LYAPUNOV EXPONENTS
FOR DISCONTINUOUS DIFFERENTIAL EQUATIONS

By

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Abstract. The vector field associated with a dynamical system is assumed to be piecewise continuously differentiable. The gradient of the vector field, entering into integral expressions for the Lyapunov exponents, may therefore contain derivatives of step functions. Results from the theory of distributions are used in the integrals' evaluation.

Introduction. Differential equations have been previously derived which describe the evolution of vectors, area tensors, and Lyapunov exponents along a trajectory. These equations have been integrated by various means to give the spectrum of Lyapunov exponents, L. E., [6,10]. For discontinuous differential equations, D. D. E., $\exists$ a set of points, $c \in \mathbb{R}$, for which the integrands of the evolution equations, gradients of the vector field, will contain distributions. It is assumed that each point of the set $c_i \in (a_i, b_i)$, where the intervals are disjoint. Integrals for the evolution of vectors, phase space volume, and L. E. are evaluated over the intervals $[a_i, b_i]$. The integration methods of [6,10] are applicable to the complement of $(a_i, b_i)$.

D.D.E. occur in a variety of applications including relay control systems [2,3] and the oscillation of mechanical systems with Coulomb friction. Such systems may exhibit intermittent motion. A numerical study indicates the utility of the derived algorithms.

Lyapunov exponents. Consider a dynamical system

$$\dot{x} = f(x)$$

(1)

where $x$ are spatial curvilinear coordinates in an $n$-dimensional Euclidean phase space, $E_n$, and $f(x)$ is piecewise continuously differentiable. Under general conditions the solution of (1),

$$x = x(X)$$

(2)

exists, is unique and continuous, [1,3], where $X$ are material coordinates.

A subspace, $V_m$, of dimension $m \leq n$ may be defined by the parametric equations $X = X(u)$ in material coordinates and after deformation, (2), by $x = x(u)$ in spatial coordinates, where $u$ has $m$ components. Let $da_{(m)}$ and $dA_{(m)}$ be spatial and
material area elements of \( V_m \), respectively. The \( m \)-dimensional L.E. associated with a trajectory originating at \( X \) is defined as, \cite{4,5},

\[
\chi(X, V_m) = \lim_{t \to \infty} \sup (\frac{\ln (dA_m)}{dA_n}) / q(t)
\]  

where \( q(t) \equiv t \). The one-dimensional L.E., \( m = 1 \), for the arc, \( ds \), with tangent, \( n \), associated with a trajectory and having initial values, respectively, of \( dS \), \( N \), and \( X \) is given by

\[
\chi(X, ds) = \lim_{t \to \infty} (\frac{\ln \lambda_n}{q(t)})
\]  

where stretch \( \lambda_n \equiv ds/dS \). Since \( D(\ln \lambda_n)/Dt = d_k n_k n_1 \), \( |n| = 1 \), it follows that (4) may be expressed as

\[
\chi(X, ds) = \lim_{t \to \infty} (1/q(t)) \int_0^t d_k n_k n_1 dt
\]  

where \( d_k \equiv (v_{k,1} + v_{1,k})/2 \) and \( v_k \equiv \dot{x}_k \).

The material derivative of a vector, \( n \), tangent to an arc \( ds \) is given by, \cite{6,7},

\[
\frac{D(n_k)}{Dt} = v_k
\]  

and therefore

\[
n_k(t) = n_k(0) + \int_0^t v_{k,1} n_1 dt.
\]  

In \cite{4,5,6} an algorithm is given for the computation of \( \chi_{(\alpha)}(X, ds_n) \), \( \alpha = 1, \ldots, n \). An arbitrary set of initial orthogonal unit vectors, \( \{n_1, \ldots, n_n\} \), is evolved through the numerical solution of (1) and (6). At each time step the vectors are orthonormalized. \( \chi_{(\alpha)}(X, ds_n) \) is then given by (5). In \cite{4,6} it is shown that

\[
\sum \chi_i(X, ds_i) = \chi(X, V_m), \ i = 1, m,
\]  

and that

\[
\chi(X, V_n) = \lim_{t \to \infty} (1/q(t)) \int_0^t v_{k,1} n_1 dt.
\]  

D.D.E. Since \( f(x) \) in (1) is piecewise continuously differentiable, the gradient, \( v_{i,j} \) may contain derivatives of step functions. The following results are useful in evaluating the integrals of such \( v_{i,j} \) as they occur in (5, 6, 7), \cite{8,9}: If, in some neighborhood of \( t = c \), \( g(t) \) is continuous, \( x(t) \) is continuous and monotonic, \( x(c) = 0 \), and \( \dot{x}(c^+) \) and does not vanish then

\[
\int_0^t g(t) \frac{\partial u(x(t))}{\partial x} dt = g(c)|\dot{x}(c^+)|
\]  

where \( u(x) = 1 \) for \( x > 0 \), \( u(x) = 0 \) for \( x < 0 \), and \( 0 < c < t \). Since \( \text{sgn}(x) = u(x) - u(-x) \) it follows from (8) that

\[
\int_0^t g(t) \frac{\partial \text{sgn}(x(t))}{\partial x} dt = g(c)(1/|\dot{x}(c^-)| + 1/|\dot{x}(c^+)|).
\]  

Asymmetric relay elements and relays with hysteresis and dead zones may be modeled by sums of \( u(x) \) and \( \text{sgn}(x) \), \cite{2}. The function describing the behavior of a relay with a dead zone is given by \((\text{sgn}(x-x_0) + \text{sgn}(x+x_0))/2 \) which = 1, 0, -1 for \( x > x_0 \), \( |x| < x_0 \), and \( x < -x_0 \), respectively.
An example will indicate the general application of (8) and (9). Consider an oscillator with Coulomb damping:

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 x_1 - F \text{sgn}(x_2) + f(x_3) \\
\dot{x}_3 &= 1
\end{align*} \] (10-12)

where the constant coefficients, \(k_1\) and \(F\), are \(\geq 0\). Then \(v_{1,1} = 0, \ v_{1,2} = 1, \ v_{1,3} = 0, \ v_{2,1} = -k_1, \ v_{2,2} = -F \partial \text{sgn}(x_2)/\partial x_2, \ v_{2,3} = \partial f/\partial x_3, \ v_{3,1} = 0, \ v_{3,2} = 0, \) and \(v_{3,3} = 0\). (7) and (9) imply that

\[ \chi(X, V_n) = \lim_{t \to \infty} \left( -F/t \right) \sum_i \left( 1/|\dot{x}_2(c^-_i)| + 1/|\dot{x}_2(c^+_i)| \right) \] (13)

where \(x_2(c_i) = 0\). In an interval, \(a \leq t \leq b\), with \(k = 1\), (6) implies

\[ n_i(b) - (h/2)n_i(a) \simeq n_i(a) - (h/2)n_i(a) \] (14)

where \(k = b - a\) and the trapezoidal integration approximation has been applied. For \(k = 3\) it follows from (6) that \(n_3(b) = n_3(a)\). Since the initial values of \(n_i\) are arbitrary, let \(n_3 = 0\) for \(0 \leq t \leq 1\). It then follows from (6) with \(k = 2\) that

\[ (k_1 h/2)n_1(b) + (1 + F/|\dot{x}_2(b)|)n_2(b) \simeq -(k_1 h/2)n_1(a) + (1 - F/|\dot{x}_2(a)|)n_2(a) \] (15)

where \(a < c < b\), \(\dot{x}_2(c^-) \simeq \dot{x}_2(a)\), and \(\dot{x}_2(c^+) \simeq \dot{x}_2(b)\). (14) and (15) determine \(n(b), \ An = d, \) and \(\det |A| > 1\).

Integration of \(D(D\lambda_n)/Dt\), (4), and (5) for \(a \leq t \leq b\) gives

\[ \ln \lambda_n(b) = \ln \lambda_n(a) + (h(1 - k_1)/2)(n_1(b)n_2(b) + n_1(a)n_2(a)) = F(n_2^2(b)/|\dot{x}_2(b)| + n_2^2(a)/|\dot{x}_2(a)|). \] (16)

If \(h \ll 1\) and \(k_1 h \ll 1\) then terms involving products with \(h\) may be omitted in (14), (15), and (16).

Solutions of D.D.E. may be intermittent in the sense that for some time interval, \(\alpha < t < \beta\), components of \(x = 0\). If the system (10), (11), and (12) stops at \(t = \alpha\) then \(\dot{x}_1(\alpha) = 0, \ \dot{x}_2(\alpha^-) \neq 0, \ \dot{x}_2(\alpha^+) = 0,\) and \(x_3(t) \neq 0\). In (9) the term containing \(|\dot{x}(c^+)|\) is therefore omitted.

Computation of L.E. The oscillator with Coulomb damping is subsequently used as a test case. Equations (10), (11), and (12) were solved simultaneously assuming \(f(x_3) \equiv 0, \ F = 5.886, \ k_1 = 60, \ x_1 = 40,\) and \(x_2 = 0\) at \(x_3 = 0\). \(\chi(X, V_2)\), (7), was evolved through (13) and is shown in Fig. 2 for \(0 \leq t \leq 50\). Two vectors, \(n^1\) and \(n^2\), were evolved with (6) and in intervals containing points \(c_i\) by (14) and (15). These were orthonormalized after each time step. The evolution of \(\chi_1\) and \(\chi_2\) associated with \(n^1\) and \(n^2\) was realized through (5) and (16). \(\chi_1\) and \(\chi_2\) are shown in Fig. 1 for \(0 \leq t \leq 50\). The numerical error associated with the equality \(\chi(X, V_2) = \chi_1(X, ds^1) + \chi_2(X, ds^2)\) was \(< 1\%\). Since this system stops, it follows that 0 is the asymptotic limit of \(\chi_1, \ \chi_2,\) and \(\chi\) which is approached as \(1/t\).
Conclusions. Integral expressions for L.E. appear to be particularly appropriate for vector fields with gradients containing distributions. An approximation of (7) was studied for (10), (11), and (12), in which sgn(\dot{x}) was approximated by \( a(\dot{x}) = +1 \).
\[ \dot{x} > \varepsilon, \quad a(x) = -1, \quad \dot{x} < \varepsilon, \quad \text{and} \quad a(x) = \varepsilon \dot{x} \quad \text{for} \quad |\dot{x}| \leq 1. \]

The results were in very close agreement with those of (13) for small \( \varepsilon \).

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**References**


