1. Introduction. Let

\[ Hu \equiv u_{xx} - u_t, \]
\[ \Omega \equiv (0, a) \times (0, T), \]
\[ \Gamma \equiv ([0, a] \times \{0\}) \cup \{0\} \times (0, T), \]
\[ S \equiv \{a\} \times (0, T), \]

where \( T \leq \infty \). Also, let \( u \) be a solution of the problem:

\[ Hu = -f(u) \quad \text{in} \, \Omega, \quad u = 0 \quad \text{on} \, \Gamma, \quad u = 0 \quad \text{on} \, S, \quad (1.1) \]

where \( f(u) \) tends to infinity as \( u \) approaches \( c^- \) for some positive constant \( c \). The length \( a^* \) is said to be the critical length for the problem (1.1) if \( u \) exists globally for \( a < a^* \), and for \( a > a^* \) there exists a finite time \( T \) such that

\[ \max\{u(x, t) : 0 < x < a\} \to c^- \quad \text{as} \quad t \to T^- . \quad (1.2) \]

This finite time \( T \) is called the quenching time. In the special case that \( f(u) = (1 - u)^{-1} \), Kawarada [9] showed that (1.2) occurred for \( a > 2^{3/2} \). Acker and Walter [2] showed that under appropriate conditions on the forcing term \( f(u) \), there existed a unique critical length \( a^* \) for the problem (1.1). This result was then extended to forcing terms of the type \( g(u, u_x) \) by Acker and Walter [3], and to \( h(x, u, u_x) \) by Chan and Kwong [7]. Results on the behavior of the solution of the problem (1.1) with \( a = a^* \) were given by Levine and Montgomery [10]. Existence of the critical length \( a^* \) and its determination by computational methods were given by Chan and Chen [4] for a more general parabolic singular operator; they studied the problem:

\[ Lu = -u_x^2 \quad \text{in} \, \Omega, \quad u = 0 \quad \text{on} \, \Gamma, \quad u = 0 \quad \text{on} \, S, \]

where \( Lu \equiv Hu + bu_x / x \) with \( b \) a constant less than 1; in particular, \( a^* = 1.5303 \) (to five significant figures) for \( b = 0 \). Similar results were given by Chan and Kaper [6] for the problem:

\[ Lu = -f(u) \quad \text{in} \, \Omega, \quad u = 0 \quad \text{on} \, \Gamma, \quad u_x = 0 \quad \text{on} \, S. \quad (1.3) \]
This includes the problem (1.1) as a special case since the solution of that problem is symmetric with respect to the line $x = a/2$. We refer to the papers of Chan and Chen [4] and Chan and Kaper [6] for the significance of the expression $Lu$. Critical lengths for global existence of solutions for a coupled system of two semilinear parabolic equations subject to zero initial-boundary data were given by Chan and Chen [5]. Existence of the critical size for the multidimensional version of the problem (1.1) was studied by Acker and Kawohl [1].

The main purpose here is to study the critical length for the following problem:

$$Lu = -f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma, \quad Bu = 0 \text{ on } S,$$

where $Bu \equiv u_x + ku$. Here, $b$ is a constant less than 1; $k$ is a positive constant; $f$ is nondecreasing and continuously differentiable on $[0, c)$ for some constant $c$ such that $f(0) > 0$; and $\lim_{u \to c} f(u) = \infty$. As in the papers by Chan and Chen [4] and Chan and Kaper [6], we assume existence of a solution $u$ before its quenching time. In the problem (1.3), $u$ attains its maxima with respect to $x$ at $x = a$; unlike the problem (1.1), the singular term $bu_x/x$ as well as the third boundary condition in our present problem destroys the symmetry of the solution $u$ about the line $x = a/2$, and shifts the points where $u$ attains its maxima with respect to $x$ from the line $x = a/2$. Thus, they make the problem more difficult both theoretically and numerically.

In Sec. 2, we establish existence of a critical length $a^*$, and give a computational method to determine $a^*$. In Sec. 3, a method is given to determine an upper bound of the quenching time for a given $a$ greater than $a^*$. An algorithm is given in Sec. 4 to compute $a^*$. For illustration, a numerical example is given by taking $f(u)$ to be $(1 - u)^{-1}$.

2. Critical length. Let us first establish the following results.

**Lemma 1.** Let $u$ be a solution of the problem (1.4).

(a) There exists at most one solution.
(b) The solution $u$ is positive in $\Omega \cup S$.
(c) The solution $u$ is a strictly increasing function of $t$ for each $x \in (0, a]$.
(d) There exists a curve $\phi(t)$ such that for each $t \in (0, T)$, $u$ is strictly decreasing in $x$ on $(\phi(t), a]$, and nondecreasing in $x$ on $[0, \phi(t)]$, where $\phi(t) \in (0, a)$.

**Proof.** (a) Let $u_1$ and $u_2$ be two distinct solutions, and $w \equiv u_1 - u_2$. Then by the mean value theorem,

$$[L + f'(\eta)]w = 0 \text{ in } \Omega,$$

where $\eta$ lies between $u_1$ and $u_2$. Without loss of generality, let $w > 0$ somewhere. Since $f'(\eta)$ is bounded above, it follows from the strong maximum principle (cf. Protter and Weinberger [12, pp. 168–169, 172, and 175]) that $w$ attains its positive maximum somewhere on $S$. At this point, $w_x > 0$ by the parabolic version of Hopf's lemma (cf. Protter and Weinberger [12, pp. 170–172]). This contradicts $Bw = 0$ on $S$. Thus, there exists at most one solution.
(b) Since \( f(0) > 0 \), we have \( Lu + f(u) < f(0) \). By the mean value theorem, \([L + f'(\eta)]u < 0\), where \( \eta \) lies between \( u \) and \( 0 \). The assertion then follows from the strong maximum principle and the parabolic version of Hopf's lemma.

(c) For any \( h > 0 \), let
\[
w(x, t) = u(x, t + h) - u(x, t).
\]
By the mean value theorem, \([L + f(\eta)]w = 0\), where \( \eta \) lies between \( u(x, t + h) \) and \( u(x, t) \). Since \( w(x, 0) > 0 \) for \( 0 < x \leq a \), \( w(0, t) = 0 \), and \( Bw = 0 \) on \( S \), it follows from the strong maximum principle and the parabolic version of Hopf's lemma that \( w > 0 \) on \( \Omega \cup S \). The assertion is then proved.

(d) It follows from Lemma 1(b) that \( u_x(a, t) = -ku(a, t) < 0 \) for \( 0 < t < T \); by the parabolic version of Hopf's lemma, \( u_x(0, t) > 0 \) for \( 0 < t < T \). For any fixed \( t \) and any positive \( x_0 \) (\( \leq a \)) such that \( u_x(x_0, t) < 0 \), it follows from the mean value theorem that for any positive \( \epsilon \) (\( \leq x_0 \)),
\[
0 < u_\epsilon(x, t) - u_\epsilon(0, t) = u_x(\eta, t)\epsilon \quad \text{for some} \quad \eta \in (0, \epsilon).
\]
Thus for each \( t \) (\( > 0 \)), there exists a point \( x \in (0, x_0) \) such that \( u_x(x, t) = 0 \).

Differentiating the differential equation in (1.4) with respect to \( x \), we obtain
\[
(L + f'(u) - b/x^2)u_x = 0.
\]
Let \( G \) be the component containing \( S \) such that \( u_x < 0 \) in \( G \). Since \( G \) does not intersect the line \( x = 0 \), it follows by applying the strong maximum principle that \( G \) is simply connected with \( u_x = 0 \) on \( \partial G \cap \Omega \), where \( \partial G \) denotes the boundary of \( G \). If \( u_x(x_1, t_1) < 0 \) somewhere in \( \Omega \setminus G^- \), where \( G^- \) denotes the closure of \( G \), then by the continuity of \( u_x \), there exists a neighborhood \( N \) of \((x_1, t_1)\) such that \( u_x < 0 \) in \( N \) and \( u_x = 0 \) on \( \partial N \cap (\Omega \setminus G) \), but this contradicts the strong maximum principle. Thus, \( u_x \geq 0 \) in \( \Omega \setminus G^- \), and \( \partial G \cap \Omega = \phi(t) \).

Let
\[
\nu U = U'' + \frac{b}{x}U', \quad \beta U = U' + kU.
\]
With slight modification of the proof of Theorem 3 of Chan and Kaper [6], we obtain the following result.

**Theorem 2.** If \( T = \infty \) and \( u(x, t) \leq C < c \) for some constant \( C \), then \( u \) converges uniformly on \([0, a]\) from below to a solution \( U \) of the singular nonlinear two-point boundary-value problem:
\[
\nu U = -f(U), \quad U(0) = 0 = \beta U(a).
\]
Furthermore, \( u < U \) in \((0, a) \times [0, \infty)\).

In order to show that beyond the critical length there exists a finite time \( T \) such that (1.2) holds, the following result is crucial.

**Theorem 3.** \( Bu(x, t) \geq 0 \) in \( \Omega \).

**Proof.** For any \( \epsilon \in (0, a) \), let
\[
\Omega_\epsilon = (\epsilon, a) \times (0, T),
\]
\[
\Gamma_\epsilon = ([\epsilon, a] \times \{0\}) \cup (\{\epsilon\} \times (0, T)).
\]
Let $u_\epsilon$ denote the solution of the (regular) problem:

$$
Lu_\epsilon = -f(u_\epsilon) \quad \text{in } \Omega_\epsilon,
$$

$$
u_\epsilon = 0 \quad \text{on } \Gamma_\epsilon, \quad Bu_\epsilon = 0 \quad \text{on } S.
$$

(2.2)

An argument as in the proofs of Lemma 1(b) and (c) shows that $u_\epsilon > 0$ in $\Omega_\epsilon \cup S$, and $u_\epsilon$ is a strictly increasing function of $t$ for each $x \in (\epsilon, a]$. It follows from the strong maximum principle and the parabolic version of Hopf's lemma that $u_\epsilon$ strictly increases as $\epsilon$ decreases. In particular, we have $0 < u_\epsilon < u$ in $\Omega_\epsilon$. Let us differentiate (2.2) with respect to $x$, and denote the partial derivative of $u_\epsilon$ with respect to $x$ by $u_{\epsilon,x}$. We obtain

$$
[L + f'(u_\epsilon) - b/x^2]u_{\epsilon,x} = 0 \quad \text{in } \Omega_\epsilon.
$$

Now,

$$
u_{\epsilon,x}(x, 0) = 0 \quad \text{for } \epsilon \leq x \leq a.
$$

For any $\tau \in (0, T)$,

$$
u_{\epsilon,x}(\epsilon, t) \geq 0 \quad \text{and} \quad \nu_{\epsilon,x}(a, t) = -k\nu_\epsilon(a, t) < 0 \quad \text{for } 0 < t < \tau.
$$

Let $\Omega_{\epsilon,\tau} = [\epsilon, a] \times [0, \tau]$. By the strong maximum principle, $\nu_{\epsilon,x}$ attains its negative minimum somewhere on $\Omega_{\epsilon,\tau}$ at $x = a$. Since $\nu_{\epsilon,x}(a, t)$ increases as $t$ increases, it follows that $\nu_{\epsilon,x}(x, t) \geq -k\nu_\epsilon(a, t)$ on $\Omega_{\epsilon,\tau}$. An argument as in the proof of Lemma 1(d) shows that there exists a curve $\psi(t)$ such that for each $t \in (0, T)$, $\psi(t) \in (\epsilon, a)$ and $u_\epsilon$ is strictly decreasing in $x$ on $(\psi(t), a)$ and nondecreasing in $x$ on $[\epsilon, \psi(t)]$. Thus for $x \in (\psi(t), a)$, $Bu_\epsilon(x, \tau) > 0$. Because $u_\epsilon(x, \tau) > 0$ for $x \in (\epsilon, \psi(t))$, $Bu_\epsilon(x, \tau) > 0$ there. Since $\tau$ is arbitrary, we have

$$
Bu_\epsilon(x, t) > 0 \quad \text{in } \Omega_\epsilon.
$$

(2.3)

Since $u_\epsilon$ is bounded, $\lim_{\epsilon \to 0} u_\epsilon$ exists. Let us denote this limit by $Z$. Then in $\Omega_\epsilon$, $0 < u_\epsilon \leq Z \leq u$ and $BZ \geq 0$.

To prove that $Z = u$, let $\sigma \in (\epsilon, a)$ and $u_\sigma$ be the unique solution of the (regular) problem:

$$
Lu_\sigma = -f(u_\sigma) \quad \text{in } \Omega_\sigma,
$$

$$
u_\sigma(x, 0) = 0, \quad u_\sigma(\sigma, t) = u_\epsilon(\sigma, t), \quad Bu_\sigma = 0 \quad \text{on } S.
$$

The adjoint $L^*$ (cf. Friedman [8, p. 26]) of $L$ in $\Omega_\sigma$ is given by

$$
L^*v = v_{xx} - (bv/x)_x + v_t
$$

with adjoint boundary conditions (cf. Polozhiy [11, p. 413]) given by

$$
v(\sigma, t) = 0 = v_x(a, t) + (k - b/a)v(a, t).
$$

Let $R^*(\xi, \tau; x, t)$ denote its Green's function (cf. Friedman [8, pp. 82-84 and 155]). In Green's identity (cf. Friedman [8, p. 27]),

$$
Lv u - uL^*v = (vu_x - uv + buv/x)x - (uv)_t,
$$

(2.4)
let \( u = u_\epsilon \) and \( v(\xi, \tau) = R^*(\xi, \tau; x, t) \). Let us integrate this over the domain \( (\sigma, a) \times (0, t - \delta) \), where \( \delta \) is a small positive constant less than \( t \). By letting \( \delta \) tend to zero, we obtain

\[
u_\epsilon(x, t) = \int_0^t \int_\sigma^a R^*(\xi, \tau; x, t) f(u_\epsilon(\xi, \tau)) d\xi d\tau + \int_0^t R_\epsilon^*(\tau; x, t) u_\epsilon(\sigma, \tau) d\tau \quad \text{in } \Omega_\sigma.
\]

Since \( R^*(\xi, \tau; x, t) > 0 \) for \( (\xi, \tau) \in (\sigma, a) \times (0, t) \) (cf. Friedman \[8, p. 84\]), it follows that \( R_\epsilon^*(\tau; x, t) > 0 \). As \( \epsilon \) decreases, \( u_\epsilon \) and \( f(u_\epsilon) \) are nondecreasing. By the monotone convergence theorem (cf. Royden \[13, p. 84\]),

\[
Z(x, t) = \int_0^t \int_\sigma^a R^*(\xi, \tau; x, t) f(Z(\xi, \tau)) d\xi d\tau + \int_0^t R_\epsilon^*(\tau; x, t) Z(\sigma, \tau) d\tau \quad \text{in } \Omega_\sigma.
\]

Thus, \( LZ = -f(Z) \) in \( \Omega_\sigma \). Since \( \sigma \) is arbitrary, it follows that \( LZ = -f(Z) \) in \( \Omega \). Now, \( Z(x, 0) = 0 \) and \( BZ = 0 \) on \( S \). From \( 0 \leq u_\epsilon \leq Z \leq u \) in \( \Omega \), we have \( Z(0, t) = 0 \). Since \( u \) is unique, it follows that \( u = Z \). From (2.3), \( Bu \geq 0 \) in \( \Omega \).

Let \( u(x, t; a) \) denote the solution \( u(x, t) \) of the problem (1.4). Then for any positive constant \( \alpha \), let \( h \) be a nonnegative constant such that \( h < \alpha \).

**Theorem 4.** If \( \lim_{t \to \infty} u(\phi(t), t; a) = c \), then there exists a finite time \( T \) such that

\[
\max\{u(x, t; a + \alpha) : 0 \leq x \leq a + \alpha\} \to c^- \quad \text{as } t \to T^- .
\]

**Proof.** Let us assume that there does not exist a finite time \( T \) such that (2.4) holds. Let

\[
w(x, t) = u(x + h, t; a + \alpha) - u(x, t; a) .
\]

By the mean value theorem,

\[
[L + f'(\eta)]w = 0 \quad \text{in } \Omega,
\]

where \( \eta \) lies between \( u(x + h, t; a + \alpha) \) and \( u(x, t; a) \). By Theorem 3, \( Bw \geq 0 \) on \( S \). Since \( w(x, 0) = 0 \) and \( w(0, t) \geq 0 \), it follows from the strong maximum principle and the parabolic version of Hopf’s lemma that \( w \geq 0 \) on \( \Omega \cup S \). That is,

\[
u(x + h, t; a + \alpha) \geq u(x, t; a) \quad \text{on } \Omega \cup S .
\]

(2.5)

Let us choose positive numbers \( \epsilon \ < c \) and \( t_0 \) such that

\[
f(z) \geq \frac{8\epsilon}{\alpha^2} \left( \frac{2}{\alpha} + \frac{2|b|}{\phi(t_0) + \alpha/4} \right) + \alpha^2
\]

for \( \leq [c - \epsilon, c] \) and \( u(\phi(t_0), t_0; a) \geq c - \epsilon \). Also, let

\[
E = (\phi(t_0) + \alpha/4, \phi(t_0) + \alpha) \times (t_0, \infty) .
\]

By assumption, \( u(x, t; a + \alpha) \) exists for all \( t > 0 \), and hence \( u(x, t; a + \alpha) < c \) in \( E \). From (2.5) and Lemma 1(c), \( u(x, t; a + \alpha) \geq c - \epsilon \) on the parabolic boundary \( \partial E \) of \( E \). Let

\[
z(x, t) = c - \epsilon + [x - \phi(t_0) - \alpha/4][\phi(t_0) + \alpha - x](t - t_0) \quad \text{in } E .
\]
On $\partial E$, $z = c - \epsilon$. By direct computation,

$$Lz = -2(t - t_0) + \frac{b}{x} \{2[\phi(t_0) - x] + 5\alpha/4\}(t - t_0)$$

$$- \{x - \phi(t_0) - \alpha/4\}[(\phi(t_0) + \alpha - x)].$$

In the domain

$$(\phi(t_0) + \alpha/4, \phi(t_0) + \alpha) \times (t_0, t_0 + 8\epsilon/\alpha^2),$$

denoted by $D$, we have for $z \in [c - \epsilon, c)$,

$$Lz + f(z) \geq 0 \quad \text{in } D.$$

By the strong maximum principle, $u(x, t; a + \alpha) > z$ in $D$. Since

$$z(\phi(t_0) + \alpha/2, t_0 + 8\epsilon/\alpha^2) = c,$$

it follows that

$$2w((\phi(t_0) - a/2, t_0 + 8\epsilon/\alpha^2) \geq c.$$

This contradiction proves the theorem.

We remark that Theorem 2 shows that there exists a critical length $a^*$ such that $u$ exists globally if $a < a^*$. This critical length is determined as the supremum of all $a$ for which a solution $U$ of the problem (2.1) exists; if $U(a^*)$ exists, then $u(a^*, t)$ exists also. Theorem 4 shows that (1.2) holds for some finite time $T$ when $a > a^*$.

To compute $a^*$, let us construct a sequence $\{U_n\}$ for $a < a^*$ by $U_0 = 0$ for $0 \leq x \leq a$, and for $n = 1, 2, 3, \ldots$,

$$lU_n + f(U_{n-1}) = 0, \quad U_n(0) = 0 = (W(a). \quad (2.6)$$

In terms of Green's function $G(x; \xi)$ corresponding to $l$, we have

$$U_n(x) = \int_0^a \xi \frac{b} G(x; \xi) f(U_{n-1}(\xi)) d\xi \quad \text{for } n = 1, 2, 3, \ldots, \quad (2.7)$$

where

$$G(x; \xi) = \begin{cases} (1 - \xi)^{1-b})x^{1-b}/(1-b) & \text{for } 0 \leq x \leq \xi, \\
(1 - a)^{1-b})a^{1-b}/(1-b) & \text{for } \xi \leq x \leq a, \end{cases}$$

with $q = k[(1 - b)/a^b + ka^{1-b}]^{-1}$. The sequence is well defined. From (2.7) and the positivity of Green's function, $U_n(x) > 0$ for $n \geq 1$ and $0 < x \leq a$. Since $U_1'(a) < 0$, it follows that $U_n(x)$ attains its positive maximum somewhere in $(0, a)$. With slight modification of the proof of Theorem 5 of Chan and Kaper [6], we obtain the following result.

**Theorem 5.** The sequence $\{U_n\}$ converges monotonically upwards to the minimal solution $U$ ($< c$) of the problem (2.1); furthermore,

$$0 < U_n < U_{n+1} < U, \quad 0 < x \leq a, \quad n = 1, 2, 3, \ldots.$$

The results established in the rest of this section are useful for computational purposes. To obtain an upper bound $a^*_u$ for $a^*$, let us use $U_1(x)$, which is a lower
bound of the solution $U$ of the problem (2.1). From (2.7),

$$U_1(x) = f(0) \left( \frac{a^{1+b} - qa^2}{1 + b} \right) x^{1-b} + \left( \frac{1}{2 - \frac{a}{1+b}} \right) \frac{x^2}{1-b} \quad \text{for } b \neq -1,$$

$$U_1(x) = f(0) \left( \frac{1-qa^2}{4}x^2 + \frac{x^2}{2} \ln \frac{a}{x} \right) \quad \text{for } b = -1.$$  

Differentiating (2.8) with respect to $x$ yields

$$U_1'(x) = f(0) \left( \frac{a^{1+b} - qa^2}{1 + b} \right) x^{b-1} + \left( \frac{1}{2 - \frac{a}{1+b}} \right) \frac{x}{1-b} \quad \text{for } b \neq -1,$$

$$U_1'(x) = f(0) \left( -\frac{qa^2}{2}x + x \ln \frac{a}{x} \right) \quad \text{for } b = -1,$$

from which $U_1'(x) = 0$ occurs at

$$x_c = \left\{ \frac{2a^{1+b} - qa^2(1+b)}{2} \right\}^{1/(1+b)} \quad \text{for } b \neq -1,$$

$$x_c = ae^{-qa^2/2} \quad \text{for } b = -1,$$

where $U''_1 = -f(0) < 0$. This implies that the (absolute) maximum of $U_1(x)$ occurs at the value $x_c$. Thus, an upper bound $a_u$ for $a^*$ is determined by $U_1(x_c) = c$, which yields

$$2a_u^{1+b} - q(1+b)a_u^2 = 2[2(1-b)c/f(0)]^{1+b/2} \quad \text{for } b \neq -1,$$

$$4c = f(0)a_u^2e^{-ka_u/(2+kka_u)}[1 + ka_u/(2 + kka_u)] \quad \text{for } b = -1.$$  

(2.9)

To show that (2.9) determines exactly one $a_u$ for a given $b$, let us differentiate (2.8) with respect to $a$:

$$\frac{\partial U_1}{\partial a} = \frac{q^2f(0)x^{1-b}}{k^2} \left( (1-b)a^{-b} + \frac{k^2}{2}a^{2-b} + \frac{k(2+b-b^2)}{2(1+b)}a^{1-b} \right) \quad \text{for } b \neq -1,$$

$$\frac{\partial U_1}{\partial a} = \frac{f(0)x^2(4 + 3ka + k^2a^2)}{2a(4 + 4ka + k^2a^2)} \quad \text{for } b = -1.$$

In either case, $\partial U_1/\partial a > 0$. Thus, $U_1$ increases as $a$ increases. Hence for a given $b$, $a_u$ is determined uniquely by (2.9). We obtain the following result.

**Lemma 6.** $0 < a^* < a_u$, where $a_u$ is determined uniquely by (2.9) for each given $b$.

Our next result is useful in stopping the computation of successive iterates.

**Lemma 7.** For $0 < x < a$, if $f'$ is strictly increasing and $U_{n+1} - U_n > U_n - U_{n-1}$ for some positive integer $n$, then $U_{m+1} - U_m > U_m - U_{m-1}$ for $m = n + 1, n + 2, n + 3, \ldots$. 

Proof. The sequences \( \{U_n\} \) and \( \{f(U_n)\} \) are strictly increasing. For some \( \eta \) between \( U_{n+1} \) and \( U_n \), and some \( \zeta \) between \( U_n \) and \( U_{n-1} \), we have

\[
U_{n+2}(x) - U_{n+1}(x) = \int_0^a \xi^b G(x; \xi)[f(U_{n+1}(\xi)) - f(U_n(\xi))] \, d\xi
\]

\[
= \int_0^a \xi^b G(x; \xi)f'(\eta)[U_{n+1}(\xi) - U_n(\xi)] \, d\xi
\]

\[
> \int_0^a \xi^b G(x; \xi)f'(\zeta)[U_n(\xi) - U_{n-1}(\xi)] \, d\xi
\]

\[
= U_{n+1}(x) - U_n(x).
\]

The lemma then follows by using mathematical induction.

We now show that each iterate is a unimodal function.

**Lemma 8.** For \( a < a^* \), and each \( n \geq 1 \), the function \( U_n(x) \) has a unique (positive) maximum.

**Proof.** Let \( h \) be a critical point of \( U_n(x) \) \((n \geq 1)\) in the interval \((0, a)\). From (2.6),

\[
U''(h) = -f'(U_n(h)) < 0,
\]

which shows that all critical points of \( U_n(x) \) give relative maxima. Hence, there is exactly one (positive) maximum.

Since \( l(U_{n+1} - U_n) \leq 0 \), a proof similar to Lemma 8 gives the following result.

**Lemma 9.** For \( a < a^* \) and each \( n \geq 0 \), the difference \( U_{n+1}(x) - U_n(x) \) has a unique (positive) maximum.

3. Quenching time. To obtain an upper bound for the quenching time, we may consider the singular Sturm-Liouville problem:

\[
lw = -\lambda^2 w, \quad w(0) = 0, \quad \beta w(a) = 0.
\]

Its eigenvalues \( \lambda^2 \) are determined by

\[
\lambda J_{\nu-1}(\lambda a) + k J_{\nu}(\lambda a) = 0,
\]

where \( \nu = (1 - b)/2 \) and \( J_{\nu}(x) \) is the Bessel function of the first kind of order \( \nu \). The eigenfunction corresponding to the smallest positive eigenvalue \( \mu^2 \) is \( x^\nu J_{\nu}(\mu x) \). Following the argument of Sec. 4 of Chan and Kaper [6], the upper bound \( t_1 \) for the quenching time is determined by

\[
\left[ \max_{0 \leq x \leq a} x^\nu J_{\nu}(\mu x) \right] g(t_1) = c,
\]

where \( g(t) \) is given by the problem

\[
g'(t) + \mu^2 g(t) = G(g(t)), \quad g(0) = 0;
\]

here,

\[
G(g(t)) \leq \inf \left\{ \frac{f(x^\nu J_{\nu}(\mu x) g(t))}{x^\nu J_{\nu}(\mu x)} : x \in [0, a] \right\}.
\]
In particular, for \( f(u) = (1 - u)^{-1} \),
\[
t_1 = \mu^{-1}(4 - \mu^2)^{-1/2}\tan^{-1}\left[\mu(4 - \mu^2)^{-1/2}\right] - (2\mu^2)^{-1}\ln[(4 - \mu^2)/4] + (\ln 2)(4 - \mu^2)^{-1}.
\]

4. Numerical algorithm. By Lemma 6, an upper bound \( a_u \) of \( a^* \) can be determined for each given \( b \) by using the subroutine DZREAL (to find, to double precision, the real zeros of a real function using Muller’s method) from the IMSL MATH/LIBRARY (Version 1.1, January, 1989; MALB-USM-PERFCT-EN8901-1.1). Since \( 0 \) can be taken as a lower bound of \( a^* \), we can use the method of bisection to approximate \( a^* \) by \( a^{**} = a_u/2 \). We use the representation formula (2.7) to compute \( U_n(x) \) with \( n \geq 1 \) by using the following steps:

1. We divide the interval \([0, a^{**}]\) into 20 equal subintervals with end points \( x_i \) satisfying \( 0 = x_1 < x_2 < x_3 < \cdots < x_{21} = a^{**} \).

2. At the 19 interior subdivision points, we evaluate
\[
y_1(x) = x^{1-b}/(1-b), \quad y_2(x) = (1-q)x^{1-b}/(1-b);
\]
we also compute \( y_2(x_{21}) \). These values are stored in the memory of the computer for future use.

3. Let
\[
F_{n1}(j, k) = \int_{x_j}^{x_k} \xi f(U_{n-1}(\xi)) d\xi, \quad F_{n2}(j, k) = \int_{x_j}^{x_k} (\xi^b - q\xi)f(U_{n-1}(\xi)) d\xi.
\]
To save computer time, we evaluate \( U_n(x_{11}) \) first. From (2.7),
\[
U_n(x_{11}) = y_2(x_{11})F_{n1}(1, 11) + y_1(x_{11})F_{n2}(11, 21).
\]
To obtain \( U_n(x_{10}) \), we only need to compute \( F_{n1}(10, 11) \) and \( F_{n2}(10, 11) \) since
\[
U_n(x_{10}) = y_2(x_{10})[F_{n1}(1, 11) - F_{n1}(10, 11)] + y_1(x_{10})[F_{n2}(11, 21) - F_{n2}(10, 11)].
\]
In this way, we can successively compute \( U_n \) at \( x_{10}, x_9, x_8, \ldots, x_2 \). Similarly,
\[
U_n(x_{12}) = y_2(x_{12})[F_{n1}(1, 11) + F_{n1}(11, 12)] + y_1(x_{12})[F_{n2}(11, 21) - F_{n2}(11, 12)].
\]
Proceeding in this way, we obtain successively \( U_n \) at \( x_{12}, x_{13}, x_{14}, \ldots, x_{21} \).

To use a computer to calculate \( U_n(x) \), we use three subroutines from the IMSL MATH/LIBRARY: DCSINT (to compute, to double precision, the cubic spline interpolant with the ‘not-a-knot’ condition) and DQDAG (to integrate, to double precision, a function using a globally adaptive scheme based on Gauss-Kronrod rules) with DCSVAL (to evaluate, to double precision, a cubic spline).

4. We use the subroutine DUVMGS (to find, to double precision, the minimum point of a nonsmooth (unimodal) function of a single variable) to determine \( \max_{0 < x < a^{**}} U_n(x) \) without any initial guesswork of where its critical point is since, by Lemma 8, \( U_n(x) \) is unimodal. Let us denote this maximum value by \( M \).

5. We stop the computation of \( U_n(x) \) as follows:

(a) If \( M \geq c \), then \( a^{**} > a^* \).

(b) If \( U_n - U_{n-1} > U_{n-1} - U_{n-2} \) for some \( n \), then, by Lemma 7, \( a^{**} > a^* \), provided \( f' \) is strictly increasing.
(c) If $M < c$ and (by using Lemma 9)
\[
\max_{0 \leq x \leq a} [U_n(x) - U_{n-1}(x)] < 5 \times 10^{-r+1}
\]
for some arbitrarily chosen nonnegative integer $r$, then $a^{**} < a^*$. Here, $r$ determines the error tolerance in computing the successive iterates.

If $a^{**} > a^*$, then we replace $a_u$ by $a^{**}$; otherwise $u$ exists globally, and we replace 0 by $a^{**}$. The above procedure of bisection is repeated until we reach the demanded accuracy (such as the difference between two successive approximations of $a^*$ is less than $5 \times 10^{-(r+1)}$). Since the difference between $a^*$ and the (ultimate) approximation $a^{**}$ can be made as small as we like, this value $a^{**}$ can be taken numerically to be $a^*$.

We apply the above algorithm to the case $f(u) = (1 - u)^{-1}$ and $k = 1$. We compute critical lengths $a^*$ for various given values of $b$ with the use of a computer. The results with $r = 5$ are given in Table 1.

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<th>$a^*$</th>
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</tr>
<tr>
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**Table 1. Critical lengths $a^*$ for four values of $b$.**

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**References**