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FINITELY CONDUCTING ALIGNED PLANE MAGNETOHYDRODYNAMIC CONFLUENT FLOWS

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1. Introduction. M. H. Martin [1] developed an equivalent alternate formulation of the Navier-Stokes equations for steady plane flow of an incompressible fluid having vorticity, kinetic energy and pressure functions as the dependent variables when the streamlines and an arbitrary family of curves generate a coordinate net. The motivation for this new formulation was to investigate some of the fifteen confluent flows suggested by him. One of these confluent flows, called Prandtl-Meyer flows, was studied by Martin [2] in one of his earlier works. In this paper, we study the flow of electrically conducting fluids of finite electrical conductivity by following Martin's and Govindaraju's approach. Since the magnetic Reynolds number is very small for most liquid metals and it is also small, or at most of the order of unity, for usual (nonsuperconducting) electrically conducting fluids to which magnetohydrodynamic (MHD) (single fluid model) approximation can be applied, our accounting for finite electrical conductivity makes the flow problem realistic and attractive from both a mathematical and physical point of view.

The plan of this paper is as follows: in Sec. 2, we write the equations governing finitely conducting aligned MHD flows in a suitable form. In Sec. 3, we employ some results from differential geometry to recast these equations when the streamlines and an arbitrary family of curves generate a coordinate net. In this section, we also develop an equivalent alternate formulation of the flow equations given in Sec. 2. This new formulated system of equations, called the bar-system, has vorticity, kinetic energy, pressure and proportionality function as the dependent variables. The derivation of this system was achieved following Martin's work. In the next three sections we follow the works of Govindaraju [3, 4] and study the following three confluent flows:

1. Isoenergetic lines and isovels coincide with the streamlines in the physical plane.
2. Isocurls coincide with streamlines in the physical plane.
3. Isobars coincide with streamlines in the physical plane.

In some of the above confluent flows we find the results which are analogous to viscous

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non-MHD fluids while in other cases we find viscous non-MHD results cannot be carried through to the MHD-aligned fluid flows.

2. Flow equations. The steady plane flow of an incompressible and electrically conducting fluid of finite electrical conductivity is governed by [5]

$$\begin{gathered}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \\
\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial p}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\mu^* H_2\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right) \\
\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{\partial p}{\partial y}=\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\mu^* H_1\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right) \\
uH_2-vH_1-\frac{1}{\mu^*\sigma}\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right)=c \\
\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0
\end{gathered} \tag{2.1}$$

where $\mathbf{V}=(u(x,y),v(x,y),0)$ is the velocity vector field, $\mathbf{H}=(H_1(x,y),H_2(x,y),0)$ the magnetic field vector and $p(x,y)$ the fluid pressure function. ρ, μ, μ^*, and σ are respectively the constant fluid density, the constant coefficient of viscosity, the constant magnetic permeability and the constant electrical conductivity. The constant c, in the fourth equation in system (2.1), is an arbitrary constant of integration obtained by integrating the two component equations of the vector diffusion equation.

$$\operatorname{curl}(\mathbf{V}\times\mathbf{H})-\frac{1}{\sigma\mu^*}\operatorname{curl}(\operatorname{curl}\mathbf{H})=\mathbf{0}.$$

Taking the flow to be aligned so that the velocity and magnetic fields are everywhere parallel, we have

$$\mathbf{H}=f\mathbf{V} \tag{2.2}$$

where $f(x,y)$ is some scalar function.

Employing (2.2) and introducing ω, Ω, and h given by

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},\qquad \Omega=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y},\qquad h=\frac{1}{2}\rho q^2+p+\frac{1}{2}\mu^* H^2 \tag{2.3}$$

where $q=\sqrt{u^2+v^2}$ is the speed and $H=\sqrt{H_1^2+H_2^2}$ the magnetic intensity magnitude, the governing system (2.1) can be replaced by the following system of

equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(continuity)} \tag{2.4}$$

$$\frac{\partial h}{\partial x} + \mu\frac{\partial \omega}{\partial y} - \rho v\omega + \mu^* f v\Omega = \frac{1}{2}\mu^*\frac{\partial}{\partial x}(f^2q^2) \quad \text{(linear momentum)} \tag{2.5}$$

$$-\frac{\partial h}{\partial y} + \mu\frac{\partial \omega}{\partial x} - \rho u\omega + \mu^* f u\Omega = -\frac{1}{2}\mu^*\frac{\partial}{\partial y}(f^2q^2) \tag{2.6}$$

$$\Omega = \mu^*\sigma c = \Omega_0 \quad \text{(say) (diffusion)} \tag{2.7}$$

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0 \quad \text{(solenoidal)} \tag{2.8}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{(vorticity)} \tag{2.9}$$

$$\Omega = f\omega + v\frac{\partial f}{\partial x} - u\frac{\partial f}{\partial y} \quad \text{(current density)} \tag{2.10}$$

of seven equations for u, v, h, ω, Ω, and f as functions of x, y. The advantage of the new system lies in the reduction of order from two to one, and in getting this system we have followed M. H. Martin [1].

3. Alternate forms of equations of motion. The equation of continuity (2.4) implies the existence of the streamfunction $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \qquad \frac{\partial \psi}{\partial y} = u. \tag{3.1}$$

We take $\phi(x, y) =$ constant to be some arbitrary family of curves which generates with the streamlines $\psi(x, y) =$ constant a curvilinear net so that in the physical plane the independent variables x, y can be replaced by ϕ, ψ.

Let

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi) \tag{3.2}$$

define a curvilinear net in the (x, y)-plane with the squared element of arc length along any curve given by

$$ds^2 = E(\phi, \psi)\,d\phi^2 + 2F(\phi, \psi)\,d\phi\,d\psi + G(\phi, \psi)\,d\psi^2 \tag{3.3}$$

where

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2, \qquad F = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi}$$
$$G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2. \tag{3.4}$$

Equations (3.2) can be solved to obtain $\phi = \phi(x, y)$, $\psi = \psi(x, y)$ such that

$$\frac{\partial x}{\partial \phi} = J\frac{\partial \psi}{\partial y}, \qquad \frac{\partial x}{\partial \psi} = -J\frac{\partial \phi}{\partial y}, \qquad \frac{\partial y}{\partial \phi} = -J\frac{\partial \phi}{\partial x}, \qquad \frac{\partial y}{\partial \psi} = J\frac{\partial \phi}{\partial x} \tag{3.5}$$

provided $0 < |J| < \infty$, where J is the transformation Jacobian, and

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi} = \pm\sqrt{EG - F^2} = \pm W \quad \text{(say)}.$$

Denoting by α the local angle of inclination of the tangent to the coordinate line $\psi = \text{constant}$, directed in the sense of increasing ϕ, we have from differential geometry the following (c.f. Martin [1]):

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= \sqrt{E}\cos\alpha, & \frac{\partial y}{\partial \phi} &= \sqrt{E}\sin\alpha, \\ \frac{\partial x}{\partial \psi} &= \frac{F}{\sqrt{E}}\cos\alpha - \frac{J}{\sqrt{E}}\sin\alpha, & \frac{\partial y}{\partial \psi} &= \frac{F}{\sqrt{E}}\sin\alpha + \frac{J}{\sqrt{E}}\cos\alpha \end{aligned} \tag{3.6}$$

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E}\Gamma_{11}^2, \qquad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E}\Gamma_{12}^2 \tag{3.7}$$

$$K = \frac{1}{W}\left[\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right)\right] = 0 \tag{3.8}$$

$$\frac{\partial}{\partial \phi}\left(\frac{E}{2W^2}\right) = \frac{1}{W^2}[F\Gamma_{11}^2 - E\Gamma_{12}^2] \tag{3.9}$$

$$\frac{\partial}{\partial \psi}\left(\frac{E}{2W^2}\right) = \frac{1}{W^2}[F\Gamma_{12}^2 - E\Gamma_{22}^2] \tag{3.10}$$

$$\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right) = \frac{1}{W}[G\Gamma_{11}^2 - 2F\Gamma_{12}^2 + E\Gamma_{22}^2] \tag{3.11}$$

where

$$\begin{aligned} \Gamma_{11}^2 &= \frac{1}{2W^2}\left[-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi}\right]. \\ \Gamma_{12}^2 &= \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \phi} - F\frac{\partial E}{\partial \psi}\right] \\ \Gamma_{22}^2 &= \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \psi} - 2F\frac{\partial F}{\partial \psi} + F\frac{\partial G}{\partial \phi}\right] \end{aligned} \tag{3.12}$$

and K is the Gaussian curvature.

Having recorded the above results, we follow and employ M. H. Martin's [1] excellent pioneer work and transform Eqs. (2.4) to (2.10) governing our flow into new forms in the new variables ϕ, ψ.

First form.

Equations of continuity and vorticity. Martin [1] has obtained the necessary and sufficient condition for the flow of a fluid, along the coordinate lines $\psi = \text{constant}$ of a curvilinear coordinate system (3.2) with ds^2 given by (3.3) to satisfy the principle of conservation of mass to be

$$Wq = \sqrt{E}, \qquad u + iv = \frac{\sqrt{E}}{W}e^{i\alpha}. \tag{3.13}$$

He has also proven that

$$\omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right]. \tag{3.14}$$

In this work, we consider that the fluid flows towards higher parameter values of ϕ so that $J = W > 0$.

Linear momentum equations. On employing (3.1) in the linear momentum equations (2.5) and (2.6) and making use of (3.5) and (2.7), we have

$$\frac{\partial h}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial h}{\partial \psi}\frac{\partial y}{\partial \phi} + \mu\left(-\frac{\partial \omega}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial \omega}{\partial \psi}\frac{\partial x}{\partial \phi}\right) - (\rho\omega - \mu^* f\Omega_0)\frac{\partial y}{\partial \phi}$$
$$= \frac{1}{2}\mu^*\left[\frac{\partial(f^2q^2)}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial(f^2q^2)}{\partial \psi}\frac{\partial y}{\partial \phi}\right] \tag{3.15}$$

$$\frac{\partial h}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial h}{\partial \psi}\frac{\partial x}{\partial \phi} + \mu\left(\frac{\partial \omega}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial \omega}{\partial \psi}\frac{\partial y}{\partial \phi}\right) - (\rho\omega - \mu^* f\Omega_0)\frac{\partial x}{\partial \phi}$$
$$= \frac{1}{2}\mu^*\left[\frac{\partial(f^2q^2)}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial(f^2q^2)}{\partial \psi}\frac{\partial x}{\partial \phi}\right]. \tag{3.16}$$

Multiplying (3.15) by $\partial y/\partial \psi$, (3.16) by $\partial x/\partial \psi$ and adding, we get

$$G\left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right] - F\left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right]$$
$$= -\mu J\frac{\partial \omega}{\partial \psi}.$$

Likewise, multiplying (3.15) by $\partial y/\partial \phi$, (3.16) by $\partial x/\partial \phi$ and adding, we have

$$-F\left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right] + E\left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right]$$
$$= \mu J\frac{\partial \omega}{\partial \phi}.$$

Solenoidal equation. Using (3.1) in Eq. (2.8), we get

$$\frac{\partial \psi}{\partial y}\left[\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial x}\right] - \frac{\partial \psi}{\partial x}\left[\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial y}\right] = 0$$

which yields that

$$\frac{\partial f}{\partial \phi} = 0 \quad \text{or} \quad f = f(\psi).$$

Current density equation. Employing (3.1) and (2.7) in Eq. (2.10), transforming to (ϕ, ψ)-net and using (3.5), we have

$$\Omega_0 = f\omega + \frac{F}{J^2}\frac{\partial f}{\partial \phi} - \frac{E}{J^2}\frac{\partial f}{\partial \psi}.$$

Summing up, we have

THEOREM I. If the streamlines $\psi(x, y) =$ constant and an arbitrary family of curves $\phi(x, y) =$ constant generate a curvilinear net in the physical plane of a viscous incompressible and electrically conducting fluid, then fluid flow under study in independent variables ϕ, ψ is governed by the system

$$q = \frac{\sqrt{E}}{J} \tag{3.17}$$

$$G\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right]-F\left[\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right] = -\mu J\frac{\partial \omega}{\partial \psi} \tag{3.18}$$

$$-F\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right]+E\left[\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right] = \mu J\frac{\partial \omega}{\partial \phi} \tag{3.19}$$

$$\frac{\partial f}{\partial \phi}=0 \tag{3.20}$$

$$\omega=\frac{1}{J}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{J}\right)-\frac{\partial}{\partial \psi}\left(\frac{E}{J}\right)\right] \tag{3.21}$$

$$\Omega_0=f\omega+\frac{F}{J^2}\frac{\partial f}{\partial \phi}-\frac{E}{J^2}\frac{\partial f}{\partial \psi} \tag{3.22}$$

$$\frac{\partial}{\partial \psi}\left(\frac{J}{E}\Gamma^2_{11}\right)-\frac{\partial}{\partial \phi}\left(\frac{J}{E}\Gamma^2_{12}\right)=0 \tag{3.23}$$

of seven partial differential equations in seven unknowns E, F, G, ω, h, f, and q as functions of ϕ, ψ. Furthermore, having determined a solution of Eqs. (3.17) to (3.23), we can find x, y as functions of ϕ, ψ from

$$z=x+iy=\int \frac{e^{i\alpha}}{\sqrt{E}}[E\,d\phi+(F+iJ)\,d\psi] \tag{3.24}$$

where

$$\alpha=\int \frac{J}{E}[\Gamma^2_{11}\,d\phi+\Gamma^2_{12}\,d\psi] \tag{3.25}$$

and u and v by

$$u+iv=\frac{\sqrt{E}}{J}e^{i\alpha}. \tag{3.26}$$

Given a solution of above system, H_1, H_2, and p as functions of ϕ and ψ are determined from Eqs. (2.2) and (2.3).

Second form (bar-system). The development of the bar-system for our fluid flow in this subsection follows Martin's [1] derivation of the bar-system for the Navier-Stokes equations.

We consider one more equivalent form of the momentum equations (3.18) and (3.19). We multiply (3.18) by E, (3.19) by F and add to obtain an equation which along with (3.19) forms the set

$$F\mu\frac{\partial \omega}{\partial \phi}-E\mu\frac{\partial \omega}{\partial \psi}=J\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right] \tag{3.27}$$

$$-F\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right]+E\left[\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right]$$
$$=\mu J\frac{\partial \omega}{\partial \phi}. \tag{3.28}$$

On solving these equations for E and F, we find that

$$E\left[\mu\frac{\partial \omega}{\partial \phi}\left\{\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right\}-\mu\frac{\partial \omega}{\partial \psi}\left\{\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right\}\right]$$
$$=J\left[\mu^2\left(\frac{\partial \omega}{\partial \phi}\right)^2+\left\{\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right\}^2\right] \tag{3.29}$$

$$F\left[\mu\frac{\partial \omega}{\partial \phi}\left\{\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right\}-\mu\frac{\partial \omega}{\partial \psi}\left\{\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right\}\right]$$
$$=J\left[\mu^2\frac{\partial \omega}{\partial \phi}\frac{\partial \omega}{\partial \psi}+\left\{\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right\}\right.$$
$$\left.\times\left\{\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right\}\right]. \tag{3.30}$$

Defining

$$\tilde{E}=\mu^2\left(\frac{\partial \omega}{\partial \phi}\right)^2+\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right]^2$$

$$\tilde{F}=\mu^2\frac{\partial \omega}{\partial \phi}\frac{\partial \omega}{\partial \psi}+\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right]$$
$$\times\left[\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right]$$

$$\tilde{J}=\mu\frac{\partial \omega}{\partial \phi}\left[\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right]$$
$$-\mu\frac{\partial \omega}{\partial \phi}\left[\frac{\partial h}{\partial \phi}-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \phi}\right] \tag{3.31}$$

$$\tilde{G}=\mu^2\left(\frac{\partial \omega}{\partial \psi}\right)^2+\left[\frac{\partial h}{\partial \psi}+\rho\omega-\mu^* f\Omega_0-\frac{1}{2}\mu^*\frac{\partial(f^2q^2)}{\partial \psi}\right]^2,$$

and

$$\widetilde{W}=\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}$$

we find that Eqs. (2.29) and (2.30) can be written as

$$\frac{E}{J}=\frac{\tilde{E}}{\tilde{J}},\qquad \frac{F}{J}=\frac{\tilde{F}}{\tilde{J}} \tag{3.32}$$

and

$$\tilde{J} = \pm \widetilde{W}. \tag{3.33}$$

We denote by $\tilde{S}$ any surface in space which has

$$ds^2 = \tilde{E}\, d\phi^2 + 2\tilde{F}\, d\phi\, d\psi + \tilde{G}\, d\psi^2$$

for its fundamental differential form.

Now, since $E > 0$ and $\tilde{E} > 0$, the first equation in (3.32) shows that J and $\tilde{J}$ have the same sign, so that $\tilde{J} = \pm\widetilde{W}$ accordingly as $J = \pm W$. Consequently Eqs. (3.32) can be written as

$$\frac{E}{W} = \frac{\tilde{E}}{\widetilde{W}}, \qquad \frac{F}{W} = \frac{\tilde{F}}{\widetilde{W}},$$

from which it is easy to verify that

$$\frac{E}{\tilde{E}} = \frac{F}{\tilde{F}} = \frac{G}{\tilde{G}} = \frac{J}{\tilde{J}} = \frac{W}{\widetilde{W}}. \tag{3.34}$$

Conversely these equations imply the linear momentum equations (3.27) and (3.28) and are, therefore, equivalent to them.

Therefore, the linear momentum equations of an incompressible and electrically conducting aligned MHD fluid in the form (3.27) and (3.28) are equivalent to Eqs. (3.34) where $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, $\tilde{J}$, and $\widetilde{W}$ are defined in (3.31). This form of the momentum equations shows that a mapping of the physical plane upon a surface $\tilde{S}$ with points having the same parameter values ϕ, ψ in correspondence is conformal.

Letting k to be the kinetic energy, we have

$$k = \frac{1}{2}\rho q^2 = h - p - \frac{1}{2}\mu^* H^2 = \frac{1}{2}\rho \frac{E}{W^2}.$$

From this, we get

$$W = \frac{\rho \tilde{E}}{2k\widetilde{W}} \tag{3.35}$$

so that the common ratio in (3.34) can be evaluated as

$$\frac{E}{\tilde{E}} = \frac{F}{\tilde{F}} = \frac{F}{\tilde{G}} = \frac{W}{\widetilde{W}} = \frac{\rho \tilde{E}}{2k\widetilde{W}^2} \tag{3.36}$$

or in the form

$$E = \frac{\rho \tilde{E}^2}{2k\widetilde{W}^2}, \qquad F = \frac{\rho \tilde{E}\tilde{F}}{2k\widetilde{W}^2}, \qquad G = \frac{\rho \tilde{E}\tilde{G}}{2k\widetilde{W}^2}. \tag{3.37}$$

In view of the definitions of $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, and $\widetilde{W}$ in (3.31) we note that E, F, G become known functions of ϕ, ψ as long as ω, k, and p are established as functions of ϕ, ψ.

We use the vorticity equation (3.21) and the Gauss equation (3.23) to obtain ω, k, p as functions of ϕ, ψ and we shall consider $\tilde{J} = \widetilde{W}$, $J = W$ as we assumed that our fluid flows towards higher values of ϕ along a streamline.

Following Martin's [1] derivation of these equations and using our definitions for $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, $\tilde{J}$, and $\widetilde{W}$, we have

$$\frac{\partial}{\partial\phi}\left(\frac{\tilde{F}}{\tilde{J}}\right)-\frac{\partial}{\partial\psi}\left(\frac{\tilde{E}}{\tilde{J}}\right)=\frac{\rho\omega}{2k}\frac{\tilde{E}}{\tilde{J}}$$

and

$$\frac{\partial}{\partial\psi}\left[\frac{\tilde{F}}{\tilde{J}}\frac{1}{k}\frac{\partial p}{\partial\phi}-\frac{\tilde{E}}{\tilde{J}}\frac{1}{k}\left(\frac{\partial p}{\partial\psi}-\mu^{*}f\Omega_0\right)\right]-\frac{\partial}{\partial\phi}\left[\frac{\tilde{G}}{\tilde{J}}\frac{1}{k}\frac{\partial p}{\partial\phi}-\frac{\tilde{F}}{\tilde{J}}\frac{1}{k}\left(\frac{\partial p}{\partial\psi}-\mu^{*}f\Omega_0\right)\right]$$
$$=\mu\frac{\partial\left(\frac{1}{k},\omega\right)}{\partial(\phi,\psi)}.$$

These results are summed up in the theorem below:

THEOREM II. When the streamlines $\psi=$ constant, in aligned flow of an incompressible and electrically conducting fluid, in the presence of a magnetic field, are taken as one set of coordinate lines in a curvilinear coordinate system ϕ, ψ in the physical plane, the system (3.17) to (3.23) may be replaced by the system

$$\frac{\partial}{\partial\phi}\left(\frac{\tilde{F}}{\tilde{J}}\right)-\frac{\partial}{\partial\psi}\left(\frac{\tilde{E}}{\tilde{J}}\right)=\frac{\rho\omega}{2k}\frac{\tilde{E}}{\tilde{J}} \tag{3.38}$$

$$\frac{\partial}{\partial\psi}\left[\frac{\tilde{F}}{\tilde{J}}\frac{1}{k}\frac{\partial p}{\partial\phi}-\frac{\tilde{E}}{k\tilde{J}}\left(\frac{\partial p}{\partial\psi}-\mu^{*}f\Omega_0\right)\right]-\frac{\partial}{\partial\phi}\left[\frac{\tilde{G}}{k\tilde{J}}\frac{\partial p}{\partial\phi}-\frac{\tilde{F}}{k\tilde{J}}\left(\frac{\partial p}{\partial\phi}-\mu^{*}f\Omega_0\right)\right]$$
$$=\mu\frac{\partial\left(\frac{1}{k},\omega\right)}{\partial(\phi,\psi)} \tag{3.39}$$

$$\Omega_0=f\omega-\frac{2}{\rho}k\frac{df}{d\psi} \tag{3.40}$$

of three equations for $\omega(\phi,\psi)$, $k(\phi,\psi)$, $p(\phi,\psi)$, and $f(\psi)$. Here, $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, and $\tilde{J}$ are given by Eqs. (3.31) and k is the kinetic energy.

Given a solution

$$\omega=\omega(\phi,\psi),\qquad k=k(\phi,\psi),\qquad p=p(\phi,\psi),\qquad f=f(\psi)$$

of these equations, the flow in the physical plane is obtained from

$$z=\pm\int e^{i\alpha}\sqrt{\frac{p}{2k}}\left[\frac{\tilde{E}}{\tilde{J}}\,d\phi+\left(\frac{\tilde{F}}{\tilde{J}}+i\right)d\psi\right],\qquad \alpha=\int\left\{\frac{\partial\alpha}{\partial\phi}\,d\phi+\frac{\partial\alpha}{\partial\psi}\,d\psi\right\} \tag{3.41}$$

with $\partial\alpha/\partial\phi$ and $\partial\alpha/\partial\psi$ given by

$$\begin{aligned}\frac{\partial\alpha}{\partial\phi}&=\frac{1}{2k}\left\{\mu\frac{\partial\omega}{\partial\phi}+\frac{\tilde{F}}{\tilde{J}}\frac{\partial p}{\partial\phi}-\frac{\tilde{E}}{\tilde{J}}\left(\frac{\partial p}{\partial\psi}-\mu^{*}f\Omega_0\right)\right\}\\ \frac{\partial\alpha}{\partial\psi}&=\frac{1}{2k}\left\{\mu\frac{\partial\omega}{\partial\psi}+\frac{\tilde{G}}{\tilde{J}}\frac{\partial p}{\partial\phi}-\frac{\tilde{F}}{\tilde{J}}\left(\frac{\partial p}{\partial\psi}-\mu^{*}f\Omega_0\right)\right\}\end{aligned} \tag{3.42}$$

and the flow in the hodograph plane is given by

$$u + iv = \sqrt{\frac{2k}{\rho}} e^{i\alpha}. \tag{3.43}$$

The undetermined system (3.38) to (3.40) can be made determinate by specifying the choice of the ϕ-curves.

In the subsequent sections we consider the applications of the above results by considering flows in which certain conditions are place a priori.

4. Energy and speed constant on streamlines. Govindaraju [3] investigated non-MHD viscous fluid flows when energy is constant on each individual streamline. We study these confluent flows for the MHD aligned case when isovels coincide with streamlines also.

We assume that $h = h(\psi)$; $h'(\psi) \neq 0$ and $q = q(\psi)$. Using these assumptions in Eqs. (3.31) we have

$$\tilde{E} = \mu^2 \left(\frac{\partial \omega}{\partial \phi}\right)^2, \qquad \tilde{F} = \mu^2 \frac{\partial \omega}{\partial \phi}\frac{\partial \omega}{\partial \psi},$$

$$\tilde{G} = \mu^2 \left(\frac{\partial \omega}{\partial \psi}\right)^2 + \left[h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^* \frac{d(f^2q^2)}{d\psi}\right]^2 \tag{4.1}$$

and

$$\tilde{J} = \mu \frac{\partial \omega}{\partial \phi}\left[h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^* \frac{d(f^2q^2)}{d\psi}\right]$$

where dash denotes differentiation with respect to ψ.

Using these expressions for $\tilde{E}$, $\tilde{F}$, and $\tilde{J}$ in Eq. (3.38), we get

$$\frac{h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2)}{h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi)} = \frac{\rho\omega}{2k} \tag{4.2}$$

when $h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi) \neq 0$.

However, if $h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi) = 0$, then $\tilde{J} = 0$. Using $\tilde{J} = 0$ in Eqs. (3.29) and (3.30) we obtain that ω is a constant for our flows.

Equation (4.2) implies that $\omega = \omega(\psi)$ and, therefore, Eqs. (3.1) yield $\tilde{E} = \tilde{J} = 0$ so that the bar-system fails.

We shall now use Eqs. (3.17) to (3.23) and take the curvilinear coordinate net (ϕ, ψ) to be an orthogonal net, so that $F = 0$.

Assuming that $h = h(\psi)$; $h'(\psi) \neq 0$ and $q = q(\psi)$, Eq. (3.18) implies that $\omega = \omega(\phi)$ and Eq. (3.19) gives

$$\sqrt{\frac{E}{G}} = \frac{\mu\dot{\omega}}{h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi)} \tag{4.3}$$

where dot denotes differentiation with respect to ϕ.

Using (4.3) in Eq. (3.21), we have

$$\omega = \frac{h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2)}{G[h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi)]}. \tag{4.4}$$

If $h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2) = 0$, then $\omega = 0$. Taking the case when $h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2) \neq 0$, we use Eqs. (4.3) and (4.4) to obtain

$$E = \frac{\mu^2\dot{\omega}^2(h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2))}{\omega[h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi)]^3},$$

$$G = \frac{h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2)}{\omega[h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi)]},$$

and

$$J = \sqrt{EG} = \frac{\mu\dot{\omega}[h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2)]}{\omega[h' + \rho\omega - \mu^* f\Omega_0 - \frac{1}{2}\mu^*(d(f^2q^2)/d\psi)]^2}.$$

Using these expressions, Eq. (3.23) simplifies to

$$\dot{\omega}[b_0\omega^4 + b_1\omega^3 + b_2\omega^2 + b_3\omega + b_4] = 0 \tag{4.5}$$

where b_i are functions of ψ with b_4 given by

$$b_4 = -\frac{1}{\mu}\left[h' - \mu^* f\Omega_0 - \frac{1}{2}\mu^*\frac{d(f^2q^2)}{d\psi}\right]^4\left[h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*\frac{d^2(f^2q^2)}{d\psi^2}\right]^2.$$

From Eq. (4.5), we note that either ω is a constant or ω must satisfy

$$b_0\omega^4 + b_1\omega^3 + b_2\omega^2 + b_3\omega + b_4 = 0.$$

This equation gives $\omega = \omega(\psi)$ unless the coefficients b_i vanish. In particular, $b_4 = 0$ implies $h'' - \mu^* f'\Omega_0 - \frac{1}{2}\mu^*(d^2(f^2q^2)/d\psi^2) = 0$, contrary to our hypothesis. Hence, ω is a constant everywhere. Summing up, we have

THEOREM 4.1. If the energy and the speed are constant on streamlines in a steady two-dimensional aligned flow of an incompressible and electrically conducting fluid, the vorticity is uniform and the streamfunction is given by

$$\psi(x, y) = -\frac{\omega}{4}(x^2 + y^2) + \chi(x, y)$$

where $\chi(x, y)$ is a harmonic function.

The expression for the streamfunction is a direct consequence of

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = -\omega$$

as is stated by Berker [6].

5. Vorticity constant on streamlines. Govindaraju [3] investigated non-MHD viscous fluid flows when vorticity is constant on each individual streamline. We study these confluent flows for the MHD aligned case.

We assume that the streamlines and isobars do not coincide. We investigate flows having constant vorticity on the streamlines. We assume that $\omega = \omega(\psi)$; $\omega'(\psi) \neq 0$. Our requirement of $\omega'(\psi) \neq 0$ is due to the fact that $\tilde{J} = 0$ when ω is a constant and Eqs. (3.38) to (3.41) are valid only if $\tilde{J} \neq 0$.

Employing $\omega = \omega(\psi)$ and $\phi = p$ in Eqs. (3.31), we obtain

$$\begin{aligned} \tilde{E} &= \left(\frac{\partial k}{\partial p} + 1\right)^2, \qquad \tilde{F} = \left(\frac{\partial k}{\partial p} + 1\right)\left(\frac{\partial k}{\partial \psi} + \rho\omega - \mu^* f\Omega_0\right) \\ \tilde{G} &= \mu^2\omega'^2(\psi) + \left(\frac{\partial k}{\partial \psi} + \rho\omega - \mu^* f\Omega_0\right)^2, \qquad \tilde{J} = -\mu\omega'(\psi)\left(\frac{\partial k}{\partial p} + 1\right). \end{aligned} \tag{5.1}$$

Using (5.1), Eq. (3.38) yields

$$\left(\frac{\partial k}{\partial p} + 1\right)\left[\frac{\rho\omega}{2k} - \frac{\omega''(\psi)}{\omega'(\psi)}\right] = 0.$$

Since $\tilde{J} \neq 0$ and $\omega'(\psi) \neq 0$, it follows from (5.1) that $\partial k/\partial p + 1 \neq 0$ and, therefore, the above equation gives

$$\frac{\rho\omega}{2k} - \frac{\omega''(\psi)}{\omega'(\psi)} = 0. \tag{5.2}$$

From Eq. (5.2) we obtain that $k = k(\psi)$ so that $q = q(\psi)$ also. Employing the results and assumptions in Eq. (3.39) we have

$$\frac{d}{d\psi}\left(\frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)k}\right) = 0$$

which upon integration yields

$$k'(\psi) + \rho\omega = B_0\mu\omega'(\psi)k \tag{5.3}$$

where B_0 is an arbitrary constant.

Equations (5.2) and (5.3) form a system of two equations in two unknowns ω and k as functions of ψ. We consider two cases $B_0 \neq 0$ and $B_0 = 0$.

Case I. $B_0 \neq 0$. Using Eq. (5.2) in (5.3), we get

$$\frac{k'(\psi)}{k} + \frac{2\omega''(\psi)}{\omega'(\psi)} = B_0\mu\omega'(\psi)$$

which upon integration gives

$$k\omega'^2(\psi) = \rho B_1 e^{B_0\mu\omega}$$

where $B_1 \neq 0$ is an arbitrary constant.

Employing Eq. (5.2) again and integrating twice with respect to ψ, we obtain

$$-2B_1\psi - B_2\omega - \frac{\omega}{B_0^2\mu^2}e^{-B_0\mu\omega} - \frac{2}{B_0^3\mu^3}e^{-B_0\mu\omega} = B_3. \tag{5.4}$$

Hence, if $B_0 \neq 0$, the function $\omega(\psi)$ is defined implicitly by Eq. (5.4).

To obtain the flow in the physical plane we use Eq. (3.42) together with (5.3) and find that

$$\alpha(p, \psi) = \alpha_0 + \frac{B_0}{2}(p + k + \beta(\psi)) \tag{5.5}$$

where $\beta'(\psi) = \rho\omega(\psi) - \mu^* f(\psi)\Omega_0$ and α_0 is an arbitrary constant. Making use of (5.5) and (5.3) in the first equation of (3.42), we get

$$z - z_0 = \mp\frac{2i}{B_0}\sqrt{\frac{\rho}{2k}}\frac{1}{\mu\omega'(\psi)}e^{i\alpha} \tag{5.6}$$

where $z_0 = x_0 + iy_0$ is an arbitrary complex constant.

From Eq. (5.6), we have

$$x - x_0 = \pm\frac{2}{B_0}\sqrt{\frac{\rho}{2k}}\frac{1}{\mu\omega'(\psi)}\sin\alpha$$

and

$$y - y_0 = \mp\frac{2}{B_0}\sqrt{\frac{\rho}{2k}}\frac{1}{\mu\omega'(\psi)}\cos\alpha.$$

Eliminating α, we get

$$(x - x_0)^2 + (y - y_0)^2 = \frac{2\rho}{B_0^2\mu^2 k\omega'^2(\psi)}. \tag{5.7}$$

This equation shows that the streamlines are concentric circles with center at z_0 and radius

$$\frac{\sqrt{2\rho}}{\sqrt{k}\mu|B_0\omega'(\psi)|}.$$

Case II. $B_0 = 0$. On eliminating k between Eqs. (5.2) and (5.3) with $B_0 = 0$ and integrating we find that the function $\omega(\psi)$ is defined implicitly by

$$\omega^3 - 6B_2\omega + (12B_1\psi - 3B_3) = 0 \tag{5.8}$$

where B_1, B_2, and B_3 are arbitrary constants. Using $B_0 = 0$ in Eqs. (3.42), we find that

$$\alpha = \alpha_0$$

where α_0 is an arbitrary constant and the flow proceeds along parallel lines.

$\omega = \omega(\psi)$ is given by Eq. (5.4) for the vortex flow and by (5.8) for the parallel flows. Having obtained $\omega = \omega(\psi)$, the kinetic energy $k(\psi)$ can be determined by

$$k(\psi) = \frac{\rho\omega\omega'(\psi)}{2\omega''(\psi)}. \tag{5.9}$$

Finally, using (5.2) in Eq. (3.41), we find that $f(\psi)$ is given by

$$f(\psi) = -\Omega_0\omega'(\psi)\omega_1(\psi) + f_0 \tag{5.10}$$

where $\omega_1'(\psi) = \omega''(\psi)/\omega\omega'^2(\psi)$ and f_0 is an arbitrary constant.

Thus, we can sum up to get:

THEOREM 5.1. If the streamlines and isobars do not coincide and the vorticity in a two-dimensional aligned MHD flow of an incompressible and electrically conducting fluid of finite electrical conductivity is constant on streamlines, then the streamlines are either parallel straight lines or concentric circles.

6. Streamlines and isobars coincide. Govindaraju [4] investigated non-MHD viscous fluid flows when streamlines and isobars coincide. We study these confluent flows for the MHD-aligned case.

In this section, along with our assumption that the streamlines and isobars coincide, we also assume that $k = k(\omega)$. We choose the (ϕ, ψ) coordinate net to be an orthogonal net so that Eq. (3.30) for the assumed flows gives

$$\tilde{F} = \frac{\partial \omega}{\partial \phi}[(\mu^2 + k'^2)\frac{\partial \omega}{\partial \psi} + k'(\dot{p} + \rho\omega - \mu^* f\Omega_0)] = 0 \tag{6.1}$$

where dash and dot denote differentiation with respect to ω and ψ respectively.

Equations governing such flows are

$$-\frac{\partial}{\partial \psi}\left(\frac{\tilde{E}}{\tilde{J}}\right) = \frac{\rho\omega}{2k}\frac{\tilde{E}}{\tilde{J}} \tag{6.2}$$

$$-\frac{\partial}{\partial \psi}\left[\frac{\tilde{E}}{\tilde{J}}\frac{1}{k}(\dot{p} - \mu^* f\Omega_0)\right] = 0 \tag{6.3}$$

$$\Omega_0 = f\omega - \frac{2}{\rho}k\dot{f} \tag{6.4}$$

along with Eq. (6.1). Here,

$$\tilde{E} = (\mu^2 + k'^2)\left(\frac{\partial \omega}{\partial \phi}\right)^2, \qquad \tilde{J} = \mu\frac{\partial \omega}{\partial \phi}(\dot{p} + \rho\omega - \mu^* f\Omega_0). \tag{6.5}$$

We consider the following two cases:

Case I: $\partial\omega/\partial\phi = 0$. In this case $\omega = \omega(\psi)$ and therefore $k = k(\psi)$ and $h = h(\psi)$. Therefore, the vorticity remains constant as we have already proved in Sec. 3 that the vorticity must remain constant throughout the flow when h and k are constant on each individual streamline.

Case II: $\partial\omega/\partial\phi \neq 0$. In this case, Eq. (6.1) gives

$$\frac{\partial \omega}{\partial \psi} = -\frac{k'(\dot{p} + \rho\omega - \mu^* f\Omega_0)}{\mu^2 + k'^2}. \tag{6.6}$$

Integrating the Gauss equation (6.3) with respect to ψ, we get

$$\frac{(\mu^2 + k'^2)\partial\omega/\partial\phi}{\mu(\dot{p} + \rho\omega - \mu^* f\Omega_0)}\frac{1}{k}(\dot{p} - \mu^* f\Omega_0) = g(\phi) \tag{6.7}$$

where $g(\phi)$ is an arbitrary function of ϕ.

Assuming that $\dot{p} - \mu^* f\Omega_0 \neq 0$, Eq. (6.7) gives

$$\frac{\partial \omega}{\partial \phi} = \frac{\mu k g(\phi)(\dot{p} + \rho\omega - \mu^* f\Omega_0)}{(\mu^2 + k'^2)(\dot{p} - \mu^* f\Omega_0)}. \tag{6.8}$$

Using (6.6) and (6.8) in the integrability condition $(\partial^2\omega/\partial\phi\partial\psi) = (\partial^2\omega/\partial\psi\partial\phi)$ we get

$$(kk'' - k'^2)(\dot{p} - \mu^* f\Omega_0 + \rho\omega)^2(\dot{p} - \mu^* f\Omega_0) - \rho\omega k(\mu^2 + k'^2)(\ddot{p} - \mu^* \dot{f}\Omega_0) = 0. \tag{6.9}$$

On the other hand, employing (6.6) and (6.8), Eq. (6.2) reduces to

$$2k'^2(\dot{p}+\rho\omega-\mu^* f\Omega_0)(\dot{p}-\mu^* f\Omega_0)+2k(\mu^2+k'^2)(\ddot{p}-\mu^*\dot{f}\Omega_0)$$
$$-\rho\omega(\mu^2+k'^2)(\dot{p}-\mu^* f\Omega_0)=0. \tag{6.10}$$

Eliminating $\ddot{p}-\mu^*\dot{f}\Omega_0$ from (6.9) and (6.10) we have

$$(\dot{p}-\mu^* f\Omega_0)\{2k'^2(\dot{p}+\rho\omega-\mu^* f\Omega_0)$$
$$+\frac{2}{\rho\omega}(kk''-k'^2)(\dot{p}-\mu^* f\Omega_0+\rho\omega)^2-\rho\omega(\mu^2+k'^2)\}=0.$$

Since we assumed that $\dot{p}-\mu^* f\Omega_0\neq 0$, then the above equation gives

$$2(kk''-k'^2)(\dot{p}+\rho\omega-\mu^* f\Omega_0)^2+2k'^2(\dot{p}+\rho\omega-\mu^* f\Omega_0)\rho\omega$$
$$-\rho^2\omega^2(\mu^2+k'^2)=0. \tag{6.11}$$

Equation (6.11) is a quadratic for $\dot{p}+\rho\omega-\mu^* f\Omega_0$ whose coefficients are functions of ω and is not an identity. Hence, we can conclude that $\dot{p}-\mu^* f\Omega_0+\rho\omega$ must be a function of ω. Since $p=p(\psi)$, $f=f(\psi)$, and $\omega=\omega(\phi,\psi)$, then we must have $\dot{p}-\mu^* f\Omega_0$ is a constant and Eq. (6.9) gives

$$(kk''-k'^2)(\dot{p}-\mu f\Omega_0+\rho\omega)^2(\dot{p}-\mu^* f\Omega_0)=0.$$

Since $(\dot{p}-\mu^* f\Omega_0+\rho\omega)^2(\dot{p}-\mu^* f\Omega_0)\neq 0$, then the equation above yields

$$kk''-k'^2=0,$$

which upon integration gives

$$k=k_0e^{a\omega},$$

where k_0 and a are arbitrary constants.

Using this value of k in (6.11), we get

$$2a^2k_0^2e^{2a\omega}\left(\dot{p}-\mu^* f\Omega_0+\frac{\rho\omega}{2k}\right)-\rho\mu^2\omega=0,$$

which implies that ω is a constant. But this contradicts our assumption that $\partial\omega/\partial\phi\neq 0$. Hence $\dot{p}-\mu^* f\Omega_0=0$.

Thus, we have the following theorem.

THEOREM 6.1. If the streamlines and isobars coincide and the vorticity remains constant along the isovels in the steady two-dimensional aligned flow of an incompressible and electrically conducting viscous fluid, then either (i) the vorticity is constant everywhere or (ii) $p=p(\psi)$ such that $\dot{p}=\mu^* f\Omega_0$.

If $\dot{p}-\mu^* f\Omega_0=0$, then the Gauss equation is identically satisfied. However, Eq. (6.2) gives

$$(\mu^2+k'^2)\frac{\partial^2\omega}{\partial\phi\,\partial\psi}+2k'k''\frac{\partial\omega}{\partial\phi}\frac{\partial\omega}{\partial\psi}+\rho\left[k'+\frac{\omega}{2k}(\mu^2+k'^2)\right]\frac{\partial\omega}{\partial\phi}=0.$$

Using Eq. (6.7) to eliminate $\partial^2\omega/\partial\phi\,\partial\psi$ and $\partial\omega/\partial\psi$, we get

$$2kk''=\mu^2+k'^2$$

which upon integration yields

$$k = \frac{c^2}{4}(\omega + b)^2 + \frac{\mu^2}{c^2} \tag{6.12}$$

where b and $c \neq 0$ are arbitrary constants. Using (6.12) in (6.1) and integrating, we have ω as a function of ϕ and ψ given implicitly as

$$\begin{aligned} &\omega + b \ln|\omega| + \frac{4\mu^2}{c^4} \ln\left|\frac{\omega}{\omega + b}\right| + \frac{2\rho}{c^2}\psi = \chi(\phi); \qquad b \neq 0 \\ &\frac{c^2}{2}\omega - \frac{2\mu^2}{c^2\omega} + \rho\psi = \chi(\phi); \qquad b = 0 \end{aligned} \tag{6.13}$$

where χ is an arbitrary function of ϕ.

To find the flow in the physical plane, we have from the second equation in (3.42)

$$\alpha(\phi, \psi) = \int \frac{\mu}{2k}\, d\omega$$

so that, by (5.13), we get

$$\alpha - \alpha_0 = \tan^{-1}\left[\frac{c^2}{2\mu}(\omega + b)\right] \tag{6.14}$$

where α_0 is an arbitrary constant.

Employing (6.1), (6.12), and (6.14) in the first equation of (3.42), we get

$$z = z_0 + \frac{ce^{i\alpha_0}}{2\mu\sqrt{2\rho}}\{2\mu \ln|\omega| + i[2\rho\psi + c^2(\rho + b\ln|\omega|)]\}$$

where $z_0 = x_0 + iy_0$ is an arbitrary complex constant.

From this, we have ω as a function of x and y given by

$$\omega = \exp\left\{\frac{\sqrt{2\rho}}{c}[A(x - x_0) + B(y - y_0)]\right\}$$

and the streamfunction is given by

$$\begin{aligned} \psi = {} & \frac{\mu\sqrt{2\rho}}{\rho c}\left[\left(-B - \frac{bc^2}{2\mu}A\right)(x - x_0) + \left(A - \frac{bc^2}{2\mu}B\right)(y - y_0)\right] \\ & - \frac{c^2}{2\rho}\exp\left\{\frac{\sqrt{2\rho}}{c}[A(x - x_0) + B(y - y_0)]\right\} \end{aligned}$$

where $A = \cos\alpha_0$ and $B = \sin\alpha_0$.

References

[1] M. H. Martin, *The flow of a viscous fluid* I, Arch. Rational Mech. Anal. **41**, 266–286 (1971)

[2] M. H. Martin, *Plane rotational Prandtl-Meyer flows*, J. Math. and Phys. **XXIX**, 76–89 (1950)

[3] K. V. Govindaraju, *The flow of a viscous fluid* II, Arch. Rational Mech. Anal. **45**, 66–80 (1972)

[4] K. V. Govindaraju, *Confluent flows of a viscous fluid*, Quart. Appl. Math. **39**, 97–108 (1981)

[5] S. I. Pai, *Magnetogasdynamics and Plasma Physics*, Prentice-Hall, Englewood Cliffs, N. J., 1962

[6] R. Berker, *Integration des equations de mouvement d'un fluide visqueux incompressible*, Stromungs Mechanik II, Handbuch der Physik, VIII/2, 1963, pp. 1–41