and the presence of $R_{\mu s}^{mn}(\alpha,\beta,\gamma)$ limit the effective range of s.

The rule of selection of the index s and the definition of s_0 have previously been given by MacPhie, Dalmas, and Deleuil in Ref. [2], p. 741. Briefly, this rule is: $\overline{R}_{\mu\nu}^{mn} = 0$ if and only if $|n-\nu|$ is odd, $s_0 = \mathrm{MAX}(|m|,|\mu|)$ and the lowest value of s becomes s_0+1 if and only if $(n+\nu-s_0)$ is odd. As in the spherical case, and for the same reasons, $\mathbf{N}^{(i)r}(h;\eta,\xi,\phi)$ satisfies a rotational addition theorem obtained by substituting $\mathbf{N}$ for $\mathbf{M}$ into Eq. (15).

Turning now to the Cartesian case ($a = x,y,z$), we use the well-known Euler angles ([19], p. 8) to express the unprimed unit vectors in terms of the primed:

$$\begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{x'} \\ \mathbf{u}_{y'} \\ \mathbf{u}_{z'} \end{bmatrix}. \tag{17}$$

The coefficients $\alpha_i,\beta_i,\gamma_i$ for $i = 1,2,3$ are listed in Appendix B. Since both the gradient and the vector cross-product are invariant with respect to a rotation we have, for $\mathbf{a} = \mathbf{u}_x = \alpha_1\mathbf{u}_{x'} + \beta_1\mathbf{u}_{y'} + \gamma_1\mathbf{u}_{z'}$:

$$\begin{aligned}{}^{\cdot}\mathbf{M}_{mn}^{(i)x}(r,\theta,\phi) = \sum_{\mu=-n}^{n} R_{\mu n}^{mn}(\alpha,\beta,\gamma) \Big\{\alpha_1\,{}^{\cdot}\mathbf{M}_{\mu n}^{(i)x'}(r',\theta',\phi') \\ +\beta_1\,{}^{\cdot}\mathbf{M}_{\mu n}^{(i)y'}(r',\theta',\phi') + \gamma_1\,{}^{\cdot}\mathbf{M}_{\mu n}^{(i)z'}(r',\theta',\phi')\Big\}.\end{aligned} \tag{18}$$

A similar expression is obtained for ${}^{\cdot}\mathbf{N}_{mn}^{(i)x}(r,\theta,\phi)$ by replacing ${}^{\cdot}\mathbf{M}$ by ${}^{\cdot}\mathbf{N}$ in Eq. (18). Moreover, by replacing $(\alpha_1,\beta_1,\gamma_1)$ by $(\alpha_2,\beta_2,\gamma_2)$ and $(\alpha_3,\beta_3,\gamma_3)$ in Eq. (18), one obtains the two other rotational addition theorems for ${}^{\cdot}\mathbf{M}_{mn}^{(i)y}(r,\theta,\phi)$ and ${}^{\cdot}\mathbf{M}_{mn}^{(i)z}(r,\theta,\phi)$, respectively, and similar expressions for ${}^{\cdot}\mathbf{N}_{mn}^{(i)y}(r,\theta,\phi)$ and ${}^{\cdot}\mathbf{N}_{mn}^{(i)z}(r,\theta,\phi)$. This completes the task for the case of *spherical* Cartesian vector wave functions.

For *spheroidal* vector wave functions, one simply uses Eqs. (4), (18), and (5) in succession to obtain, for $\mathbf{a} = \mathbf{u}_x$:

$$\mathbf{M}_{mn}^{(i)x}(h;\eta,\xi,\phi) = \sum_{\mu=-\infty}^{\infty}\sum_{\nu=|\mu|}^{\infty} \overline{R}_{\mu\nu}^{mn}(h,h';\alpha,\beta,\gamma) \cdot \left\{\alpha_1 \mathbf{M}_{\mu\nu}^{(i)x'}(h';\eta',\xi',\phi') + \beta_1 \mathbf{M}_{\mu\nu}^{(i)y'}(h';\eta',\xi',\phi') + \gamma_1 \mathbf{M}_{\mu\nu}^{(i)z'}(h';\eta',\xi',\phi')\right\} \tag{19}$$

in which $\overline{R}_{\mu\nu}^{mn}$ is still given by Eq. (16).

The same substitutions as those mentioned for Eq. (18) allow us to deduce rotational addition theorems for $\mathbf{M}_{mn}^{(i)y}, \mathbf{M}_{mn}^{(i)z}, \mathbf{N}_{mn}^{(i)x}, \mathbf{N}_{mn}^{(i)y}$, and $\mathbf{N}_{mn}^{(i)z}$.

Since the gradient operator is invariant with respect to a rotation, the rotational-addition theorem concerning the *spherical* vector wave function $\mathbf{L}$ is easily deduced from Eq. (10). This yields

$$^{\cdot}\mathbf{L}_{mn}^{(i)}(r,\theta,\phi) = \sum_{\mu=-n}^{n} R_{\mu n}^{mn}(\alpha,\beta,\gamma)^{\cdot}\mathbf{L}'^{(i)}_{\mu n}(r',\theta',\phi') \tag{20}$$

with $r' = r$. Then, starting from Eq. (8), in view of Eqs. (20) and (9) and, just as for $\mathbf{M}$, the rotational-addition theorem for the *spheroidal* vector wave function $\mathbf{L}$ is obtained:

$$\mathbf{L}_{mn}^{(i)}(h;\eta,\zeta,\phi) = \sum_{\mu=-\infty}^{\infty}\sum_{\nu=|\mu|}^{\infty} \overline{R}_{\mu\nu}^{mn}(h,h';\alpha,\beta,\gamma)\mathbf{L}'^{(i)}_{\mu\nu}(h';\eta',\zeta',\phi') \tag{21}$$

in which $\overline{R}_{\mu\nu}^{mn}$ is still given by Eq. (16).

4.3. *Translational addition theorems for spherical and spheroidal vector wave functions.* With the notations used in Ref. [2] and in the case of scalar spherical wave functions, we have

$$^{\cdot}\psi_{mn}^{(i)}(r,\theta,\phi) = \sum_{\mu=-\infty}^{\infty}\sum_{\nu=|\mu|}^{\infty} {}^{(i)}a_{\mu\nu}^{mn}(\mathbf{d})^{\cdot}\psi'^{(1)}_{\mu\nu}(r',\theta',\phi') \tag{22}$$

valid for $r' \leq d$, and

$$^{\cdot}\psi_{mn}^{(i)}(r,\theta,\phi) = \sum_{\overline{\mu}=-\infty}^{\infty}\sum_{p=|\overline{\mu}|}^{\infty} {}^{(1)}b_{\overline{\mu}p}^{mn}(\mathbf{d})^{\cdot}\psi'^{(i)}_{\overline{\mu}p}(r',\theta',\phi') \tag{23}$$

valid for $r' \geq d$.

The expressions for the coefficients ${}^{(i)}a_{\mu\nu}^{mn}$ and ${}^{(1)}b_{\overline{\mu}p}^{mn}$ can be found in Sec. 3 of Ref. [2]. As can be seen, Eq. (22) is identical to the (B-1) form provided by Cruzan [5] but Eq. (23), which differs from the Cruzan (B-2) form and was first given by Danos and Maximon (Ref. [20], Eq. (34)), is more convenient for calculations.

Translational addition theorems for vector wave functions $^{\cdot}\mathbf{M}^r$ and $^{\cdot}\mathbf{M}^r$ have had a direct application in studies on multiple scattering of electromagnetic waves from spheres [24] and afterwards from prolate spheroids in parallel alignment [16, 18].

To be precise, the translational addition theorems of vector wave functions ${}^{\cdot}\mathbf{M}^r$ and ${}^{\cdot}\mathbf{N}^r$ or $\mathbf{M}^r$ and $\mathbf{N}^r$, obtained from Eq. (22) with $i = 3$ or $i = 4$ according to the time-dependence $e^{-j\omega t}$ or $e^{j\omega t}$, transform the field scattered by one scatterer (an outgoing wave) into an incident field (an ingoing wave) on the other scatterers. We note that the *translation* under consideration can have an arbitrary direction.

With the notations used in Refs. [5, 21] and [2], we get, in the case $r' \le d$:

$$ {}^{\cdot}\mathbf{M}^{(i)r}_{mn}(r,\theta,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} \left\{ {}^{(i)}A^{mn}_{\mu\nu}(\mathbf{d}) {}^{\cdot}\mathbf{M}^{(1)r'}_{\mu\nu}(r',\theta',\phi') + {}^{(i)}B^{mn}_{\mu\nu}(\mathbf{d}) {}^{\cdot}\mathbf{N}^{(1)r'}_{\mu\nu}(r',\theta',\phi') \right\} \tag{24} $$

and a similar expression for ${}^{\cdot}\mathbf{N}^{(i)r}$ is deduced from Eq. (24) by substituting $\mathbf{N}$ for $\mathbf{M}$. For calculations of the coefficients ${}^{(i)}A^{mn}_{\mu\nu}$ and ${}^{(i)}B^{mn}_{\mu\nu}$ related to ${}^{(i)}a^{mn}_{\mu\nu}$, the reader is referred to the works by Cruzan [5] and by Dalmas and Deleuil [16].

The case $r' \ge d$ is easily treated by starting from Eq. (23) instead of the form (B-2) of the work by Cruzan [5] and we obtain

$$ {}^{\cdot}\mathbf{M}^{(i)r}_{mn}(r,\theta,\phi) = \sum_{\overline{\mu}=-\infty}^{\infty} \sum_{p=|\overline{\mu}|}^{\infty} \left\{ {}^{(1)}C^{mn}_{\overline{\mu}p}(\mathbf{d}) {}^{\cdot}\mathbf{M}^{(i)r'}_{\overline{\mu}p}(r',\theta',\phi') + {}^{(1)}D^{mn}_{\overline{\mu}p}(\mathbf{d}) {}^{\cdot}\mathbf{N}^{(i)r'}_{\overline{\mu}p}(r',\theta',\phi') \right\}. \tag{25} $$

A similar expression is available for ${}^{\cdot}\mathbf{N}^{(i)r}(r,\theta,\varphi)$; it is deduced from Eq. (25) by simply replacing $\mathbf{M}$ by $\mathbf{N}$ because ${}^{\cdot}\mathbf{N}^r = (1/K)\mathrm{curl}\,{}^{\cdot}\mathbf{M}^r$ is a translationally invariant operator [22]. The coefficients ${}^{(1)}C^{mn}_{\overline{\mu}p}$ and ${}^{(1)}D^{mn}_{\overline{\mu}p}$ are related to the ${}^{(1)}b^{mn}_{\overline{\mu}p}(\mathbf{d})$ just as the coefficients ${}^{(i)}A^{mn}_{\mu\nu}$ and ${}^{(i)}B^{mn}_{\mu\nu}$ are to ${}^{(i)}a^{mn}_{\mu\nu}(\mathbf{d})$. To realize this, it is sufficient to examine the calculations developed in the work by Cruzan [5].

Translational addition theorems for spheroidal vector wave functions $\mathbf{M}^a$ and $\mathbf{N}^a$ have previously been established with $a = x, y, z$ by Sinha and MacPhie [23] and with $a = r$ by Dalmas and Deleuil (Ref. [16], Sec. 2.2 and Ref. [21]).

As a matter of fact, to deduce vector addition theorems from scalar addition theorems involves various difficulties: when a rotation is concerned, the scalar to vectorial way is much easier for $a = r$ than for $a = x, y, z$ (Ref. [4], p. 17) but it is the contrary for a translation (Ref. [23], p. 152 and Refs. [16] and [21]). Simply stated, a rotation preserves the radius vector $\mathbf{r}$ while a translation does not change the three Cartesian unit vectors.

For sake of completeness, the translational-addition theorems concerning the spheroidal vector wave function $\mathbf{L}$ are now established. As the gradient operator is translationally invariant, we immediately get, from Eq. (22),

$$ {}^{\cdot}\mathbf{L}^{(i)}_{mn}(r,\theta,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}^{(i)}a^{mn}_{\mu\nu}(\mathbf{d}) {}^{\cdot}\mathbf{L}'^{(1)}_{\mu\nu};(r',\theta',\phi') \tag{26} $$

valid for $r' \le d$ and, from Eq. (23),

$$ {}^{\cdot}\mathbf{L}^{(i)}_{mn}(r,\theta,\phi) = \sum_{\overline{\mu}=-\infty}^{\infty} \sum_{p=|\overline{\mu}|}^{\infty} {}^{(1)}b^{mn}_{\overline{\mu}p}(\mathbf{d}) {}^{\cdot}\mathbf{L}'^{(i)}_{\overline{\mu}p}(r',\theta',\phi') \tag{27} $$

valid for $r' \ge d$.

Then, starting from Eq. (8) in which each msth spherical vector wave function is translated by means of Eq. (26) or Eq. (27) according to the magnitude of r' with respect to d and, finally in view of Eq. (9), the translational-addition theorems for the spheroidal vector wave function $\mathbf{L}$ are obtained for the case $r' \leq d$:

$$\mathbf{L}^{(i)}(h;\eta,\zeta,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{l=|\mu|}^{\infty} {}^{(i)}U_{\mu l}^{mn}(h,h';\mathbf{d})\mathbf{L}'^{(1)}_{\mu l}(h';\eta',\zeta',\phi') \tag{28}$$

with

$$^{(i)}U_{\mu l}^{mn}(h,h';\mathbf{d}) = \sum_{s=|m|,|m|+1}^{\infty}{}' \sum_{\nu=|\mu|,|\mu|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)^{(i)}a_{\mu\nu}^{ms}(\mathbf{d})\Gamma_{\mu l}^{\mu\nu}(h') \tag{29}$$

and for the case $r' \geq d$:

$$\mathbf{L}_{mn}^{(i)}(h;\eta,\zeta,\phi) = \sum_{\overline{\mu}=-\infty}^{\infty} \sum_{l=|\overline{\mu}|}^{\infty} {}^{(1)}V_{\overline{\mu}l}^{mn}(h,h';\mathbf{d})\, \mathbf{L}'^{(i)}_{\overline{\mu}l}(h';\eta',\zeta',\phi') \tag{30}$$

with

$$^{(1)}V_{\overline{\mu}l}^{mn}(h,h';\mathbf{d}) = \sum_{s=|m|,|m|+1}^{\infty}{}' \sum_{p=|\overline{\mu}|,|\overline{\mu}|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h)^{(1)}b_{\overline{\mu}p}^{ms}(\mathbf{d})\Gamma_{\overline{\mu}l}^{\overline{\mu}p}(h'). \tag{31}$$

In Eqs. (29) or (31), it is to be noted that $|s-n|$, $|\nu-l|$, or $|\nu-p|$ must be even.

4.4. *Rotational-translational addition theorems for spherical and spheroidal vector wave functions.* The case of a rotation (first transformation) followed by a translation (second transformation) is now treated and the above established results are utilized. As previously, the transformations always operate on spherical coordinates. In order to avoid the use of a great number of symbols denoting the coefficients of addition theorems as calculations are going forward, the superscript caret ($\hat{}$) denotes the coefficients of an addition theorem concerning spherical vector wave functions, and the subscript rt indicates that the corresponding transformation is a rotation followed by a translation. Moreover, the parameters upon which the coefficients depend clearly indicate the transformation in question. For example, (α,β,γ) shows a rotation and $\mathbf{d}$ (d,θ_0,ϕ_0) a translation. In addition, the semi-interfocal distances h and h' of the initial and final prolate spheroidal coordinate systems are indicated when they occur.

In view of Eqs. (12) and (13) (spherical rotational theorems) and of Eqs. (24) and (25) (spherical translational theorems), we finally obtain, for $r' \leq d$,

$$\begin{aligned} {}^{\cdot}\mathbf{M}_{mn}^{(i)r}(r,\theta,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} \Big\{ {}_{rt}^{(i)}\hat{A}_{\mu\nu}^{mn}(\alpha,\beta,\gamma;\mathbf{d})^{\cdot}\mathbf{M}_{\mu\nu}^{(1)r'}(r',\theta',\phi') \\ + {}_{rt}^{(i)}\hat{B}_{\mu\nu}^{mn}(\alpha,\beta,\gamma;\mathbf{d})^{\cdot}\mathbf{N}_{\mu\nu}^{(1)r'}(r',\theta',\phi') \Big\} \end{aligned} \tag{32}$$

with

$$\begin{aligned} {}_{rt}^{(i)}\hat{A}_{\mu\nu}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) &= \sum_{\tilde{\mu}=-n}^{n} R_{\tilde{\mu}n}^{mn}(\alpha,\beta,\gamma)^{(i)}A_{\mu\nu}^{\tilde{\mu}n}(\mathbf{d}), \\ {}_{rt}^{(i)}\hat{B}_{\mu\nu}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) &= \sum_{\tilde{\mu}=-n}^{n} R_{\tilde{\mu}n}^{mn}(\alpha,\beta,\gamma)^{(i)}B_{\mu\nu}^{\tilde{\mu}n}(\mathbf{d}), \end{aligned} \tag{33}$$

and for $r' \geq d$,

$$
{}^{\cdot}\mathbf{M}_{mn}^{(i)r}(r,\theta,\phi) = \sum_{\bar{\mu}=-\infty}^{\infty} \sum_{p=|\bar{\mu}|}^{\infty} \left\{ {}_{rt}^{(1)}\hat{C}_{\bar{\mu}p}^{mn}(\alpha,\beta,\gamma;\mathbf{d})\,{}^{\cdot}\mathbf{M}_{\bar{\mu}p}^{(i)r'}(r',\theta',\phi') + {}_{rt}^{(1)}\hat{D}_{\bar{\mu}p}^{mn}(\alpha,\beta,\gamma;\mathbf{d})\,{}^{\cdot}\mathbf{N}_{\bar{\mu}p}^{(i)r'}(r',\theta',\phi') \right\} \tag{34}
$$

with

$$
\begin{aligned}
{}_{rt}^{(1)}\hat{C}_{\bar{\mu}p}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) &= \sum_{\tilde{\mu}=-n}^{n} R_{\tilde{\mu}n}^{mn}(\alpha,\beta,\gamma)\,{}^{(1)}C_{\bar{\mu}p}^{\tilde{\mu}n}(\mathbf{d}),\\
{}_{rt}^{(1)}\hat{D}_{\bar{\mu}p}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) &= \sum_{\tilde{\mu}=-n}^{n} R_{\tilde{\mu}n}^{mn}(\alpha,\beta,\gamma)\,{}^{(1)}D_{\bar{\mu}p}^{\tilde{\mu}n}(\mathbf{d}).
\end{aligned} \tag{35}
$$

Expressions of rotational-translational addition theorems for ${}^{\cdot}\mathbf{N}_{mn}^{(i)r}$ can easily be deduced from those of ${}^{\cdot}\mathbf{M}_{mn}^{(i)r}$ by substituting $\mathbf{N}$ for $\mathbf{M}$ in Eqs. (32) and (34), because both rotation and translation preserve the curl operator.

For the case of ${}^{\cdot}\mathbf{L}_{mn}^{(i)}$, a similar procedure leads to

$$
{}^{\cdot}\mathbf{L}_{mn}^{(i)}(r,\theta,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=|\mu|}^{\infty} {}_{rt}^{(i)}\hat{U}_{\mu\nu}^{mn}(\alpha,\beta,\gamma;\mathbf{d})\,{}^{\cdot}\mathbf{L}'^{(1)}_{\mu\nu}(r',\theta',\phi'), \tag{36}
$$

valid for $r' \leq d$, and

$$
{}^{\cdot}\mathbf{L}_{mn}^{(i)}(r,\theta,\phi) = \sum_{\bar{\mu}=-\infty}^{\infty} \sum_{p=|\bar{\mu}|}^{\infty} {}_{rt}^{(1)}\hat{V}_{\bar{\mu}p}^{mn}(\alpha,\beta,\gamma;\mathbf{d})\,{}^{\cdot}\mathbf{L}'^{(i)}_{\bar{\mu}p}(r',\theta',\phi'), \tag{37}
$$

valid for $r' \geq d$, with

$$
{}_{rt}^{(i)}\hat{U}_{\mu\nu}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) = \sum_{\tilde{\mu}=-n}^{n} R_{\tilde{\mu}n}^{mn}(\alpha,\beta,\gamma)\,{}^{(i)}a_{\mu\nu}^{\tilde{\mu}n}(\mathbf{d}), \tag{38}
$$

$$
{}_{rt}^{(1)}\hat{V}_{\bar{\mu}p}^{mn}(\alpha,\beta,\gamma;\mathbf{d}) = \sum_{\tilde{\mu}=-n}^{n} R_{\tilde{\mu}n}^{mn}(\alpha,\beta,\gamma)\,{}^{(1)}b_{\bar{\mu}p}^{\tilde{\mu}n}(\mathbf{d}). \tag{39}
$$

In the *spheroidal* case, we start from Eq. (14) in which we replace μ by $\tilde{\mu}$ and (r',θ',ϕ') by (r'',θ'',ϕ''). Then, we translate the spherical vector wave function ${}^{\cdot}\mathbf{M}^{(i)r''}$ by means of Eq. (24) or (25), according to the magnitude of r' with respect to d. Finally, we convert spherical into spheroidal vector wave functions by means of Eq. (5). This yields, for the case $r' \leq d$,

$$
\mathbf{M}_{mn}^{(i)r}(h;\eta,\xi,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{l=|\mu|}^{\infty} \left\{ {}_{rt}^{(i)}A_{\mu l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h')\mathbf{M}_{\mu l}^{(1)r'}(h';\eta',\xi',\phi') + {}_{rt}^{(i)}B_{\mu l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h')\mathbf{N}_{\mu l}^{(1)r'}(h';\eta',\xi',\phi') \right\} \tag{40}
$$

with

$$
{}_{rt}^{(i)}A_{\mu l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') = {\sum_{s=|m|,|m|+1}^{\infty}}' \sum_{\tilde{\mu}=-s}^{s} {\sum_{\nu=|\mu|,|\mu|+1}^{\infty}}' \Lambda_{ms}^{mn}(h) \cdot R_{\tilde{\mu}s}^{ms}(\alpha,\beta,\gamma)\,{}^{(i)}A_{\mu\nu}^{\tilde{\mu}s}(\mathbf{d})\Gamma_{\mu l}^{\mu\nu}(h'), \tag{41}
$$

$$\cdot{}_{rt}^{(i)}B_{\mu l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') = \sum_{s=|m|,|m|+1}^{\infty}{}' \sum_{\tilde{\mu}=-s}^{s} \sum_{\nu=|\mu|,|\mu|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h) \cdot R_{\tilde{\mu}s}^{ms}(\alpha,\beta,\gamma)^{(i)}B_{\mu\nu}^{\tilde{\mu}s}(\mathbf{d})\Gamma_{\mu l}^{\mu\nu}(h'), \tag{42}$$

and a similar expression for $\mathbf{N}^{(i)r}$ is deduced from Eq. (40) by substituting $\mathbf{M}$ for $\mathbf{N}$.

For the case $r' \geq d$, we get

$$\mathbf{M}_{mn}^{(i)r}(h;\eta,\xi,\phi) = \sum_{\bar{\mu}=-\infty}^{\infty} \sum_{l=|\bar{\mu}|}^{\infty} \Big\{ {}_{rt}^{(1)}C_{\bar{\mu}l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h')\mathbf{M}_{\bar{\mu}l}^{(i)r'}(h';\eta',\xi',\phi') + {}_{rt}^{(1)}D_{\bar{\mu}l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h')\mathbf{N}_{\bar{\mu}l}^{(i)r'}(h';\eta',\xi',\phi') \Big\} \tag{43}$$

with

$${}_{rt}^{(1)}C_{\bar{\mu}l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') = \sum_{s=|m|,|m|+1}^{\infty}{}' \sum_{\tilde{\mu}=-s}^{s} \sum_{p=|\bar{\mu}|,|\bar{\mu}|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h) \cdot R_{\tilde{\mu}s}^{ms}(\alpha,\beta,\gamma)^{(1)}C_{\bar{\mu}p}^{\tilde{\mu}s}(\mathbf{d})\Gamma_{\bar{\mu}l}^{\bar{\mu}p}(h'), \tag{44}$$

$${}_{rt}^{(1)}D_{\bar{\mu}l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') = \sum_{s=|m|,|m|+1}^{\infty}{}' \sum_{\tilde{\mu}=-s}^{s} \sum_{p=|\bar{\mu}|,|\bar{\mu}|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h) \cdot R_{\tilde{\mu}s}^{ms}(\alpha,\beta,\gamma)^{(1)}D_{\bar{\mu}p}^{\tilde{\mu}s}(\mathbf{d})\Gamma_{\bar{\mu}l}^{\bar{\mu}p}(h'). \tag{45}$$

A similar expression for $\mathbf{N}^{(i)r}$ is deduced from Eq. (45) by substituting $\mathbf{M}$ for $\mathbf{N}$.

A virtually identical sequence of calculations for the cases $a = x, y, z$ permit us to deduce the corresponding rotational-translational addition theorems for both the spherical and spheroidal vector wave functions. For brevity, we present only the final expressions for the spheroidal case with $a = x$:

$$\mathbf{M}_{mn}^{(i)x}(h;\eta,\xi,\phi) = \sum_{\mu=-\infty}^{\infty} \sum_{l=|\mu|}^{\infty} {}_{rt}^{(i)}\tilde{A}_{\mu l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') \cdot \{\alpha_1\mathbf{M}_{\mu l}^{(1)x'}(h';\eta',\xi',\phi') + \beta_1\mathbf{M}_{\mu l}^{(1)y'}(h';\eta',\xi',\phi') + \gamma_1\mathbf{M}_{\mu l}^{(1)z'}(h';\eta',\xi',\gamma')\} \tag{46}$$

if $r' \leq d$, with

$${}_{rt}^{(i)}\tilde{A}_{\mu l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') = \sum_{s=|m|,|m|+1}^{\infty}{}' \sum_{\tilde{\mu}=-s}^{s} \sum_{\nu=|\mu|,|\mu|+1}^{\infty}{}' \Lambda_{ms}^{mn}(h) \cdot R_{\tilde{\mu}s}^{ms}(\alpha,\beta,\gamma)^{(i)}a_{\mu\nu}^{\tilde{\mu}s}(\mathbf{d})\Gamma_{\mu l}^{\mu\nu}(h') \tag{47}$$

and

$$\mathbf{M}_{mn}^{(i)x}(h;\eta,\xi,\phi) = \sum_{\bar{\mu}=-\infty}^{\infty} \sum_{l=|\bar{\mu}|}^{\infty} {}_{rt}^{(i)}\tilde{B}_{\bar{\mu}l}^{mn}(\alpha,\beta,\gamma;\mathbf{d};h,h') \cdot \{\alpha_1\mathbf{M}_{\bar{\mu}l}^{(1)x'}(h';\eta',\xi',\phi') + \beta_1\mathbf{M}_{\bar{\mu}l}^{(1)y'}(h';\eta',\xi',\phi') + \gamma_1\mathbf{M}_{\bar{\mu}l}^{(1)x'}(h';\eta',\xi',\phi')\} \tag{48}$$

if $r' \geq d$, with

$$
{}^{(1)}_{rt}\tilde{B}^{mn}_{\overline{\mu}l}(\alpha, \beta, \gamma; \mathbf{d}; h, h') = \sideset{}{'}\sum_{s=|m|,|m|+1}^{\infty} \sum_{\tilde{\mu}=-s}^{s} \sideset{}{'}\sum_{p=|\overline{\mu}|,|\overline{\mu}|+1}^{\infty} \Lambda^{mn}_{ms}(h) \cdot \tilde{R}^{ms}_{\tilde{\mu}s}(\alpha, \beta, \gamma)^{(i)} b^{\tilde{\mu}s}_{\overline{\mu}p}(\mathbf{d}) \Gamma^{\overline{\mu}p}_{\overline{\mu}l}(h'). \tag{49}
$$

Similar equations, giving $\mathbf{M}^{(i)y}$ and $\mathbf{M}^{(i)z}$ for $r' \leq d$ and $r' \geq d$, must be deduced. For that, in Eqs. (46) and (48), $(\alpha_1, \beta_1, \gamma_1)$ is respectively replaced by $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$. Three other equations, giving $\mathbf{N}^{(i)x}, \mathbf{N}^{(i)y}$, and $\mathbf{N}^{(i)z}$, are also obtained when $\mathbf{M}$ is replaced by $\mathbf{N}$ in the equations giving $\mathbf{M}^{(i)x}, \mathbf{M}^{(i)y}$, and $\mathbf{M}^{(i)z}$.

Finally, in order to be complete, the rotational-translational addition theorems for the vector wave function $\mathbf{L}$ are established by means of Eqs. (8), (20), (26) or (27), and (9). They can be written, for $r' \leq d$,

$$
\mathbf{L}^{(i)}_{mn}(h; \eta, \xi, \phi) = \sum_{\mu=-\infty}^{\infty} \sum_{l=|\mu|}^{\infty} {}^{(i)}_{rt}\mathit{tildeU}^{mn}_{\mu l}(\alpha, \beta, \gamma; \mathbf{d}; h, h') \mathbf{L}'^{(1)}_{\mu l}(h'; \eta', \xi', \phi') \tag{50}
$$

and, for $r' \geq d$,

$$
\mathbf{L}^{(i)}_{mn}(h; \eta, \xi, \phi) = \sum_{\overline{\mu}=-\infty}^{\infty} \sum_{l=|\overline{\mu}|}^{\infty} {}^{(1)}_{rt}\tilde{V}^{mn}_{\overline{\mu}l}(\alpha, \beta, \gamma; \mathbf{d}; h, h') \mathbf{L}'^{(i)}_{\overline{\mu}l}(h'; \eta', \xi', \phi') \tag{51}
$$

with

$$
{}^{(i)}_{rt}\tilde{U}^{mn}_{\mu l}(\alpha, \beta, \gamma; \mathbf{d}; h, h') = \sideset{}{'}\sum_{s=|m|,|m|+1}^{\infty} \sum_{\tilde{\mu}=-s}^{s} \sideset{}{'}\sum_{\nu=|\mu|,|\mu|+1}^{\infty} \Lambda^{mn}_{ms}(h) \cdot R^{ms}_{\tilde{\mu}s}(\alpha, \beta, \gamma)^{(i)} a^{\tilde{\mu}s}_{\mu\nu}(\mathbf{d}) \Gamma^{\mu\nu}_{\mu l}(h') \tag{52}
$$

and

$$
{}^{(1)}_{rt}\tilde{V}^{mn}_{\overline{\mu}l}(\alpha, \beta, \gamma; \mathbf{d}; h, h') = \sideset{}{'}\sum_{s=|m|,|m|+1}^{\infty} \sum_{\tilde{\mu}=s}^{s} \sideset{}{'}\sum_{p=|\overline{\mu}|,|\overline{\mu}|+1}^{\infty} \Lambda^{mn}_{ms}(h) \cdot R^{ms}_{\tilde{\mu}s}(\alpha, \beta, \gamma)^{(1)} b^{\tilde{\mu}s}_{\overline{\mu}p}(\mathbf{d}) \Gamma^{\overline{\mu}p}_{\overline{\mu}l}(h'). \tag{53}
$$

As in Eq. (29) or in Eq. (31), $|s-n|$, $|\nu - l|$, or $|\nu - p|$ must be even.

5. Conclusion. These expressions of rotational-translational addition theorems for spheroidal vector wave functions make it possible to theoretically treat the multiple scattering of electromagnetic waves from two (or more) prolate spheroids with various orientations of their axes of revolution and various eccentricities. Numerical investigations of various configurations of practical interest can then be performed. Initial results have been recently presented [25] in the particular case of two identical prolate spheroids having their axes of revolution mutually perpendicular. To support this numerical implementation, experiments on a microwave range will also be attempted.

Appendix A. The noteworthy vectorial properties, satisfied by the radius vector $\mathbf{r}$, are the following (Ref. [7], Appendix II and Ref. [24]):

$$\begin{aligned}
&\operatorname{curl}\mathbf{r} = \mathbf{0},\\
&\operatorname{div}\mathbf{r} = 3,\\
&(\operatorname{grad}\psi\cdot\operatorname{grad})\mathbf{r} = \operatorname{grad}\psi,\\
&\operatorname{curl}(\mathbf{r}\psi) = \operatorname{grad}\psi\times\mathbf{r},\\
&\operatorname{curl}\operatorname{curl}(\mathbf{r}\psi) = \operatorname{grad}(\mathbf{r}\cdot\operatorname{grad}\psi+\psi) - \mathbf{r}\Delta\psi,\\
&\operatorname{curl}\operatorname{curl}\operatorname{curl}(\mathbf{r}\psi) = -\operatorname{curl}(\mathbf{r}\Delta\psi).
\end{aligned}$$

Now, let $\mathbf{u}$ be one of the three Cartesian unit vectors. The vectorial properties, satisfied by $\mathbf{u}$, are

$$\begin{aligned}
&\operatorname{curl}\mathbf{u} = \mathbf{0},\\
&\operatorname{div}\mathbf{u} = 0,\\
&(\operatorname{grad}\psi\cdot\operatorname{grad})\mathbf{u} = \mathbf{0},\\
&\operatorname{curl}(\mathbf{u}\psi) = \operatorname{grad}\psi\times\mathbf{u},\\
&\operatorname{curl}\operatorname{curl}(\mathbf{u}\psi) = \operatorname{grad}(\mathbf{u}\cdot\operatorname{grad}\psi) - \mathbf{u}\Delta\psi,\\
&\operatorname{curl}\operatorname{curl}\operatorname{curl}(\mathbf{u}\psi) = -\operatorname{curl}(\mathbf{u}\Delta\psi).
\end{aligned}$$

The similitude of vectorial properties between $\mathbf{r}$ and $\mathbf{u}$ must be noted. In addition, if $\mathbf{a}$ denotes either $\mathbf{r}$ or $\mathbf{u}$, we have

$$\operatorname{curl}\operatorname{curl}\mathbf{M}^a = -\operatorname{curl}(\mathbf{a}\Delta\psi) \quad \text{with} \quad \mathbf{M}^a = \operatorname{curl}(\mathbf{a}\psi).$$

Finally, it may be seen that $\mathbf{M}^a$ satisfies the vector wave equation $\operatorname{curl}\operatorname{curl}\mathbf{M}^a - K^2\mathbf{M}^a = \mathbf{0}$, provided that the scalar function ψ is a solution to the scalar wave equation $\Delta\psi + K^2\psi = 0$. It is tacitly understood that the medium is isotropic and homogeneous.

Appendix B. The elements of the rotation matrix, appearing in Eq. (16), are given as functions of the Euler angles (α,β,γ) according to the following equations:

$$\begin{aligned}
\alpha_1 &= \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma,\\
\alpha_2 &= \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma,\\
\alpha_3 &= -\sin\beta\cos\gamma,\\
\beta_1 &= -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma,\\
\beta_2 &= -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\beta,\\
\beta_3 &= \sin\beta\sin\gamma,\\
\gamma_1 &= \cos\alpha\sin\beta,\\
\gamma_2 &= \sin\alpha\sin\beta,\\
\gamma_3 &= \cos\beta.
\end{aligned}$$

These elements are independent of the space coordinates.

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