QUARTERLY OF APPLIED MATHEMATICS
VOLUME L, NUMBER 1
MARCH 1992, PAGES 57-71

FORCED OSCILLATIONS OF ELASTIC STRINGS WITH NONLINEAR DAMPING

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1. Introduction. In this paper we discuss the question of existence and numerical approximation of periodic solutions of nonlinear elastic string vibrations in the presence of damping. Starting with an analysis of free undamped motion in [2] there had been a significant amount of work on related questions of forced and free vibrations in the engineering literature. In [6] the authors presented an analysis which leads to a pair of coupled nonlinear partial differential equations for the two components of the transverse motion of the string. As is well known the classical theory of small amplitude vibration of stretched strings neglects the change in tension of the string. In [6] the authors discuss how this is not proper in the context of forced oscillations near resonant frequencies. The authors in [6] then proceed to obtain approximate solutions to describe the resonance response of the string, in particular of the jump and hysteresis phenomena and of the tubular motion.

We will consider the equations of motion derived in [6] to describe the two components of transverse motion from the point of view of existence and numerical approximation of solutions with the same period as the forcing, i.e., the resonant solutions. While the method of approximation of solutions in [6] is more in the spirit of Fourier expansions and harmonic balance, we will present a method based on describing the steady state or stationary periodic solution as the evolution from arbitrary initial data.

Thus we will look at the sequence of initial-boundary value problems as approximations to the periodic problem. We will then use a finite difference scheme to approximate the system of partial differential equations and use the nice structure of the nonlinearities to solve the discrete equations at each time step.

It must be noted here that we are allowing for damping to be present in the system as proposed in [6]. We however allow the damping to be a general nonlinearity which allows us to consider forms of damping besides the viscous damping studied in [6]. If we disregard the second component in transverse motion all of the equations referred to in [2, 6] reduce the equation derived in [9]. We would like to point out here that a different equation for planar motion was derived in [8]. Reference must be made

Received August 9, 1990.

here to the interesting article [8] in which approximate equations to the full problem were obtained by using a perturbation argument. We also mention the work of [4] where the stability theory for these problems is discussed.

2. Equations of motion. We first derive, for the sake of completeness, the equations of motion as in [6]. Thus let T_0 be the tension in the string in equilibrium position, A the area of cross section, $m = CA$ the mass per unit length, and E the Young's modulus. Before we derive the equations of [6], let us first recall Carrier's general equations of motion [2]. Thus if the element of length dx in the equilibrium position has a vertical displacement $v = v(x, t)$ at end A and displacements $u + u_x dx$ and $v + v_x dx$ at B then the local tension is

$$T(x) = T_0 + EA\left(\frac{ds - dx}{dx}\right), \tag{2.1}$$

where ds is the new length of the element dx. Thus

$$T(x) = T_0 + EA\left\{\sqrt{(1 + v_x)^2 + u_x^2} - 1\right\} \tag{2.2}$$

and the equations of motion are

$$CAu_{tt} = \frac{\partial}{\partial x}(T \sin\theta) \tag{2.3}$$

and

$$CAv_{tt} = \frac{\partial}{\partial x}(T \cos\theta), \tag{2.4}$$

where

$$\theta = \tan^{-1}\left[\frac{u_x}{1 + v_x}\right]. \tag{2.5}$$

Assuming $v = 0$ and u_x small, but allowing for tension variation we would have

$$T(x) = T_0 + \frac{EA}{2}u_x^2 \quad \text{and} \quad \sin\theta = u_x;$$

substituting these expressions for $T(x)$ and $\sin\theta$ in (2.3) we get

$$u_{tt} = \frac{T_0}{m}u_{xx} + \frac{3}{2}\frac{EA}{m}u_x^2 u_{xx},$$

which was the equation studied in [2].

In [6] the authors assume, as in [2] above, that $v = 0$ and $\tan\theta$ is small. However, considering a deformation of the string into a space curve we proceed as follows.

Let dx be an element that is deformed into the space element ds. Assuming longitudinal displacements of the ends to be negligible

$$ds = (dx^2 + dy^2 + dz^2)^{1/2} = dx[1 + \tfrac{1}{2}(y_x^2 + z_x^2) - \tfrac{1}{8}(y_x^2 + z_x^2)^2 + \cdots].$$

Thus $ds - dx = [\frac{1}{2}(y_x^2 + z_x^2) - \frac{1}{8}(y_x^2 + z_x^2)^2 + \cdots]dx$. The local tension T is given by

$$\begin{aligned} T &= T_0 + EA\left(\frac{ds - dx}{dx}\right) = T_0 + EA\lambda \\ &= T_0 + EA[\tfrac{1}{2}(y_x^2 + z_x^2) - \tfrac{1}{8}(y_x^2 + z_x^2)^2 + \cdots]. \end{aligned}$$

The potential energy of the element dx is thus

$$\left(T_0 + \frac{1}{2}EA\lambda\right)\lambda\, dx = \left[\frac{1}{2}T_0(y_x^2 + z_x^2) + \frac{EA - T_0}{8}(y_x^2 + z_x^2)^2\right].$$

Usually T_0 is very small compared to EA and is thus dropped from the second term on the right. Let $f(x)\cos\omega t$ be the external force per unit length acting in the y direction. Then the potential energy of the string for an arbitrary deformation is

$$U = \int_0^l \left[\frac{1}{2}T_0(y_x^2 + z_x^2) + \frac{EA}{8}(y_x^2 + z_x^2)^2 + yf(x)\cos\omega t\right] dx.$$

The kinetic energy is

$$T = \frac{1}{2}m\int_0^l (y_t^2 + z_t^2)\, dx.$$

By Hamilton's principle, the first variation $\delta\int_0^t (T - U)\, dt = 0$ and one can see that $y(x, t)$ and $z(x, t)$ must satisfy the Euler equations given by

$$y_{tt} - c_0^2 y_{xx} - \frac{3}{2}c_1^2 y_{xx}y_x^2 - \frac{1}{2}c_1^2\frac{\partial}{\partial x}(y_x z_x^2) = \frac{1}{m}f(x)\cos\omega t, \tag{2.6}$$

$$z_{tt} - c_0^2 z_{xx} - \frac{3}{2}c_1^2 z_{xx}z_x^2 - \frac{1}{2}c_1^2\frac{\partial}{\partial x}(z_x y_x^2) = 0, \tag{2.7}$$

where

$$c_0 = \left(\frac{T_0}{m}\right)^{1/2} \quad \text{and} \quad c_1 = \left(\frac{EA}{m}\right)^{1/2}.$$

Finally, it must be remarked, as noted in [6], that near resonance y_x and z_x are appreciable and the nonlinear terms cannot be ignored. Further the longitudinal velocity c_1 is much larger than the transverse velocity c_0.

3. Existence of periodic solutions to the coupled equations. Following the ideas in [3], we now demonstrate the existence of a unique periodic solution to the nonlinearly coupled system of equations analogous to (2.6), (2.7) in the presence of damping. Thus we have the system of equations

$$u_{tt} - u_{xx} - u_x^2 u_{xx} - (u_x v_x^2)_x + \beta(u_t) = p(x, t), \tag{3.1}$$

$$v_{tt} - v_{xx} - v_x^2 v_{xx} - (v_x u_x^2)_x + \gamma(u_t) = q(x, t), \tag{3.2}$$

with the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad u(x, t) = u(x, t + 2\pi), \tag{3.3}$$

$$v(0, t) = v(\pi, t) = 0, \quad v(x, t) = v(x, t + 2\pi). \tag{3.4}$$

The hypotheses on p, q, β, and γ are as follows:

(i) $p(x, t)$ and $q(x, t)$ are defined for $0 < x < \pi$, $-\infty < t < \infty$ and are 2π-periodic in t; $p, q \in L^2([0, \pi] \times [0, 2\pi])$;

(ii) β and γ are strictly monotone increasing continuous functions;

(iii) there exist constants a and b such that

$$\xi\beta(\xi) \geq a|\xi|^{m+1}, \qquad \xi\gamma(\xi) \geq b|\xi|^{m+1} \tag{3.5}$$

for some $m > 1$ and $\xi \in \mathbf{R}$;

(iv) there exist constants c_i and d_i such that

$$|\beta(\xi)| \le c_1 + d_1|\xi|^m, \qquad |\gamma(\xi)| \le c_2 + d_2|\xi|^m \tag{3.6}$$

for $\xi \in \mathbf{R}$.

Let D denote the set of all functions $u(x, t)$, $0 \le x \le \pi$, $-\infty < t < \infty$, which are of class C^∞ and satisfy $u(x, t+2\pi) = u(x, t)$, $u(0, t) = u(\pi, t) = 0$. Further let A_{11} denote the completion of D by means of the norm

$$||u||^2_{A_{11}} = ||u_t||^2 + ||u_x||^2,$$

where $||\cdot||$ denotes the norm in $L_2([0, \pi] \times [0, 2\pi])$ (we will use L_2 for the rest of this paper).

The space L_2 has a complete orthonormal system given by

$$\phi_{kl} = 2^{-1/2}\pi^{-1}\cos kt \sin lx, \qquad \psi_{kl} = 2^{-1/2}\pi^{-1}\sin kt \sin lx,$$
$$\lambda_l(x) = 2^{-1/2}\pi^{-1/2}\sin lx$$

for $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$.

Let S_0 be the subspace generated by $\lambda_l(x)$, $l = 1, 2, \ldots$, and S_1 the orthogonal complement of S_0 in L_2 generated by ϕ_{kl} and ψ_{kl}, $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$.

Any pair of elements $u, v \in L_2$ has a Fourier series given by

$$u(x, t) = u_1(x, t) + w_1(x), \qquad v(x, t) = v_1(x, t) + w_2(x), \tag{3.7}$$

where $u_1, v_1 \in S_1$ and $w_1, w_2 \in S_0$.

We can now define $P: L_2 \to S_0$ to be the projection operator defined by $Pu = w$ for $u \in L_2$ and w given by (3.7).

Finally let $R_n: L_2 \to L_2$ be the truncation operator which truncates the Fourier series for any $u \in L_2$ by deleting all the terms with $k > n$ or $l > n$ or both. If we now consider the truncated problems

$$R_n(u_{tt} - u_{xx} - u_x^2 u_{xx} - (u_x v_x^2)_x + \beta(u_t) - p) = 0,$$
$$R_n(v_{tt} - v_{xx} - v_x^2 v_{xx} - (v_x u_x^2)_x + \gamma(v_t) - q) = 0,$$

we get the corresponding equivalent system of equations:

$$(I - P)R_n[u_{tt} - u_{xx} - u_x^2 u_{xx} - (u_x v_x^2)_x + \beta(u_t) - p] = 0, \tag{3.8}$$

$$PR_n[u_{tt} - u_{xx} - u_x^2 u_{xx} - (u_x v_x^2)_x + \beta(u_t) - p] = 0, \tag{3.9}$$

$$(I - P)R_n[v_{tt} - v_{xx} - v_x^2 v_{xx} - (v_x u_x^2)_x + \gamma(v_t) - q] = 0, \tag{3.10}$$

$$PR_n[v_{tt} - v_{xx} - v_x^2 v_{xx} - (v_x u_x^2)_x + \gamma(v_t) - q] = 0. \tag{3.11}$$

We now outline the various steps in proving the existence of a periodic solution to (3.1)–(3.2).

Step I. Using the notations of (3.7) and dropping the subscript n for the sake of simplicity, we first consider the system of auxiliary equations, in order to apply the Leray-Schauder principle, given by

$$u_{1tt} - u_{1xx} + (1-\lambda)u_{1t} + \lambda(I-P)R[\beta(u_{1t}) - u_x^2 u_{xx} - (u_x v_x^2)_x - p] = 0, \tag{3.12}$$

$$-w_{1xx} + \lambda PR[\beta(u_{1t}) - u_x^2 u_{xx} - (u_x v_x^2)_x - p] = 0, \tag{3.13}$$

$$v_{1tt} - v_{1xx} + (1-\lambda)v_{1t} + \lambda(I-P)R[\gamma(v_{1t}) - v_x^2 v_{xx} - (u_x v_x^2)_x - q] = 0, \tag{3.14}$$

$$-w_{2xx} + \lambda PR[\gamma(v_{1t}) - v_x^2 v_{xx} - (u_x^2 v_x)_x - q] = 0. \tag{3.15}$$

Here u_1, u_t are 2π-periodic in t and all of u_1, w_t, w_1, and w_2 are zero at $x = 0$ and $x = \pi$.

For $\lambda \in (0, 1)$ we first show that all possible solutions of (3.12)–(3.15) are bounded in A_{11}-norm independent of λ.

Step II. We note that when $\lambda = 0$, (3.12)–(3.15) has only the trivial solution $u_1 = v_1 = w_1 = w_2 = 0$.

Step III. By the Leray-Schauder principle we conclude that (3.12)–(3.15) has a solution for $\lambda = 1$ and in this case (3.12)–(3.15) coincides with (3.8)–(3.11) (which depends on n).

Step IV. Having solved (3.8)–(3.11) for each n, we now show that the corresponding solutions $u_n = u_{1n} + w_{1n}$, $v_n = v_{1n} + w_{2n}$ have a subsequence that converges to a solution (u, v) of (3.1)–(3.2).

Step V. We now show that if (3.1)–(3.2) has a solution it must be unique.

We now outline the proofs of Steps I, IV, and V. Multiplying (3.12) and (3.14) by u_{1t} and v_{1t} respectively and integrating over $[0, \pi] \times [0, 2\pi]$ we get (using the notation $\langle\ ,\ \rangle$ for the inner product),

$$(1-\lambda)\|u_{1t}\|^2 + \lambda(I-P)R\langle \beta(u_{1t}) - u_x^2 u_{xx} - (u_x v_x^2)_x - p, u_{1t}\rangle = 0, \tag{3.16}$$

$$(1-\lambda)\|v_{1t}\|^2 + \lambda(I-P)R\langle \gamma(v_{1t}) - v_x^2 v_{xx} - (v_x u_x^2)_x - q, v_{1t}\rangle = 0. \tag{3.17}$$

(It is easy to see that $\langle u_{1tt}, u_{1t}\rangle = \langle u_{1xx}, u_{1t}\rangle = 0$ and similarly for v.) Using (3.5) in (3.16) and (3.17) we get

$$a\|u_{1t}\|_{m+1}^{m+1} - \langle u_x^2 u_{xx}, u_{1t}\rangle - \langle (u_x v_x^2)_x, u_{1t}\rangle \le \langle p, u_{1t}\rangle, \tag{3.18}$$

$$b\|v_{1t}\|_{m+1}^{m+1} - \langle v_x^2 v_{xx}, v_{1t}\rangle - \langle (v_x v_x^2)_x, v_{1t}\rangle \le \langle q, v_{1t}\rangle. \tag{3.19}$$

Adding two equations and using Dirichlet boundary conditions in x and periodic boundary conditions in t, it follows that

$$\|u_{1t}\|_{m+1} \quad \text{and} \quad \|v_{1t}\|_{m+1} \tag{3.20}$$

are bounded independent of n. (Note that we dropped the subscript n at the beginning of Step I.) This implies that

$$\|u_{1t}\| \quad \text{and} \quad \|v_{1t}\| \tag{3.21}$$

are bounded independent of n.

For every x, $u_1(x, t)$ and $v_1(x, t)$ have mean value zero as a function of t. Hence

$$u_1(x, t) = \int_\xi^t u_{1t}(x, \tau)\, d\tau, \qquad 0 < x < \pi,\ 0 \le \xi, t \le 2\pi, \tag{3.22}$$

and a similar equality for v_1.

Using (3.22) in conjunction with (3.20) we get

$$\|u_1\|_{m+1}^{m+1} = \int_0^{2\pi}\int_0^{\pi} |u_1(x, t)|^{m+1}\, dx\, dt \le \int_0^{2\pi}\int_0^{\pi}\left[\int_0^{2\pi} |u_{1t}(x, \tau)|^{m+1}\, d\tau\right] dx\, dt.$$

This implies

$$\|u_1\|_{m+1} \quad \text{and similarly} \quad \|v_1\|_{m+1} \tag{3.23}$$

are bounded independent of n.

Proceeding as we did from (3.21) to (3.22) we have

$$\|u_1\| \text{ and } \|v_1\| \text{ are bounded independent of } n. \tag{3.24}$$

Note that in (3.20) and hence in all subsequent estimates the bound is dependent only on $p(x, t)$ and $q(x, t)$.

We now obtain bounds on u_{1x} and v_{1x}. To this end we multiply (3.12) by u_1 and (3.13) by w_1 and integrate both of the equations over $[0, 2\pi] \times [0, \pi]$. Recall that

$$\langle u_{1tt}, u_1\rangle = -\|u_{1t}\|^2, \qquad \langle u_{1xx}, u_1\rangle = -\|u_{1x}\|^2,$$
$$\langle w_{1xx}, w_1\rangle = -\|w_{1x}\|^2, \qquad -\langle w_{1xx}, w_1\rangle \ge \|w_1\|^2.$$

Also from (3.21) and (3.24), $\|u_{1t}\|$, $\|v_{1t}\|$, $\|u_1\|$, and $\|v_1\|$ are bounded. Then from (3.12) and (3.13) we get

$$\begin{aligned} &-\|u_{1t}\|^2 + \|u_{1x}\|^2 + (1-\lambda)\langle u_{1t}, u_1\rangle + \lambda(I-P) \\ &\qquad \times R[\langle \beta(u_{1t}), u_1\rangle - \langle u_x^2 u_{xx}, u_1\rangle - \langle (u_x v_x^2)_x, u_1\rangle - \langle p, u_1\rangle] = 0, \end{aligned} \tag{3.25}$$

$$\|w_1\|^2 + \lambda P R[\langle \beta(u_{1t}), w_1\rangle - \langle u_x^2 u_{xx}, w_1\rangle - \langle (u_x v_x^2)_x, w_1\rangle - \langle p, w_1\rangle] \le 0, \tag{3.26}$$

with similar equations resulting from (3.14) and (3.15) when multiplied by v_1 and w_2 and integrated.

Adding (3.25) and (3.26), we obtain

$$\|u_{1x}\|^2 + \|w_1\|^2 - \|u_{1t}\|^2 + \lambda\langle \beta(u_{1t}), u\rangle - \langle u_x^2 u_{xx}, u\rangle - \langle (u_x v_x^2)_x, u\rangle - \langle p, u\rangle] \le 0, \tag{3.27}$$

where $u = u_1 + w_1$.

Now

$$-\langle u_x^2 u_{xx}, u\rangle = \langle u_x^3, u_x\rangle \ge 0,$$
$$-\langle (u_x v_x^2)_x, u\rangle = \langle u_x v_x^2, u_x\rangle \ge 0.$$

Using that $\|u_{1t}\|$ and $\|u_1\|$ are bounded and the growth condition (3.6) on β we can conclude from (3.27) that

$$\|u_{1x}\| \text{ and } \|w_1\| \text{ are bounded independent of } n. \tag{3.28}$$

Also from (3.13) and $\langle -w_{1xx}, w_1\rangle = \|w_{1x}\|^2$ we could have obtained, instead of (3.26),

$$\|w_{1x}\|^2 + \lambda PR[\langle \beta(u_{1t}), w_1\rangle - \langle u_x^2 u_{xx}, w_1\rangle - \langle (u_x v_x^2)_x, w_1\rangle - \langle p, w_1\rangle] = 0. \quad (3.29)$$

Adding (3.25) and (3.29) and using that $\|u_{1t}\|$, $\|u_1\|$, and $\|w_1\|$ are bounded we conclude that

$$\|w_{1x}\| \text{ is bounded independent of } n. \quad (3.30)$$

From (3.21), (3.24), (3.28), and (3.30) it follows that

$$\|u\|_{A_{11}} \le \|u_1\|_{A_{11}} + \|w_1\|_{A_{11}} = \{\|u_{1t}\|^2 + \|u_{1x}\|^2\}^{1/2} + \{\|w_{1t}\|^2 + \|w_{1x}\|^2\}^{1/2}$$

and hence

$$\|u\|_{A_{11}} \text{ is bounded independent of } n,$$

and similarly

$$\|v\|_{A_{11}} \text{ is bounded independent of } n.$$

This proves Step I, i.e., all possible solutions of (3.12)–(3.15) are bounded independent of n.

By the Leray-Schauder principle (3.8)–(3.11) has solutions and all such solutions are bounded independent of n and $\{v_n\}$.

A passage to the limit argument now ensures that there exist periodic solutions to (3.1) and (3.2) which are limit points of the sequences $\{u_n\}$ and $\{v_n\}$. This step follows standard arguments of passing through subsequences and closure type properties. The key assumption is that β is strictly monotone and continuous.

Finally we establish the uniqueness of the solution to (3.1)–(3.2). Thus, let (u^i, v^i), $i = 1, 2$, be two sets of solutions to (3.1)–(3.2), i.e.,

$$u^i_{tt} - u^i_{xx} - u^{i^2}_x u^i_{xx} - (u^i_x v^{i^2}_x)_x + \beta(u^i_t) = p(x, t), \quad (3.31)$$

$$v^i_{tt} - v^i_{xx} - v^{i^2}_x v^i_{xx} - (v^i_x u^{i^2}_x)_x + \gamma(v^i_t) = q(x, t). \quad (3.32)$$

Adding these two equations and setting $A^i = u^i + v^i$, we get

$$A^i_{tt} - A^i_{xx} - \{u^{i^2}_x u^i_{xx} + v^{i^2}_x v^i_{xx} + (u^{i^2}_x v^i_x)_x + (u^i_x v^{i^2}_x)_x\} + \beta(u^i_t) + \gamma(v^i_t) = p + q$$

or

$$A^i_{tt} - A^i_{xx} - A^i_{xx} A^{i^2}_x + \beta(u^i_t) + \gamma(v^i_t) = p + q. \quad (3.33)$$

Similarly subtracting (3.32) from (3.31) and setting $B^i = u^i - v^i$ we get

$$B^i_{tt} - B^i_{xx} - B^{i^2}_x B^i_{xx} + \beta(u^i_t) - \gamma(v^i_t) = p - q. \quad (3.34)$$

We now write (3.33) for $i = 1$ and $i = 2$, subtract these two equations, multiplying the difference by $A^1_t - A^2_t$, and integrate over $[0, \pi] \times [0, 2\pi]$. Using the periodicity of A^i with respect to t and the property that the A^i satisfy Dirichlet boundary conditions in x we get

$$-\langle A^1_{xx} A^{1^2}_x - A^2_{xx} A^{2^2}_x, A^1_t - A^2_t\rangle + \langle \beta(u^1_t) + \gamma(v^1_t) - \beta(u^2_t) - \gamma(v^2_t), A^1_t - A^2_t\rangle = 0.$$

Simplifying

$$\langle \beta(u_t^1) + \gamma(v_t^1) - \beta(u_t^2) - \gamma(v_t^2), u_t^1 - u_t^2 + v_t^1 - v_t^2 \rangle = 0. \tag{3.35}$$

Similarly, from (3.34), we get

$$\langle \beta(u_t^1) - \gamma(v_t^1) - \beta(u_t^2) + \gamma(v_t^2), u_t^1 - u_t^2 - v_t^1 + v_t^2 \rangle = 0. \tag{3.36}$$

Adding (3.35) and (3.36) we have

$$\langle \beta(u_t^1) - \beta(u_t^2), u_t^1 - u_t^2 \rangle = 0, \tag{3.37}$$

$$\langle \gamma(v_t^1) - \gamma(v_t^2), v_t^1 - v_t^2 \rangle = 0. \tag{3.38}$$

The strict increasing nature of β and γ as assumed at the beginning of this section implies that $u_t^1 = u_t^2$ and $v_t^1 = v_t^2$. Now repeating the above steps with (3.33) and (3.34), excepting that we take inner products with A_x^i, B_x^i and so on, we conclude that the solution to (3.1)–(3.2) is unique.

4. Discrete approximations. In this section, we develop discrete approximations of the system

$$u_{tt} - c_0^2 u_{xx} - \tfrac{3}{2} c_1^2 u_x^2 u_{xx} - \tfrac{1}{2} c_1^2 \frac{\partial}{\partial x}(u_x v_x^2) + \beta(u_t) = f_1(x, t), \tag{4.1}$$

$$v_{tt} - c_0^2 v_{xx} - \tfrac{3}{2} c_1^2 v_x^2 v_{xx} - \tfrac{1}{2} c_1^2 \frac{\partial}{\partial x}(u_x^2 v_x) + \gamma(u_t) = f_2(x, t), \tag{4.2}$$

subject to

$$u(0, t) = u(\pi, t) = 0, \qquad u(x, t) = u(x, t + 2\pi), \tag{4.3}$$

$$v(0, t) = v(\pi, t) = 0, \qquad v(x, t) = v(x, t + 2\pi). \tag{4.4}$$

We begin our presentation of the numerical method by discussing finite difference approximations for the derivatives which occur. We employ the standard second-order centered differences in t. Thus, at the point (x_i, t_j) we have

$$u_{tt} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2} + O(k^2)$$

and

$$u_t = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2),$$

where $u_{ij} = u(x_i, t_j) = u(ih, (j-1)k)$. For the derivatives in x, we employ averaged centered differences to create an implicit system at each time level. To fix the ideas, consider the term

$$-c_0^2 u_{xx} - \tfrac{3}{2} c_1^2 u_x^2 u_{xx} = -\frac{\partial}{\partial x}[c_0^2 u_x + \tfrac{1}{2} c_1^2 (u_x)^3].$$

We may approximate $u_x(x, t)$ by

$$u_x(x_i, t_j) = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{h} + O(h^2).$$

Thus

$$-\frac{\partial}{\partial x}\left[c_0^2 u_x + \frac{1}{2}c_1^2 (u_x)^3\right]$$
$$= -\frac{\partial}{\partial x}\left[c_0^2 \frac{u_{i+1/2,j} - u_{i-1/2,j}}{h} + \frac{1}{2}c_1^2\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{h}\right)^3\right] + O(h^2). \quad (4.5)$$

The outer derivative in (4.5) is approximated in a similar way. We see that

$$\begin{aligned}\frac{\partial}{\partial x}\left[c_0^2 \frac{u_{i+1/2,j} - u_{i-1/2,j}}{h}\right] &= \frac{c_0^2}{h}\left[\frac{\partial}{\partial x}(u_{i+1/2,j} - u_{i-1/2,j})\right] \\ &= \frac{c_0^2}{h}\left[\frac{u_{i+1,j} - u_{ij}}{h} - \frac{u_{ij} - u_{i-1,j}}{h}\right] + O(h^2) \\ &= c_0^2\left[\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}\right] + O(h^2). \qquad (4.6)\end{aligned}$$

Note that the final expression in (4.6) is just the standard second-order centered difference $c_0^2 u_{xx} = c_0^2 \frac{\partial}{\partial x}(u_x)$. Similarly,

$$\begin{aligned}\frac{\partial}{\partial x}\left[\frac{1}{2}c_1^2\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{h}\right)^3\right] &= \frac{c_1^2}{2h^3}\left[\frac{\partial}{\partial x}(u_{i+1/2,j} - u_{i-1/2,j})^3\right] \\ &= \frac{c_1^2}{2h^3}\left[\frac{(u_{i+1/2,j} - u_{ij})^3 - (u_{ij} - u_{i-1,j})^3}{h}\right] + O(h^2) \\ &= \frac{c_1^2}{2h^4}[(u_{i+1/2,j} - u_{ij})^3 - (u_{ij} - u_{i-1,j})^3] + O(h^2). \qquad (4.7)\end{aligned}$$

Finally, we approximate $\frac{\partial}{\partial x}(u_x v_x^2)$ by

$$\begin{aligned}\frac{\partial}{\partial x}(u_x v_x^2) &= \frac{\partial}{\partial x}\left[\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{h}\right)\left(\frac{v_{i+1/2,j} - v_{i-1/2,j}}{h}\right)^2\right] + O(h^2) \\ &= \frac{1}{h^3}\frac{\partial}{\partial x}[(u_{i+1/2,j} - u_{i-1/2,j})(v_{i+1/2,j} - v_{i-1/2,j})^2] + O(h^2) \\ &= \frac{1}{h^3}\left[\frac{(u_{i+1,j} - u_{ij})(v_{i+1,j} - v_{ij})^2 - (u_{ij} - u_{i-1,j})(v_{ij} - v_{i-1,j})^2}{h}\right] + O(h^2) \\ &= \frac{1}{h^4}[(u_{i+1,j} - u_{ij})(v_{i+1,j} - v_{ij})^2 - (u_{ij} - u_{i-1,j})(v_{ij} - v_{i-1,j})^2] + O(h^2). \qquad (4.8)\end{aligned}$$

If we replace the derivatives in (4.1) by the differences given above (and with differences in (4.6), (4.7), and (4.8) averaged between time lines $j+1$ and $j-1$) we

obtain

$$
\begin{aligned}
&\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{k^2} \\
&\quad -\frac{c_0^2}{2}\left[\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}+\frac{u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}}{h^2}\right] \\
&\quad -\frac{c_1^2}{4h^4}\{[(u_{i+1,j+1}-u_{i,j+1})^3-(u_{i,j+1}-u_{i-1,j+1})^3] \\
&\qquad\qquad +[(u_{i+1,j-1}-u_{i,j-1})^3-(u_{i,j-1}-u_{i-1,j-1})^3]\} \\
&\quad -\frac{c_1^2}{4h^4}\{[(u_{i+1,j+1}-u_{i,j+1})(v_{i+1,j+1}-v_{i,j+1})^2 \\
&\qquad\qquad -(u_{i,j+1}-u_{i-1,j+1})(v_{i,j+1}-v_{i-1,j+1})^2] \\
&\qquad\qquad +[(u_{i+1,j-1}-u_{i,j-1})(v_{i+1,j-1}-v_{i,j-1})^2 \\
&\qquad\qquad\qquad -(u_{i,j-1}-u_{i-1,j-1})(v_{i,j-1}-v_{i-1,j-1})^2]\} \\
&\quad +\beta\left(\frac{u_{i,j+1}-u_{i,j-1}}{2k}\right)=f_{1ij}. \qquad (4.9)
\end{aligned}
$$

After multiplication of (4.9) by h^2, rearrangement yields

$$
\begin{aligned}
&-\frac{c_0^2}{2}u_{i+1,j+1}+\left(c_0^2+\frac{h^2}{k^2}\right)u_{i,j+1}-\frac{c_0^2}{2}u_{i-1,j+1}+h^2\beta\left(\frac{u_{i,j+1}-u_{i,j-1}}{2k}\right) \\
&\quad -\frac{c_1^2}{4h^2}[(u_{i+1,j+1}-u_{i,j+1})^3-(u_{i,j+1}-u_{i-1,j+1})^3] \\
&\quad -\frac{c_1^2}{4h^2}[(u_{i+1,j+1}-u_{i,j+1})(v_{i+1,j+1}-v_{i,j+1})^2 \\
&\qquad\qquad -(u_{i,j+1}-u_{i-1,j+1})(v_{i,j+1}-u_{i-1,j+1})^2] \\
&\quad -\left\{2\frac{h^2}{k^2}u_{ij}-\left(c_0^2+\frac{h^2}{k^2}\right)u_{i,j-1}+\frac{c_0^2}{2}(u_{i+1,j-1}+u_{i-1,j-1})\right. \\
&\qquad +\frac{c_1^2}{4h^2}[(u_{i+1,j-1}-u_{i,j-1})^3-(u_{i,j-1}-u_{i-1,j-1})^3] \\
&\qquad +\frac{c_1^2}{4h^2}[(u_{i+1,j-1}-u_{i,j-1})(v_{i+1,j-1}-v_{i,j-1})^2 \\
&\qquad\qquad \left.-(u_{i,j-1}-u_{i-1,j-1})(v_{i,j-1}-v_{i-1,j-1})^2]+h^2f_{1ij}\right\}=0. \qquad (4.10)
\end{aligned}
$$

To obtain the discrete analog of (4.2), modify (4.10) by interchanging u and v,

replacing β with γ, and replacing f_{1ij} with f_{2ij}. Thus, we obtain

$$
\begin{aligned}
&-\frac{c_0^2}{2}v_{i+1,j+1}+\left(c_0^2+\frac{h^2}{k^2}\right)v_{i,j+1}-\frac{c_0^2}{2}v_{i-1,j+1}+h^2\gamma\left(\frac{v_{i,j+1}-v_{i,j-1}}{2k}\right)\\
&\quad-\frac{c_1^2}{4h^2}[(v_{i+1,j+1}-v_{i,j+1})^3-(v_{i,j+1}-v_{i-1,j+1})^3]\\
&\quad-\frac{c_1^2}{4h^2}[(v_{i+1,j+1}-v_{i,j+1})(u_{i+1,j+1}-v_{i,j+1})^2\\
&\qquad\qquad-(v_{i,j+1}-v_{i-1,j+1})(u_{i,j+1}-u_{i-1,j+1})^2]\\
&\quad-\left\{2\frac{h^2}{k^2}v_{ij}-\left(c_0^2+\frac{h^2}{k^2}\right)v_{i,j-1}+\frac{c_0^2}{2}(v_{i+1,j-1}+v_{i-1,j-1})\right.\\
&\qquad\qquad+\frac{c_1^2}{4h^2}[(v_{i+1,j-1}-v_{i,j-1})^3-(v_{i,j-1}-v_{i-1,j-1})^3]\\
&\qquad\qquad+\frac{c_1^2}{4h^2}[(v_{i+1,j-1}-v_{i,j-1})(u_{i+1,j-1}-u_{i,j-1})^2\\
&\qquad\qquad\left.-(v_{i,j-1}-v_{i-1,j-1})(u_{i,j-1}-u_{i-1,j-1})^2]+h^2f_{2ij}\right\}=0. \quad (4.11)
\end{aligned}
$$

We may now write $g_{1ij}=0$ and $g_{2ij}=0$, where g_{1ij} is defined to be the left-hand side of (4.10) and g_{2ij} is defined to be the left-hand side of (4.11). We assume that u and v are known at the grid points (x_i, t_{j-1}) and (x_i, t_j) $(i=1,2,\dots,N)$. Therefore, the unknowns at each time step are the values $u_{i,j+1}$ and $v_{i,j+1}$ $(i=1,2,\dots,N)$ and all terms inside the "curly brackets" in (4.10) and (4.11) are considered known. We may solve for the $u_{i,j+1}$ and $v_{i,j+1}$ at each time step by the generalized Newton method. That is, the system can be solved iteratively by

$$u_{i,j+1}^{(n+1)}=u_{i,j+1}^{(n)}-\omega\frac{g_{1ij}}{\partial g_{1ij}/\partial u_{i,j+1}}, \tag{4.12}$$

$$v_{i,j+1}^{(n+1)}=v_{i,j+1}^{(n)}-\omega\frac{g_{2ij}}{\partial g_{2ij}/\partial v_{i,j+1}}, \tag{4.13}$$

where ω is a relaxation parameter $(0<\omega<2)$,

$$
\begin{aligned}
\frac{\partial g_{1ij}}{\partial u_{i,j+1}}&=\left(c_0^2+\frac{h^2}{k^2}\right)+\frac{h^2}{2k}\beta'\left(\frac{u_{i,j+1}-u_{i,j-1}}{2k}\right)\\
&\quad+\frac{3c_1^2}{4h^2}[(u_{i+1,j+1}-u_{i,j+1})^2+(u_{i,j+1}-u_{i-1,j+1})^2]\\
&\quad+\frac{c_1^2}{4h^2}[(v_{i+1,j+1}-v_{i,j+1})^2+(v_{i,j+1}-v_{i-1,j+1})^2]
\end{aligned}
$$

and

$$\frac{\partial g_{2ij}}{\partial v_{i,j+1}} = \left(c_0^2 + \frac{h^2}{k^2}\right) + \frac{h^2}{2k}\gamma'\left(\frac{v_{i,j+1} - v_{i,j-1}}{2k}\right)$$
$$+ \frac{3c_1^2}{4h^2}[(v_{i+1,j+1} - v_{i,j+1})^2 + (v_{i,j+1} - v_{i-1,j+1})^2]$$
$$+ \frac{c_1^2}{4h^2}[(u_{i+1,j+1} - u_{i,j+1})^2 + (u_{i,j+1} - u_{i-1,j+1})^2].$$

Note that $\partial g_{1ij}/\partial u_{i,j+1} > 0$ and $\partial g_{2ij}/\partial v_{i,j+1} > 0$ since $\beta'(\cdot) \geq 0$ and $\gamma'(\cdot) \geq 0$.

Also, we have

$$\frac{\partial g_{1ij}}{\partial u_{i+1,j+1}} = \frac{-c_0^2}{2} - \frac{3c_1^2}{4h^2}(u_{i+1,j+1} - u_{i,j+1})^2 - \frac{c_1^2}{4h^2}(v_{i+1,j+1} - v_{i,j+1})^2,$$
$$\frac{\partial g_{1ij}}{\partial u_{i-1,j+1}} = \frac{-c_0^2}{2} - \frac{3c_1^2}{4h^2}(u_{i,j+1} - u_{i-1,j+1})^2 - \frac{c_1^2}{4h^2}(v_{i,j+1} - v_{i-1,j+1})^2,$$
$$\frac{\partial g_{2ij}}{\partial v_{i+1,j+1}} = \frac{-c_0^2}{2} - \frac{3c_1^2}{4h^2}(v_{i+1,j+1} - v_{i,j+1})^2 - \frac{c_1^2}{4h^2}(u_{i+1,j+1} - u_{i,j+1})^2,$$

and

$$\frac{\partial g_{2ij}}{\partial v_{i-1,j+1}} = \frac{-c_0^2}{2} - \frac{3c_1^2}{4h^2}(v_{i,j+1} - v_{i-1,j+1})^2 - \frac{c_1^2}{4h^2}(u_{i,j+1} - u_{i-1,j+1})^2.$$

These last four partial derivatives are strictly negative and satisfy

$$\left|\frac{\partial g_{1ij}}{\partial u_{i,j+1}}\right| > \left|\frac{\partial g_{1ij}}{\partial u_{i+1,j+1}}\right| + \left|\frac{\partial g_{1ij}}{\partial u_{i-1,j+1}}\right|$$

and

$$\left|\frac{\partial g_{2ij}}{\partial v_{i,j+1}}\right| > \left|\frac{\partial g_{2ij}}{\partial v_{i+1,j+1}}\right| + \left|\frac{\partial g_{2ij}}{\partial v_{i-1,j+1}}\right|.$$

Thus we see that in the iterations for $u_{i,j+1}$ and $v_{i,j+1}$, the Jacobians are tridiagonal, symmetric, and strictly diagonally dominant. Since, in addition, the Jacobians have positive terms on the diagonals and negative terms off the diagonals, they are positive definite. The positive definiteness in turn implies that the Jacobians may be viewed as second derivatives of strictly convex functions ϕ_1 and ϕ_2 ($\phi_i : \mathbf{R}^n \to \mathbf{R}^1$) for which the minimizing equations are $g_{1ij} = 0$ and $g_{2ij} = 0$. Finally, the strict convexity of ϕ_i implies that (4.12) and (4.13) are convergent for any starting point in $\mathbf{R}^n$.

So far, we have discussed the solution for u and v for any given time line. Now suppose that u_{i1}, u_{i2}, v_{i1}, and v_{i2} are known for $i = 1, 2, \ldots, N$. The values of u_{i3} and v_{i3} for $i = 1, 2, \ldots, N$ may be calculated from (4.12) and (4.13) provided that an initial guess is given. In this regard, we note that the difference schemes can be guaranteed to be valid for functions with bounded fourth derivatives. Since such functions should change relatively little for a small change in t, it is reasonable to let $u_{i,2}$ be the initial approximation for u_{i3} (and v_{i2} for v_{i3}) for $i = 1, 2, \ldots, N$.

Having calculated u_{i3} and v_{i3} by the generalized Newton method, we use them as initial guesses to calculate u_{i4} and v_{i4}. This process is continued until the values $u(x_i, 2\pi)$ and $v(x_i, 2\pi)$ have been calculated and all u_{ij} and v_{ij} values have been stored. In order to save computer storage, the (x, t) grid on $[0, \pi] \times [0, 2\pi]$ is then used again to calculate values of u_{ij} and v_{ij} for $[0, \pi] \times [2\pi, 4\pi]$. If the values of u_{ij} and v_{ij} on $[0, \pi] \times [2\pi, 4\pi]$ are within a precribed tolerance of the stored values in $[0, \pi] \times [0, 2\pi]$, the process terminates. If not, values on $[0, \pi] \times [2\pi, 4\pi]$ are stored and later compared with calculated values on $[0, \pi] \times [4\pi, 6\pi]$. Calculations continue until the tolerance is met over the entire grid.

It is now seen that there are really two iterative processes occurring. One is an "inner" iteration, the generalized Newton. The other, or "outer" iteration, is essentially a Picard iteration in t on $[0, \pi] \times [0, 2\pi]$.

We close this section with a numerical example. Consider

$$\begin{aligned} u_{tt} - u_{xx} - u_x^2 u_{xx} - \frac{1}{3}\frac{\partial}{\partial x}(u_x v_x^2) - u_t|u_t| \\ = 2(\cos x \sin t)^2 \sin x \sin t + \sin x \cos t|\sin x \cos t|, \\ v_{tt} - v_{xx} - v_x^2 v_{xx} - \frac{1}{3}\frac{\partial}{\partial x}(u_x^2 v_x) + v_t|v_t| \\ = 2(\cos x \sin t)^2 \sin x \sin t + \sin x \cos t|\sin x \cos t|, \end{aligned}$$

$$\begin{aligned} u(0, t) = u(\pi, t) = 0, \qquad u(x, t) = u(x, t + 2\pi), \\ v(0, t) = v(\pi, t) = 0, \qquad v(x, t) = v(x, t + 2\pi). \end{aligned}$$

This corresponds to (4.1)–(4.4) with $c_0 = 1$, $c_1 = \sqrt{2/3}$, $\beta(r) = \gamma(r) = r|r|$, and $f_1(x, t) = f_2(x, t) = 2(\cos x \sin x)^2 \sin x \sin t + \sin x \cos t|\sin x \cos t|$. The exact solution to our sample problem is known to be $u(x, t) = v(x, t) = \sin x \sin t$. We calculate using $h = \pi/12$, $k = \pi/24$, $u_{i1} = u_{i2} = v_{i1} = v_{i2} = 0$ for $i = 1, 2, \ldots, N$, and the error tolerances for the outer and the inner iterations are set equal to 0.001. The periodic solution is reached in six outer iterations (i.e., the values on $[0, \pi] \times [10\pi, 12\pi]$ are within 0.001 of those on $[0, \pi] \times [8\pi, 10\pi]$). For comparison with the known exact solution the Fourier coefficients are computed from the approximate solutions using a two-dimensional trapezoidal rule. The approximate solutions are thus found to be

$$u(x, t) = 1.006 \sin x \sin t + \varepsilon_1(x, t)$$

and

$$v(x, t) = 1.009 \sin x \sin t + \varepsilon_2(x, t),$$

where coefficients in ε_1 are smaller than 0.0174 in magnitude and coefficients in ε_2 are smaller than 0.0409 in magnitude. We should note that a refinement of the grid produces more accurate answers.

5. Remarks. Finally we note that more general equations governing nonlinear stretched strings can be solved by the method outlined above. Thus to solve

$$\begin{aligned} &u_{tt} - [A(u_x)]_x + \beta(u_t) = f(x, t), \\ &u(0, t) = u(\pi, t) = 0, \\ &u(x, t) = u(x, t + 2\pi) \end{aligned} \tag{5.1}$$

we proceed as follows. Using central differences, we obtain

$$A(u_x) = A\left(\frac{u_{i+1/2} - u_{i-1/2}}{\Delta x}\right)$$

and

$$[A(u_x)]_x = \left[\frac{A((u_{i+1,j} - u_{i,j})/\Delta x) - A((u_{i,j} - u_{i-1,j})/\Delta x)}{\Delta x}\right].$$

With an averaged difference in x, the discrete analog of (5.1) is

$$\begin{aligned} \frac{u_{i,j+1} - 2u_{i,j} + u_{ij-1}}{\Delta t^2} + \frac{1}{2\Delta x}\Bigg\{&\left[A\left(\frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x}\right) - A\left(\frac{u_{i,j+1} - u_{i-1,j+1}}{\Delta x}\right)\right] \\ &+ \left[A\left(\frac{u_{i+1,j-1} - u_{i,j-1}}{\Delta x}\right) - A\left(\frac{u_{i,j-1} - u_{i-1,j-1}}{\Delta x}\right)\right]\Bigg\} \\ &+ \beta\left(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t}\right) = f_{ij}. \end{aligned}$$

After rearrangement, we obtain

$$\begin{aligned} g_{ij} = \frac{1}{2\Delta x}&\left[A\left(\frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x}\right) - A\left(\frac{u_{i,j+1} - u_{i-1,j+1}}{\Delta x}\right)\right] \\ &+ \frac{u_{i,j+1}}{\Delta t^2} + \beta\left(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t}\right) \\ &+ \Bigg\{\frac{1}{2\Delta x}\left[A\left(\frac{u_{i+1,j-1} - u_{i,j-1}}{\Delta x}\right) - A\left(\frac{u_{i,j-1} - u_{i-1,j-1}}{\Delta x}\right)\right] \\ &\qquad + \frac{-2u_{i,j} + u_{i,j-1}}{(\Delta t)^2} - f_{ij}\Bigg\} = 0. \end{aligned} \tag{5.2}$$

In (5.2), as in (4.10) and (4.11), boundary values are zero and all terms in "braces" are considered to be known. The symmetry present in the terms in "A" yields a symmetric Jacobian matrix, while positive definiteness will be retained if "A" is of an appropriate form. For example, if $A(u_x) = -c_0^2 u_x$ so that $[A(u_x)]_x = -c_0^2 u_{xx}$, (5.1) reduces to the wave equation with linear principal part. For $A(u_x) = -c_0^2 u_x - \frac{1}{2}c_1^2 u_x^3 - \frac{1}{2}c_1^2(u_x v_x^2)$ we have $[A(u_x)]_x = -c_0^2 u_{xx} - \frac{3}{2}c_1^2 u_x^2 u_{xx} - \frac{1}{2}c_1^2(u_x v_x^2)_x$ and so (5.1) becomes (4.1).

References

[1] G. V. Anand, *Nonlinear resonance in stretched strings with viscous damping*, J. Accoust. Soc. Amer. **40**, 1517–1528 (1966)

[2] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Quart. Appl. Math. **3**, 157–165 (1945)

[3] L. Cesari and R. Kannan, *Solutions of nonlinear wave equations with nonlinear damping*, Comm. Partial Differential Equations **8**, 199–211 (1983)
[4] R. W. Dickey, *Stability theory for quasilinear wave equations with linear damping*, Proc. Roy. Soc. Edinburgh Sect. A **88**, 25–41 (1981)
[5] J. W. Miles, *Stability of forced oscillations of a vibrating string*, J. Accoust. Soc. Amer. **38**, 855–861 (1965)
[6] G. S. Srinivasa Murthy and B. S. Ramakrishna, *Nonlinear character of resonance in stretched strings*, J. Accoust. Soc. Amer. **38**, 461–471 (1965)
[7] R. Narasimha, *Nonlinear vibration of an elastic string*, J. Sound Vibration **8**, 134–146 (1968)
[8] D. W. Oplinger, *Frequency response of a nonlinear stretched string*, J. Accoust. Soc. Amer. **32**, 1529–1538 (1960)
[9] E. W. Lee, *Nonlinear forced vibration of a stretched string*, British J. Appl. Phys. **8**, 411–413 (1957)