HEAT TRANSFER FOR THE FLOW THROUGH A PIPE

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Abstract. The heat flux per unit length through the wall of a straight pipe of arbitrary but uniform cross section is shown to be the product of the constant pressure gradient and the volume flux, when a steady Poiseuille flow of a viscous incompressible fluid is maintained through it, and its wall is kept at a constant temperature. Bounds on the heat flux are obtained using the methods of isoperimetric inequalities.

1. Introduction. Consider the steady Poiseuille flow of a viscous incompressible fluid through a straight pipe of uniform but arbitrary cross section, when its wall is maintained at a constant temperature $T_0$. Taking the $z$-axis along the length of the pipe, neglecting the variations of the coefficient of viscosity $\mu$ and the conductivity $k$ of the fluid with temperature, but taking into account the dissipation of energy due to viscosity, the equations for the velocity $w(x, y)$ along the pipe, and the temperature $T(x, y)$ in it, are [2, p.39]:

$$\nabla^2 w = -P/\mu, \quad (1.1)$$

$$\nabla^2 T = -(\mu/k)(w_x^2 + w_y^2) \quad (1.2)$$

in $S$, while

$$w = 0, \quad T = T_0 \quad \text{on } \partial S. \quad (1.3)$$

Here $S$ is the cross sectional region of the pipe bounded by a closed curve $\partial S$ and $-P$ ($P > 0$) is the constant pressure gradient along the pipe.

It is possible (i) to express the mean temperature in $S$, the mean temperature gradient over $\partial S$, the mean Nusselt number, and the heat flux $H$ across the wall per unit length of the pipe in terms of certain integrals of $w$, without requiring a pointwise solution of (1.2), (1.3), (ii) to obtain the bounds on them, and thus (iii) to develop a qualitative theory of heat transfer. To illustrate this, we obtain an expression for the heat flux $H$ in the next section.

2. The heat flux $H$. This is the most important quantity. It is given by

$$H = \left| \int_{\partial S} k(\partial T/\partial n) \, ds \right|. \quad (2.1)$$
Using (1.1)–(1.3) and the Green’s identities, we obtain

\[ H = k \left| \int_S \nabla^2 T \, ds \right| = \mu \left| \int_S (w_x^2 + w_y^2) \, dS \right| \\
= \mu \left| \int_{\partial S} w (\partial w / \partial n) \, ds - \int_S w \nabla^2 w \, dS \right|. \]

Therefore

\[ H = PQ, \quad (2.2) \]

where

\[ Q = \int_S w \, dS \quad (2.3) \]

is the volume flux of fluid through \( S \). Equation (2.2) is an expression for an obvious energy balance.

3. General results on heat transfer. Since \( w \) and hence \( Q \) is proportional to \( P/\mu \), we may write (2.2) in the form

\[ H = (P^2/\mu)Q, \quad (3.1) \]

where \( q \) is a purely geometric quantity. When \( S \) is simply connected, \( q \) is \( 1/4 \) of the torsional rigidity [1, p. 64]. Thus several standard properties and bounds on torsional rigidity (which is proportional to \( q \)), for example, [1, pp. 64, 67, 150, 152], are directly applicable to (3.1) when \( P^2/\mu \) is fixed. Some of the results thus obtained when \( P, \mu \) are fixed, are given below.

(i) \( H \) is independent of \( k \), the conductivity.
(ii) For a given area of cross section, the circular pipe offers the maximum \( H \).
(iii) \( H \) is an increasing functional of of the domain \( S \) (i.e., \( S_1 \subseteq S_2 = \Rightarrow H_1 \leq H_2 \)).
(iv)

\[ H \geq (P^2 S^2/8 \pi \mu)[1 - 2\beta^2(1 - \beta^2)^{-1} - 4\beta^4(1 - \beta^2)^{-2} \log \beta], \quad (3.2) \]

where \( S \) is the area of \( S \), \( L \) is the length of \( \partial S \), and \( \beta = 1 - 4\pi S/L^2 \). Inequality (3.2) becomes an equality for a circle with \( \beta = 0 \).
(v)

\[ H > P^2 S^3/3L^3 \mu. \quad (3.3) \]

Statements (ii) and (iii) hold for the torsional rigidity, and therefore, in view of (3.1), they hold for \( H \). (iv) and (v) are direct consequences of (3.1) and the corresponding results for the torsional rigidity due to Payne-Weinberger and Polya [1, p. 150]. These results give some lower bounds on \( H \) in terms of the geometric constants of the domain, and the physical constants \( P, \mu \).

3.1. Trap-domains. Since \( Q \) (and hence \( q \)) is explicitly known for several standard domains such as the regions bounded by (i) a circle, (ii) a pair of concentric or eccentric circles, (iii) an ellipse, (iv) an equilateral triangle, and (v) a rectangle [4], a semicircle, and its diameter [3]. \( H \) is known explicitly for these domains from (2.2) or (3.1) without requiring a pointwise solution of (1.2), (1.3). An arbitrary region \( S \) may be trapped between any two such best fitting standard domains \( S_1, S_2 \) in the
sense that the difference of the areas $|S - S_i|$, $i = 1, 2$, is as small as possible and
$S_1 \subseteq S \subseteq S_2$. Then using the result (iii) above upper and lower bounds for $H$ are
easily obtained. For example, when $S_i$ are circles of radii $r_i$, $i = 1, 2$, so that $2r_1$
and $2r_2$ indicate ‘the width’ and the ‘length’ of $S$ respectively,

$$(\pi P^2 / 8\mu) r_1^4 \leq H \leq (\pi P^2 / 8\mu) r_2^4.$$ (3.4)

Obviously the upper bound in (3.4) is never better than the isoperimetric bound given
by result (ii) above.

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