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PHENOMENOLOGICAL BEHAVIOR † OF MULTIPOLAR VISCOUS FLUIDS

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Abstract. A constitutive theory is formulated to describe the flow of viscous fluids: the theory of multipolar fluids exhibits nonlinear relations among the stress tensors and spatial derivatives of the velocity of order greater than one and is compatible with the basic principles of continuum mechanics and thermodynamics. For an isothermal, incompressible, dipolar fluid the velocity profiles for various steady flows, such as proper Poiseuille flow in a circular pipe, are computed; the results are compared with the velocity profiles obtained from the steady Navier-Stokes equations through an application of the Prandtl boundary-layer theory.

Introduction. The physical theory of multipolar fluids was delineated in the papers of Nečas and Šilhavý [1] and Bleustein and Green [2] and follows the general ideas of Green and Rivlin [3], [4]; the theory is compatible with the principles of thermodynamics, as well as with the principle of material frame indifference. The formulation in [2] is somewhat different than the presentation in [1] dealing as it does only with the special case of a dipolar fluid but allowing for the presence of dipolar inertia terms in the constitutive theory which are not present in [1]. As formulated in [1] and [2], the theory takes into account the possibility of nonlinear relations between the various stress tensors and spatial derivatives of the velocity vector. In [2] only a special example of a flow of linear dipolar viscous fluid was studied; for viscous, heat conducting, compressible fluids, with the assumption of linear relations between the relevant stress tensors and spatial derivatives of the velocity vector, a rigorous mathematical proof of the global existence in time of solutions to associated initial-boundary value problems may be established, the basic theorems being contained in

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the series of recent papers by Nečas, Novotny, and Šilhavý [5]–[8].

In the current work we take advantage of the full scope of the constitutive theory formulated in [1] and discuss what we feel is a major improvement over the special example of a linear dipolar viscous fluid which was introduced there and in [2] and subsequently studied in [5]–[8]. More precisely, we not only allow for nonlinear relations between the various stress tensors and spatial derivatives of the velocity vector, as is already encompassed in the general theories constructed in both [1] and [2] but we specifically study some of the implications of such nonlinearity within the context of the theory of an isothermal, incompressible, viscous dipolar fluid. The plan of the paper is as follows: In Sec. 2 we briefly summarize the general theory of multipolar viscous fluids, as formulated in [1], with particular emphasis on compatibility with both the principles of thermodynamics and the principle of framed indifference. In Sec. 3 we specialize the theory to the case of a linearly viscous fluid and, for the incompressible dipolar fluid, compare the predicted results with those obtained from the steady-state Navier-Stokes equations as applied to the standard problem of Poiseuille flow between parallel plates. We also indicate the relationship of these results to the analogous results obtained in [2] for the Poiseuille flow of a linear dipolar fluid in a circular cylinder. The specific type of nonlinear relation between stress and velocity gradient, which is subsequently studied, is then motivated by the analysis of Sec. 4; specific consequences of this nonlinearity, again within the context of an incompressible, isothermal, dipolar fluid, are carefully examined in Sec. 5 by studying the problems of plane Poiseuille flow, Poiseuille flow down a circular pipe, and plane Couette flow, i.e., flow between parallel plates when one of the plates is moving with constant velocity and the pressure is constant. The calculations in Sec. 5 are carried out in closed form by setting the single (constant) higher order viscosity coefficient μ_1 equal to zero. However, for the problem of plane Poiseuille flow, we prove in Sec. 6 several rigorous results which establish the existence of solutions for the relevant nonlinear boundary value problem, with $\mu_1 \neq 0$, as well as continuous dependence on μ_1, in the norm of $C^{1+\delta}$, $0 < \delta < 1/2$, as $\mu_1 \to 0^+$; this latter result justifies the velocity profiles calculated and depicted in Sec. 5 for the specific problem of plane Poiseuille flow. A subsequent paper will establish the existence and continuous dependence results which are required to justify the velocity profiles obtained for the problem of Poiseuille flow in a circular cylinder; a general existence theorem for the time-dependent, isothermal, incompressible, nonlinear problem has been constructed by Bellout, Bloom, and Nečas in [10]. The analysis in Sec. 5 and Sec. 6 yields a mechanism for describing the laminar profiles observed in viscous fluid flow just prior to the onset of actual turbulence, *such profiles having previously been obtained only by use of boundary-layer theory for the Navier-Stokes equations.* We also emphasize the fact that the *boundary conditions associated with the higher order multipolar stress tensors provide a mechanism for analytically studying flows in pipes, channels, etc., with rough boundaries*; *in fact, the analysis of the nonlinear dipolar fluid presented here clearly points to our ability to capture curvature variations along bounding surfaces of the flows.*

2. Multipolar fluids. We begin by defining a thermodynamic process of a multipolar fluid to be a collection of functions of position and time (written in Eulerian coordinates), namely, the velocity field $\mathbf{v}$, the field of positive absolute temperature θ, the density ρ, the specific internal energy e, the specific entropy η, the specific external body force $\mathbf{b}$, the rate of working of external heat sources r, the heat flux vector $\mathbf{q}$, and the spatial multipolar stress tensors $\tau_{ii_1\cdots i_kj}$ of order $k = 0, 1, \ldots, N-1$. The Euler (spatial) coordinates will be denoted by x_i, $i = 1, 2, \ldots, m$, and it is assumed that the $\tau_{ii_1\cdots i_kj}$ are symmetric in the indices $i_1, \ldots, i_k$.

With the notation for the convective derivative

$$(\bullet) \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}$$

(summation on repeated indices being understood throughout the remainder of the paper) the above functions satisfy

$$\dot{\rho} + \rho\frac{\partial v_i}{\partial x_i} = 0, \tag{2.1}$$

$$\rho\left(e + \frac{1}{2}|\mathbf{v}|^2\right)^{\bullet} = \frac{\partial}{\partial x_i}\left(-q_i + \sum_{k=0}^{N-1}\tau_{jj_1\cdots j_ki}\frac{\partial^{k+1}v_j}{\partial x_{j_1}\cdots\partial x_{j_k}\partial x_i}\right) + \rho b_iv_i + \rho r, \tag{2.2}$$

$$\rho\dot{v}_i = \frac{\partial\tau_{ij}}{\partial x_j} + \rho b_i, \tag{2.3}$$

$$\rho(\varepsilon_{jkp}x_kv_p)^{\bullet} = \frac{\partial}{\partial x_i}(\varepsilon_{jkp}x_k\tau_{pi} + \varepsilon_{jkp}\tau_{pki}) + \rho\varepsilon_{jkp}x_kb_p, \tag{2.4}$$

$$\rho\dot{\eta} \geq -\frac{\partial}{\partial x_i}\left(\frac{q_i}{\theta}\right) + \rho\frac{r}{\theta}, \tag{2.5}$$

which express, respectively, conservation of mass, conservation of energy, balance of momentum, conservation of angular momentum, and the Clausius-Duhem inequality, i.e., the second law of thermodynamics; as explained in [1] the rate of working of the higher order stresses enters the balance of energy equation (2.2) but only the (usual) second order stress tensor enters the balance of momentum relation (2.3).

We assume a less general situation here, vis à vis the constitutive relations, than that which was treated in [1], i.e., that e, η, $\mathbf{q}$, and the $\tau_{ii_1\cdots i_kj}$, $k = 0, 1, \ldots, N-1$, are functions of $\rho, \nabla\mathbf{v}, \ldots, \nabla^k\mathbf{v}, \theta$, and $\nabla\theta$ with $k = N-1$. Fluids that conform to the above constitutive hypotheses will be termed N-polar fluids. As is customary we require that the constitutive relations are such that the principle of material frame indifference and the Clausius-Duhem inequality (2.5) are satisfied in every sufficiently smooth process.

If we consider a change of frame of the form

$$\overline{x}_i = Q_{ij}(t)x_i + c_i(t) \tag{2.6}$$

with $Q_{ij}(t)Q_{ik}(t) = \delta_{jk}$, then the principle of material frame indifference postulates that under the change of frame (2.6) the quantities θ, ρ, e, and r are invariant while q_i and $\tau_{ii_1\cdots i_kj}$ change in the usual tensorial fashion. It then follows that, in the constitutive relations, the first spatial gradient of velocity appears only through its symmetric part $\mathbf{e}$, the rate of the deformation tensor, where $e_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$. For brevity, in what follows, we will write $\mathbf{e} = \mathscr{D}\mathbf{v}$.

We denote by $\tau^E_{ii_1\cdots i_kj} = \tau_{ii_1\cdots i_kj}(\rho, \mathbf{0}, \dots, \mathbf{0}, \theta, \mathbf{0})$ the equilibrium part of the multipolar stress $\tau_{ii_1\cdots i_kj} = \tau_{ii_1\cdots i_kj}(\rho, \mathscr{D}\mathbf{v}, \dots, \nabla^K\mathbf{v}, \theta, \nabla\theta)$ and then set $\tau^v_{ii_1\cdots i_kj} = \tau_{ii_1\cdots i_kj} - \tau^E_{ii_1\cdots i_kj}$ (the viscous part of the multipolar stress tensor). We also set $\tau_{ii_1\cdots i_Nj} \equiv 0$. We are interested in three important cases where equations (2.1), (2.3) may be solved independently of (2.2): the barotropic case, the isothermal case, and the incompressible case. If we suppose that $\tau^E_{ii_1\cdots i_kj} \equiv 0$,[1] for $k \geq 1$, that $\tau_{ij} = \tau_{ji}$, and that $\partial\tau_{pki}/\partial x_i = \partial\tau_{kpi}/\partial x_i$ then these latter two conditions, together with (2.3), imply that (2.4) is satisfied. Suppose, furthermore, that for $k = 0, 1, \dots, N-1$

(i) $\tau^v_{ii_1\cdots i_kj} = \tau^v_{ii_1\cdots i_kj}(\mathscr{D}\mathbf{v}, \dots, \nabla^K\mathbf{v})$, and

(ii) $\mathbf{q} = -\kappa\nabla\theta$, $\kappa > 0$ [Fourier Law of Heat Conduction].

It then follows from (2.5) (see [1]) that

$$\sum_{k=0}^{N-1}\left(\tau_{jj_1\cdots j_ki} + \frac{\partial}{\partial x_p}\tau_{jj_i\cdots j_kip}\right)\frac{\partial^{k+1}v_j}{\partial x_{j_1}\cdots\partial x_{j_k}\partial x_i} \geq 0, \tag{2.7}$$

and

$$\tau^E_{ji} = -p(\rho, \theta)\delta_{ji}, \tag{2.8}$$

where p is the pressure. Under the above assumptions restricting the class of multipolar fluids, the condition (2.7), together with the generalized Gibb's equation (which also follows from (2.5)), i.e.,

$$\rho\dot{\Psi} = -\rho\eta\dot{\theta} - p(\rho, \theta)\frac{\partial v_i}{\partial x_i} \tag{2.9}$$

are equivalent to (2.5); this is demonstrated in [1]. In (2.9), Ψ is the Helmholtz free energy: $\Psi(\rho, \theta) = e(\rho, \theta) - \theta\eta(\rho, \theta)$; it also follows from (2.5) that e and η are independent of the gradients of $\mathbf{v}$ and θ while

$$\eta = -\frac{\partial\Psi}{\partial\theta}, \qquad p = \rho^2\frac{\partial\Psi}{\partial\theta}. \tag{2.10}$$

3. Linear viscous multipolar fluids. If we consider (e.g., [1]) linear constitutive relations for the $\tau_{ii_1\cdots i_kj}$ with constant coefficients, then it follows from the principle of material frame indifference that the $\tau_{ii_1\cdots i_kj}$ with even indices depend only on odd-order gradients of $\mathbf{v}$ while the $\tau_{ii_1\cdots i_kj}$ with odd indices depend only on even-order gradients of $\mathbf{v}$. Also, in this case, by virtue of material frame indifference the general representation of τ^v_{ij} assumes the form

$$\begin{aligned}\tau^v_{ij} = \lambda v_{k,k} + \mu(v_{i,j} + v_{j,i}) + \sum_{l=0}^{N-2}(\alpha^{(l)}\delta_{ij}\Delta^{l+1}v_{k,k} + \beta_1^{(l)}\Delta^{l+1}v_{i,j} \\ + \beta_2^{(l)}\Delta^{l+1}v_{j,i} + \gamma^{(l)}\Delta^l v_{k,kij})\end{aligned} \tag{3.1}$$

[1]This assumption is not imposed in the more restrictive dipolar theory formulated in [2].

where $v_{i,j} = \partial v_i/\partial x_j$; analogous formulas are also valid for the higher order stress tensors. For the dipolar case $(N = 2)$, we have the general form[2]

$$\begin{aligned}\tau^v_{ijk} &= c_1\delta_{ij}\Delta v_k + c_2\delta_{ij}v_{m,mk} + c_3\delta_{ik}\Delta v_j + c_4\delta_{ik}v_{m,mj} + c_5\delta_{jk}\Delta v_i \\ &\quad + c_6\delta_{jk}v_{m,m} + c_7 v_{i,jk} + c_8 v_{k,ij} + c_9 v_{j,ki}.\end{aligned} \tag{3.2}$$

If (2.5) is taken into account then we find the set of relations

$$\begin{cases} \alpha^{(0)} + c_1 + c_2 = 0, & \beta_1^{(0)} + c_5 + c_4 = 0, \\ \beta_2^{(0)} + c_8 + c_9 = 0, & \gamma^{(0)} + c_4 + c_6 + c_8 = 0. \end{cases} \tag{3.3}$$

An example of linear constitutive relations satisfying both the principle of material frame indifference and the Clausius-Duhem inequality (2.5) is given by the following:

$$\tau_{iji_1\cdots i_m} = \sum_{l=m}^{N-1}(-1)^{l+m}\Delta^{l-m}\frac{\partial^m g^l_{ij}}{\partial x_{i_1}\cdots\partial x_{i_m}} \tag{3.4}$$

with $g^l_{ij} = \lambda_l v_{k,k}\delta_{ij} + 2\mu_l e_{ij}(\mathbf{v})$, $\mu_l \geq 0$, $\lambda_l \geq -2\mu_l/3$, and $\tau_{ii_1\cdots i_m j} \equiv 0$ for $1 \leq m \leq N-1$. In considering an ideal gas for which

$$p = R\rho\theta \tag{3.5}$$

it is necessary (e.g. [5]), to add to the previous assumptions certain conditions that guarantee appropriate a priori bounds for energy and entropy. For example, it may be assumed that for $\Omega \subseteq R^m$

$$-\int_\Omega \frac{\partial \tau_{ij}}{\partial x_j}(\mathbf{v})v_i\,d\mathbf{x} = \int_\Omega \langle \mathbf{v}, \mathbf{w}\rangle\,d\mathbf{x} + \sum_{m=1}^{N-1}\int_{\partial\Omega}\tau_{ii_1\cdots i_m j}(\mathbf{v})\frac{\partial^m w_i}{\partial x_{i_1}\cdots\partial x_{i_m}}\nu_j\,ds, \tag{3.6}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$ and where $\langle \mathbf{v}, \mathbf{w}\rangle$ is a symmetric bilinear form with constant coefficients satisfying

$$\int_\Omega \langle \mathbf{v}, \mathbf{v}\rangle\,d\mathbf{x} \geq \alpha\|\mathbf{v}\|_{W^{N,2}(\Omega)}, \qquad \alpha > 0, \tag{3.7}$$

$W^{N,2}(\Omega)$ being the usual (vector-valued) Sobolev space of L^2 functions with L^2 integrable derivatives up to the order N; it is assumed that (3.7) holds for all $\mathbf{v}$ such that $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$. Under the assumptions (3.6), (3.7), it was proven in [4] that for a dipolar isothermal gas there exists a global solution of Eqs. (2.1), (2.3) subject to appropriate initial conditions and boundary conditions of the form[3]

$$\mathbf{v} = \mathbf{0}, \qquad \tau_{ijk}(\mathbf{v})\nu_j\nu_k = 0, \qquad i = 1, 2, 3, \quad \text{on } \partial\Omega \times [0, T]. \tag{3.8}$$

We note, in particular, the way in which the higher order stresses $\tau_{ijk}(\mathbf{v})$ enter into the boundary conditions (3.8) and will comment on this further in Sec. 5.

The dipolar case allows for nonunique solutions and it is not excluded that the density $\rho = 0$ on some set of positive measure in $Q \equiv \Omega \times (0, \infty)$; however,

[2]The constitutive hypotheses (3.1), (3.2) for the linear dipolar case are, in essence, equivalent to the analogous relations formulated in [2].

[3]The first set of boundary conditions is standard in any theory of viscous fluid flow while the second set indicates that the dipolar tractions vanish on the boundary, e.g., [1], [2].

for the case $N = 3$ uniqueness of solutions, as well as noncavitation of ρ, have been established [7]. The baratropic gas with $p = p(\rho)$ was considered in [8] and the existence of a unique global solution was established there under the condition $N \geq 3$.

We now turn to the special case of an incompressible multipolar fluid, which turns out to be very important in applications; this case also serves to highlight the basic viewpoint of multipolarity. In order to be somewhat specific we will concentrate on the problem of steady flow parallel plates at $x_2 = \pm 1$.

We suppose that $p = p(x_1, x_2)$ while

$$\frac{\partial v_i}{\partial x_i} = 0, \qquad \rho = \text{const.}, \tag{3.9}$$

$$\rho \frac{\partial v_i}{\partial x_j} v_j + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}^v}{\partial x_j} = 0. \tag{3.10}$$

With the classical Stoke's law

$$\tau_{ij}^v = \lambda_0 \operatorname{div} \mathbf{v} \delta_{ij} + 2\mu_0 e_{ij}(\mathbf{v}), \tag{3.11}$$

$\lambda_0 \geq -\frac{2}{3}\mu_0$, $\mu_0 > 0$, we obtain the usual steady-state Navier-Stoke's equations. Assuming that the flow has the form $v_1 = v_1(x_2)$, $v_2 = 0$, $v_3 = 0$ (plane Poiseuille flow between fixed parallel plates), (3.10), (3.11) reduce to

$$\frac{\partial p}{\partial x_1} - \mu_0 v_1'' = 0, \qquad \frac{\partial p}{\partial x_2} = 0, \tag{3.12}$$

so that $p = p_0 + p_1 x_1$. Thus

$$p_1 - \mu_0 v_1'' = 0 \tag{3.13}$$

and as $\mathbf{v} = 0$ for $x_2 = \pm 1$ we obtain the well-known result [12]

$$v_1(x_2) = -\frac{p_1}{2\mu_0}(1 - x_2^2), \qquad -1 \leq x_2 \leq 1. \tag{3.14}$$

If we set $P = \int_{-1}^{1} v_1(x_2)\, dx_2$ then $v_1 = 3P(1 - x_2^2)/4$. A major problem with (3.14) is the fact that the form of the profile (3.14) remains parabolic as $\mu_0 \to 0^+$. From experimental observations and the Prandtl boundary-layer theory, which is in reasonable accord with such observations, we know that as $\mu_0 \to 0^+$ the resulting profiles must both flatten out with respect to the axis $x_2 = 0$ and must approach the boundaries at $x_2 = \pm 1$ in an ever increasing tangential fashion: more precisely, experimental observations and the Prandtl boundary-layer equations lead us to expect for a sequence of viscosities $\{\mu_0^n\}$ with $\mu_0^n \to 0$, as $n \to \infty$, a sequence of progressively flattened profiles such as those depicted in Fig. 1(a)–(c).

In order to gauge the effect of multipolarity on the velocity profile (3.14) we will consider special cases of (3.1), with $N = 2$, and (3.2). To facilitate the discussion

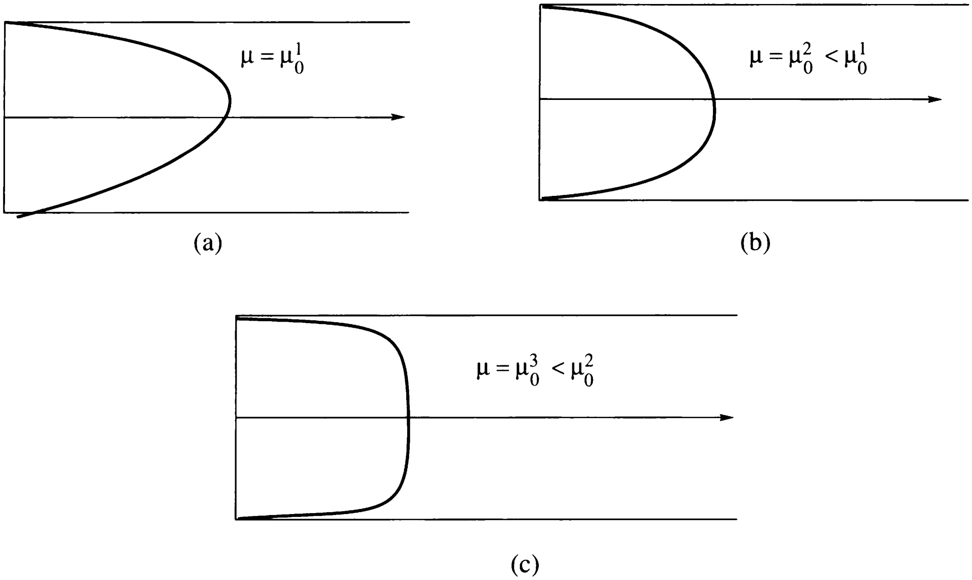

FIG. 1

let us set

$$\begin{cases} \tau^0_{ij} = \lambda_0 \delta_{ij} \operatorname{div} \mathbf{v} + 2\mu_0 e_{ij} & (\lambda_0 \geq 0\,,\ \mu_0 \geq 0)\,, \\ \tau^1_{ij} = \lambda_1 \delta_{ij} \operatorname{div} \mathbf{v} + 2\mu_1 e_{ij} & (\lambda_1 \geq 0\,,\ \mu_1 \geq 0)\,, \end{cases} \tag{3.15}$$

$$\tau^E_{ij} = -p\delta_{ij}\,, \qquad \tau^E_{ijk} = 0\,, \tag{3.16}$$

$$\tau^v_{ij} = \tau^0_{ij} - \Delta\tau^1_{ij}\,, \qquad \tau^v_{ijk} = \frac{\partial \tau^1_{ij}}{\partial x_k}\,; \tag{3.17}$$

then we easily find

$$\tau_{ij} = -p\delta_{ij} + \lambda_0 \delta_{ij} \operatorname{div} \mathbf{v} + 2\mu_0 e_{ij} - \lambda_1 \delta_{ij} \Delta \operatorname{div} \mathbf{v} - 2\mu_1 \Delta e_{ij}\,, \tag{3.18}$$

$$\tau_{ijk} = \lambda_0 \delta_{ij} v_{l,lk} + 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}\,. \tag{3.19}$$

Clearly, (3.18), (3.19) are special cases of (3.1) (with $N = 2$) and (3.2); with the assumption of incompressibility these constitutive relations for an isothermal, linear, bipolar[4] fluid reduce to

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} - 2\mu_1 \Delta e_{ij}\,, \tag{3.20}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}\,. \tag{3.21}$$

Assuming again that the flow has the form $v_1 = v_1(x_2)$, $v_2 = 0$, $v_3 = 0$, with $p = p(x_1, x_2)$, and substituting this information into (3.20), and then the viscous

[4]We will use the terms bipolar and dipolar from now on in an interchangeable fashion; we prefer the former terminology but have also referenced the latter as it has already appeared in the literature, e.g., [2].

part of τ_{ij} into (3.10), we find that

$$\mu_0 v_1''(x_2) - \mu_1 v_2''''(x_2) = p_1 \text{ (const.)} \tag{3.22}$$

while the boundary conditions (3.8), by virtue of (3.21), and the assumed form of the velocity field $\mathbf{v}$, yield

$$v_1(\pm 1) = 0; \qquad v_1''(\pm 1) = 0. \tag{3.23}$$

A simple calculation shows that the solution of the boundary value problem (3.22), (3.23) is given by

$$v_1(x_2) = \frac{P}{\frac{4}{3} - \frac{4}{\eta^2}\left(1 - \frac{1}{\eta}\frac{\sinh\eta}{\cosh\eta}\right)}\left(1 - x^2 + \frac{2}{\eta^2}\frac{\cosh\eta x_2}{\cosh\eta} - \frac{2}{\eta^2}\right), \tag{3.24}$$

where $\eta = \sqrt{\mu_0/\mu_1}$ and, again, $P = \int_{-1}^{1} v_1(x_2)\,dx$. As $\mu_1 \to 0^+$, it is not difficult to show that the profile given by (3.24) converges to that profile predicted by the classical Navier-Stokes relation (3.11), with $\operatorname{div}\mathbf{v} = 0$, i.e., to (3.14). We also note that it follows from (3.24) that for $\eta = \infty$, $v_1(0) = 3P/4$ while for $\eta \to 0^+$, $v_1(0) = 25P/32$; in fact for $\mu = 0$, $\mu_1 > 0$ we calculate that

$$v_1(x_2) = \frac{5}{32}P(5 - 6x_2^2 + x_2^4). \tag{3.25}$$

Thus, *multipolarity has only a minor perturbative effect on the velocity profile for this particular steady flow*; the profiles are still distinctly parabolic in character and *do not exhibit the "flattening out" phenomenon which is predicted by the boundary layer theory for the classical Navier-Stokes equations in the case of vanishing kinematic viscosity.* This point is reinforced by the calculations of Bleustein and Green [2] who consider the companion problem of Poiseuille flow, in a circular cylinder of radius R, of a linear dipolar fluid. In a cylindrical coordinate system r, θ, z, with associated unit vectors in the coordinate directions given by $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ the velocity field in [2] has the form $\mathbf{v} = u(r)\mathbf{e}_z$ and is governed by the ordinary differential equation (Eq. 16.6) of [2]

$$(1 - l^2 D_r^2) D_r^2 u = \frac{1}{\mu} p_1; \qquad D_r^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}, \tag{3.26}$$

where a study of the comparable constitutive relations in [2] yields the identifications:

$$\mu \to \mu_0 \quad \text{and} \quad l = \frac{1}{\eta} = \sqrt{\frac{\mu_1}{\mu_0}}, \tag{3.27}$$

and p_1 is again the constant pressure gradient; the differential equation (3.26) is just the special case of our nonlinear equation (5.38), corresponding to parameter $\alpha = 0$, which governs the same flow within the context of the specific nonlinear constitutive model that is developed in Sec. 4 of this paper. The solution of (3.26) (subject to the boundary conditions $u(R) = u''(R) = 0$, and the assumption of finite velocity along the center line $(z = 0)$ of the cylinder) is given by formula (6.15) of [2] with $M_z = \tau_{rrz}(R) = 0$; that expression, which is a complicated perturbation of the usual

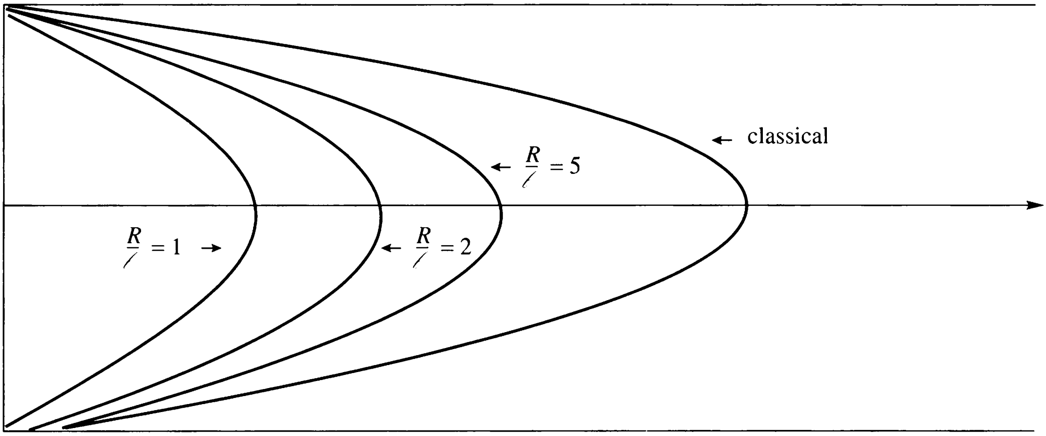

FIG. 2

parabolic profile (predicted by the Stokes relation) involving $I_0(\frac{r}{l})$, I_0 being the modified Bessel function of the first kind, is depicted in Fig. 2 which is reproduced directly from the corresponding Fig. 3 of [2]: for a fixed μ_1 the figure shows the result of numerical calculations and graphs the velocity profile $u(r)$ as a function of $R/l \equiv R\eta$.[5] As is the case with (3.24), for the problem of Poiseuille flow of a dipolar fluid between parallel plates, the profiles produced by (6.15) in [2] converge to the parabolic profile predicted by the Stokes relation as $l \to 0^+$ (i.e., $\eta \to +\infty$) for fixed R and tend to flatten only slightly in a neighborhood of $r = 0$; more importantly, in a neighborhood of the walls of the cylinder at $r = R$ the profiles are even more out of line with those observed in practice, and computed through use of the Prandtl boundary-layer theory, than that of the classical parabolic profile for small viscosity.

An excellent picture, for the case of Poiseuille flow between parallel plates, of the actual velocity profiles at Large Reynolds numbers, may be gleaned from Fig. 82 (page 310) of [13]; in this case the flattened profiles are observed in the inlet length of the channel (the Reynolds number being large, not because of the small magnitude of the kinematic viscosity but, rather, because of the small distance traversed by the fluid) and correspond precisely to those depicted in our Fig. 1 for $\mu < \mu_0^3$. The remedy to the situations described above will be to move away from the linearity built into the multipolar constitutive relations (3.18), (3.19), for the dipolar fluid, and toward a slightly nonlinear version of these constitutive relations, which is completely admissible within the context of the original [1] Nečas-Šilhavý formulation of the basic theory.

4. Nonlinear bipolar fluids. The general theory of multipolar fluids, as constructed both in [1] and [2], allows for very broad types of nonlinearity in the relationships between the stress tensors and velocity gradients. In early work on the phenomenological behavior of viscous fluids Prandtl suggested replacing the constant viscosity μ_0, in the classical Stokes law (3.11), by a viscosity function μ_0 which was defined to

[5] The results in [2] indicate a lower bound of unity for R/L or, equivalently, that $\sqrt{\mu_0/\mu_1} \geq 1/R$ so that μ_0 cannot tend to zero independently of μ_1.

be directly proportional to the first spatial velocity gradient; for the plane Poiseuille flow problem (between parallel plates), which was considered in the last section, this constitutive hypothesis is embodied in the relation $\mu_0 = \mu|v_1'|$ in which case (3.13) becomes

$$p_1 - \mu(|v_1'|v_1')' = 0. \tag{4.1}$$

Solving (4.1) subject to the boundary conditions $v_1(\pm 1) = 0$ we obtain

$$v_1(x_2) = \frac{5}{6}P(1 - |x_2|^{3/2}), \tag{4.2}$$

a profile which is even further removed from the actual profile that one expects to see in a steady flow situation, as the viscosity grows smaller, than the parabolic profile $v_1 = 3P(1-x_2^2)/4$ predicted by the classical Stokes law, or the profile (3.24) predicted by the linear theory of the dipolar fluid.

Suppose, however, that we now return to the constitutive relations (3.20), (3.21) for the isothermal, incompressible fluid. We set

$$e'_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}\operatorname{div}\mathbf{v} \tag{4.3}$$

and consider a situation in which, for some constants $\varepsilon, \lambda \geq 0$, μ_0 is a function given by

$$\mu_0 = \mu(\sqrt{2}(\varepsilon + e'_{ij}e'_{ij} + \lambda(\operatorname{div}\mathbf{v})^2)^{1/2}). \tag{4.4}$$

Of course, in the incompressible situation the value of λ is irrelevant; thus with the assumption of incompressibility (3.20), (4.3) and (4.4) combine to yield the constitutive relation

$$\tau_{ij} = -p\delta_{ij} + 2\mu(\sqrt{2}(\varepsilon + e_{ij}e_{ij})^{1/2})e_{ij} - 2\mu_1\Delta e_{ij} \tag{4.5}$$

which, together with (3.21), provides a constitutive theory for an isothermal, incompressible, bipolar fluid that is completely compatible with the principles of frame-indifference and thermodynamics, as per the analysis in [1]. We now want to look at (4.5) with both $\varepsilon = 0$ and $\mu_1 = 0$ to motivate the subsequent analysis.

If we substitute (4.5), with $\varepsilon = \mu_1 = 0$, into (3.10) and again assume that we are looking at the problem of Sec. 3, i.e., at plane Poiseuille flow between parallel plates, we obtain, in lieu of (3.12),

$$\frac{\partial p}{\partial x_1} - (\mu(|v_1'|)v_1')' = 0, \qquad \frac{\partial p}{\partial x_2} = 0 \tag{4.6}$$

so that $p = p_1x_2 + p_0$ again, and

$$p_1x_2 - \mu(|v_1'|)v_1' = 0. \tag{4.7}$$

Now, in principle, the constant pressure gradient p_1 and the velocity profile $v_1 = v_1(x_2)$ may be identified by experimental procedures; under such circumstances the functional form of μ may be obtained from (4.7) as follows: suppose that $v_1'(x_2)$ is monotone in $[-1, 1]$, odd, and nonnegative on $[-1, 0]$. Writing, in (4.7), $x_2 = v_1'^{-1}(v_1'(x_2))$ we find that, on the range of $|v_1'|$,

$$p_1\frac{(v_1')^{-1}(v_1')}{v_1'} = \mu(|v_1'|). \tag{4.8}$$

Let us suppose that our measurements give us a more realistic profile than those produced in Sec. 3 (such profiles result, for example, from applications of boundary-layer theory for the steady-state Navier-Stokes equations), say, a profile of the form

$$v_1(x_2) = \frac{P}{2}\frac{1+\delta}{\delta}(1-|x_2|^\delta), \qquad \delta \geq 2, \tag{4.9}$$

where, again, $P > 0$ is given by $\int_{-1}^{1} v_1(x_2)\,dx_2$. We look for a solution μ of (4.8) of the form

$$\mu(|v_1'|) = \beta|v_1'|^{(2-\delta)/(\delta-1)} \tag{4.10}$$

with $\beta > 0$ constant. Substituting (4.9) and (4.10) into (4.8) we find that the constitutive hypothesis (4.10) is admissible provided the constant β is given by

$$\beta = |p_1|\left(\frac{2}{P(1+\delta)}\right)^{1/(\delta-1)}. \tag{4.11}$$

For the plane Poiseuille flow governed by the classical Stokes relation we have, from the analysis of Sec. 3, that the viscosity μ_0 appearing in (3.11) satisfies

$$\mu_0 = \frac{2|p_1|}{3P} \tag{4.12}$$

and (4.11) should be viewed in the same light. If we now set $\alpha = (\delta-2)/(\delta-1)$, $\delta \geq 2$, then, clearly, $0 < \alpha < 1$.

The formal considerations described above now provide us with the motivation to study incompressible, isothermal bipolar fluids in which the stress tensors τ_{ij} and τ_{ijk} are given, respectively, by (3.20) and (3.21) with $\mu_1 > 0$, and the constant μ_0 in (3.20) is replaced by a function μ_0 of the form

$$\mu_0 = \mu^0(\varepsilon + e_{ij}e_{ij})^{-\alpha/2}, \qquad 0 < \alpha < 1, \tag{4.13}$$

with $\mu^0 > 0$, $\varepsilon \geq 0$; if we remove the condition of incompressibility then (4.13) must be replaced by

$$\mu_0 = \mu^0(\varepsilon + e_{ij}e_{ij} + \lambda(\operatorname{div}\mathbf{v})^2)^{-\alpha/2}, \qquad 0 < \alpha < 1, \tag{4.14}$$

with $\lambda \geq 0$ constant.

5. Some steady flows of isothermal, incompressible, nonlinear bipolar fluids. In this section we examine the three elementary examples of steady flows: plane Poiseuille flow between fixed parallel plates, proper Poiseuille flow down a circular pipe, and plane Couette flow over a plate which moves with constant velocity. Our constitutive equations have the form

$$\tau_{ij} = -p\delta_{ij} + 2\mu^0(\varepsilon + e_{ij}e_{ij})^{-\alpha/2} - 2\mu_1\Delta e_{ij}, \tag{5.1}$$

$$\tau_{ijk} = 2\mu_1\frac{\partial e_{ij}}{\partial x_k} \tag{5.2}$$

with $\mu^0, \mu_1 > 0$ and $0 < \alpha < 1$. In order to calculate closed form solutions in this section we will set $\varepsilon = \mu_1 = 0$; however, in Sec. 6, as previously emphasized, we prove an existence theorem and a continuous dependence result for the plane

Poiseuille flow problem with $\varepsilon \neq 0$, $\mu_1 \neq 0$ which shows, at least in this case, that the velocity profile obtained in this section is accurate, as $\mu_1 \to 0^+$, in the norm of $C^{1+\delta}$, for some δ, $0 < \delta < 1/2$. The profile is, therefore, also accurate in the C^0 norm and a similar continuous dependence and existence theorem (for $\varepsilon \neq 0$, $\mu_1 \neq 0$) is forthcoming for the problem of Poiseuille flow in a circular cylinder which we will consider next; without such theorems the results in this section, vis à vis the constitutive theory given by (5.1), (5.2), would be vacuous.

(i) *Plane Poiseuille flow.* As in Sec. 3 we assume a flow of the form

$$v_1 = v_1(x_2), \qquad v_2 = 0, \qquad v_3 = 0. \tag{5.3}$$

The fixed plates will be located at $x_2 = \pm a$ for some $a > 0$. With $\mu_1 = 0$ the boundary conditions are the same as for the steady-state Navier-Stokes equations, i.e., $v_1(\pm a) = 0$. For $\mu_1 \neq 0$ the constitutive relation (5.1) coupled with (5.3) and (3.10) yields the fourth order nonlinear ordinary differential equation

$$\mu^0 \left[\left(\varepsilon + \frac{1}{2} v_1'^2(x_2) \right)^{-\alpha/2} v_1'(x_2) \right]' - \mu_1 v_1''''(x_2) = p_1, \tag{5.4}$$

where $p_1 = \partial p / \partial x_1 = \text{const.}$; the corresponding equation for the Stokes constitutive law (3.11), with $\operatorname{div} \mathbf{v} = 0$, which results from (5.4) by setting $\alpha = \varepsilon = \mu_1 = 0$ and identifying μ_0 with μ^0, produces the parabolic profile

$$v_1(x_2) = -\left(\frac{p_1}{2\mu_0} \right) (a^2 - x_2^2), \qquad -a \leq x_2 \leq a. \tag{5.5}$$

For $\alpha = 0$, $\mu_0 = \mu^0$, $\mu_1 \neq 0$ we recover the equation produced by the linear dipolar theory, i.e., (3.22), whose solution subject to the boundary conditions (3.23) is given explicitly by the (still parabolic) profile displayed in (3.24). With $\mu_1 = 0$, (5.4) reduces to

$$\left[\left(\varepsilon + \frac{1}{2} v_1'^2(x_2) \right)^{-\alpha/2} v_1'(x_2) \right]' = \frac{p_1}{\mu^0} \tag{5.6}$$

so that for some real γ

$$\left(\varepsilon + \frac{1}{2} v_1'^2(x_2) \right)^{-\alpha/2} v_1'(x_2) = \left(\frac{p_1}{\mu^0} \right) x_2 + \gamma \equiv g(x_2). \tag{5.7}$$

If we set $w_\varepsilon = \varepsilon + v_1'^2(x_2)/2$ then it follows from (5.7) that w_ε satisfies the transcendental algebraic equation

$$w_\varepsilon^{1-\alpha} - \varepsilon w_\varepsilon^{-\alpha} = \frac{1}{2} g^2; \qquad \varepsilon > 0, \ 0 < \alpha < 1, \tag{5.8}$$

whose solutions are easily seen to depend continuously on ε as $\varepsilon \to 0^+$; we, therefore, turn our attention to (5.7) with $\varepsilon = 0$ or

$$2^{\alpha/2} |v_1'(x_2)|^{-\alpha} v_1'(x_2) = g(x_2) \tag{5.9}$$

from which it follows that $\operatorname{sgn} v_1'(x_2) = \operatorname{sgn} g(x_2)$. Thus, (5.9) yields

$$2^{\alpha/2} |v_1'(x_2)|^{1-\alpha} = \operatorname{sgn} g(x_2) \cdot g(x_2) \tag{5.10}$$

or

$$(\sqrt{2})^{\alpha/(1-\alpha)} v_1'(x_2) = \pm[\operatorname{sgn} g(x_2) \cdot g(x_2)]^{1/(1-\alpha)}, \tag{5.11}$$

where we choose the plus sign in (5.11) for $v_1'(x_2) > 0$ (equivalently, $g(x_2) > 0$) and the minus sign for $v_1'(x_2) < 0$ (equivalently, $g(x_2) < 0$). In view of the viscous behavior of the fluid we know that, besides $v_1(\pm a) = 0$, we must have $v_1'(-a) > 0$ and $v_1'(a) < 0$. Although (5.11) may be integrated for arbitrary α, $0 < x < 1$, it is instructive to proceed as follows: Consider the sequence $\{\alpha_n\}$, $0 < \alpha_n < 1$, for each positive integer n, given by $\alpha_n = (n-1)/n$; then $\alpha_n \to 1^-$ as $n \to \infty$ while $\alpha_1 = 0$. Setting $\alpha = \alpha_n$ in (5.11), and denoting the solution of the corresponding equation by $u_n(x_2)$, we have

$$2^{(n-1)/2} u_n'(x_2) = \pm(\operatorname{sgn} g(x_2))^n g^n(x_2) \tag{5.12}$$

so that

$$\begin{cases} 2^{(n-1)/2} u_n'(x_2) = \pm\, g^n(x_2), & n \text{ even}, \\ 2^{(n-1)/2} u_n'(x_2) = g^n(x_2), & n \text{ odd}, \end{cases} \tag{5.13}$$

where we have used the fact that the plus sign in (5.12) corresponds to $g(x_2) > 0$ while the minus sign corresponds to $g(x_2) < 0$. We will consider two special cases of (5.13):

(a) $n = 2$ $(\alpha_n = 1/2)$. In this case (5.13) applies and our differential equation reads

$$\sqrt{2} u_2'(x_2) = \pm\left[\left(\frac{p_1}{\mu^0}\right)^2 x_2^2 + \frac{2\gamma p_1}{\mu^0} x_2 + \gamma^2\right] \tag{5.14}$$

so that for some constant $\tilde{\gamma}$

$$\sqrt{2} u_2(x_2) = \pm\left[\left(\frac{p_1}{\mu^0}\right)^2 \frac{x_2^3}{3} + \frac{\gamma p_1}{\mu^0} x_2^2 + \gamma^2 x_2\right] + \tilde{\gamma}. \tag{5.15}$$

We now apply the boundary conditions $u_2(\pm a) = 0$, choosing the plus sign in (5.15) at $x_2 = -a$ and the minus sign at $x_2 = +a$; this follows from the fact that $u_2'(-a) > 0$ while $u_2'(a) < 0$. We obtain

$$\begin{cases} -\dfrac{1}{3}\left(\dfrac{p_1}{\mu^0}\right)^2 a^3 + \dfrac{\gamma p_1}{\mu^0} a^2 - \gamma^2 a + \tilde{\gamma} = 0, \\ -\left(\dfrac{1}{3}\left(\dfrac{p_1}{\mu^0}\right)^2 a^3 + \dfrac{\gamma p_1}{\mu^0} a^2 + \gamma^2 a\right) + \tilde{\gamma} = 0, \end{cases} \tag{5.16}$$

from which it follows that $\gamma = 0$ while $\tilde{\gamma} = (p_1/\mu^0)^2 a^3/3$. As $\gamma = 0$, for $p_1 < 0$

$$g(x_2) = \left(\frac{p_1}{\mu^0}\right) x_2 \quad \begin{cases} > 0 & (x_2 < 0), \\ < 0 & (x_2 > 0). \end{cases} \tag{5.17}$$

Thus

$$u_2(x_2) = \frac{2^{-1/2}}{3}\left(\frac{p_1}{\mu^0}\right)^2 \begin{cases} x_2^3 + a^3, & x_2 < 0, \\ -x_2^3 + a^3, & x_2 > 0. \end{cases} \tag{5.18}$$

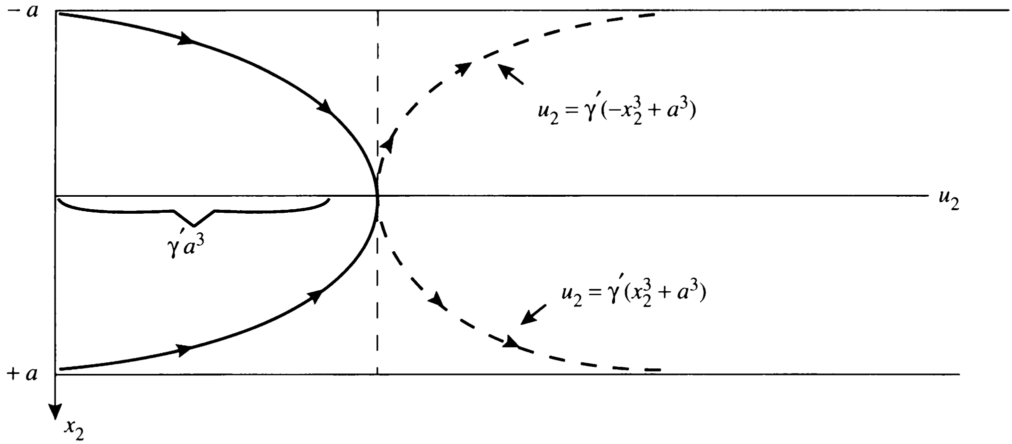

FIG. 3

A sketch of the velocity profile is depicted in Fig. 3, where we have set $\gamma' = \frac{2^{-1/2}}{3}(p_1/\mu^0)^2$.

For the velocity profile given by (5.18) we easily compute that

$$P = \int_{-a}^{a} u_2(x_2)\, dx_2 = \frac{3}{4}a^4\gamma' \equiv \frac{a^4}{2\sqrt{2}}\left(\frac{p_1}{\mu^0}\right)^2 . \tag{5.19}$$

(b) $n = 3$ $(\alpha_n = 2/3)$. In this case (5.13_2) applies and the differential equation is

$$2u_3'(x_2) = \left(\frac{p_1}{\mu^0}\right)^3 x_2^3 + 3\left(\frac{p_1}{\mu^0}\right)^2 \gamma x_2^2 + 3\frac{p_1}{\mu^0}\gamma^2 x_2 + \gamma^3 \tag{5.20}$$

so that

$$2u_3(x_2) = \left(\frac{p_1}{\mu^0}\right)^3 \frac{x_2^4}{4} + \left(\frac{p_1}{\mu^0}\right)^2 \gamma x_2^3 + \frac{3}{2}\cdot\frac{p_1}{\mu^0}\gamma^2 x_2^2 + \gamma^3 x_2 + \overline{\gamma} . \tag{5.21}$$

Applying the boundary conditions at $x_2 = \pm a$ then yields, after some algebraic manipulation, the relations

$$\begin{cases} \frac{1}{2}\left(\frac{p_1}{\mu^0}\right)^3 a^4 + \frac{3}{2}p_1\left(\frac{\gamma}{\mu^0}\right)^2 a^2 + 2\overline{\gamma} = 0 , \\ \gamma\left(2\left(\frac{p_1}{\mu^0}\right)^2 + 2\gamma^2 a\right) = 0 , \end{cases} \tag{5.22}$$

from which it is immediate that $\gamma = 0$ and $\overline{\gamma} = -(1/4)(p_1/\mu^0)^3 a^4$. Substituting these results into (5.21), we find as the explicit expression for the velocity profile in this case (again, with the assumption that $p_1 < 0$):

$$u_3(x_2) = \frac{1}{8}\left(\frac{|p_1|}{\mu^0}\right)^3 (a^4 - x_2^4) \tag{5.23}$$

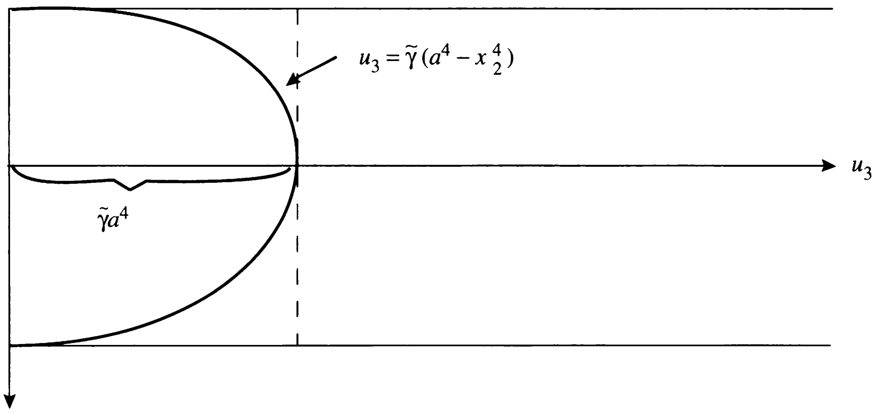

FIG. 4

a sketch of which is depicted in Fig. 4, where we have set $\tilde{\gamma} = (1/8)(|p_1|/\mu^0)^3$.

In this case we compute that

$$P = \int_{-a}^{a} u_3(x_2)\,dx_2 = \frac{8}{5}a^5\tilde{\gamma} \equiv \frac{a^5}{5}\left(\frac{|p_1|}{\mu^0}\right)^3 . \tag{5.24}$$

(ii) *Poiseuille flow in a pipe.* In this section, we look at a somewhat physically distinct problem, one which is genuinely two-dimensional in nature.

We begin by looking at steady flow in a cylinder of arbitrary cross section and take the x_1-axis parallel to the generators of the cylinder (which we will, henceforth, call a pipe). This time we ask if there exists a flow of the form

$$v_1 = v_1(x_2, x_3), \quad v_2 = 0, \quad v_3 = 0. \tag{5.25}$$

For the flow in (5.25) all components e_{ij} of the rate of deformation tensor vanish except for

$$e_{12} = e_{21} = \frac{1}{2}\frac{\partial v_1}{\partial x_2}; \qquad e_{13} = e_{31} = \frac{1}{2}\frac{\partial v_1}{\partial x_3}. \tag{5.26}$$

Then by virtue of (5.1)

$$\tau_{11} = \tau_{22} = \tau_{33} = -p; \qquad \tau_{23} = \tau_{32} = 0, \tag{5.27}$$

while

$$\tau_{12} = \mu^0\left(\varepsilon + \frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_2} - \mu_1\left(\frac{\partial^3 v_1}{\partial x_2^3} + \frac{\partial^3 v_1}{\partial x_3^2\partial x_2}\right) \tag{5.28}$$

and

$$\tau_{13} = \mu^0\left(\varepsilon + \frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_3} - \mu_1\left(\frac{\partial^3 v_1}{\partial x_3\partial x_2^2} + \frac{\partial^3 v_1}{\partial x_3^3}\right). \tag{5.29}$$

The equilibrium equations, (3.10), (5.25), (5.27), (5.28), and (5.29), then yield

$$\frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0, \tag{5.30}$$

$$
\begin{aligned}
-\frac{\partial p}{\partial x_1}+\mu^0\Bigg\{ & \frac{\partial}{\partial x_2}\left[\left(\varepsilon+\frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2+\left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_2}\right] \\
& +\frac{\partial}{\partial x_3}\left[\left(\varepsilon+\frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2+\left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_3}\right]\Bigg\} \\
-\mu_1 & \left(\frac{\partial^4 v_1}{\partial x_2^4}+2\frac{\partial^4 v_1}{\partial x_2^2\partial x_3^2}+\frac{\partial^4 v_1}{\partial x_3^4}\right)=0.
\end{aligned}
\tag{5.31}
$$

From (5.30), (5.31) it follows that $p=p(x_1)$ with $p'(x_1)=p_1=\text{const}$.

REMARKS. With $\mu^0=\mu_0$, $\varepsilon=\alpha=\mu_1=0$, we recover the Poiseuille flow predicted by the Navier-Stokes equations and governed by the Poisson equation

$$\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}=\frac{p_1}{\mu_0}. \tag{5.32}$$

With $r=\sqrt{x_2^2+x_3^2}$, and $v_1(x_2, x_3)=u(r)$, the case of (proper) Poiseuille flow in a pipe of circular cross section, (5.32) becomes

$$u''(r)+\frac{1}{r}u'(r)=\frac{p_1}{\mu_0} \tag{5.33}$$

and if the radius of a cross section is $R>0$ then by virtue of the viscous nature of the fluid $u(R)=0$; the well-known solution of this problem (e.g. Shinbrot [12]) for which $u(0)<\infty$ is given by

$$u(r)=\frac{-p_1}{4\mu_0}(R^2-r^2), \tag{5.34}$$

so that the speed varies parabolically across the pipe with a maximum at $r=0$.

We now consider (5.31) within the context of (proper) Poiseuille flow in a circular pipe whose cross section has radius R; i.e., we assume $v_1(x_2, x_3)=u(r)$ and introduce polar coordinates $x_2=r\cos\theta$, $x_3=r\sin\theta$ so that

$$\left(\frac{\partial v_1}{\partial x_2}\right)^2+\left(\frac{\partial v_1}{\partial x_3}\right)^2=u'^2(r), \tag{5.35}$$

while a lengthy but straightforward calculation yields

$$
\begin{aligned}
&\frac{\partial}{\partial x_2}\left[\left(2\varepsilon+\left(\frac{\partial v_1}{\partial x_2}\right)^2+\left(\frac{\partial v_1}{\partial x_3}\right)^2\right)^{-\alpha/2}\frac{\partial v_1}{\partial x_2}\right] \\
&\quad=\Gamma_\varepsilon(r)^{-\alpha/2}\left[u''(r)\frac{x_2^2}{r^2}+\frac{u'(r)}{r}\left\{1-\frac{x_2^2}{r^2}\right\}\right] \\
&\quad\quad-\alpha\Gamma_\varepsilon(r)^{-(\alpha/2+1)}u'^2(r)u''(r)\left(\frac{x_2^2}{r^2}\right)
\end{aligned}
\tag{5.36}
$$

with $\Gamma_\varepsilon(r) = 2\varepsilon + u'^2(r)$; an entirely analogous expression is obtained for

$$\frac{\partial}{\partial x_3}\left[\left(2\varepsilon + \left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right)^{-\alpha/2} \frac{\partial v_1}{\partial x_3}\right]$$

and addition of these expressions readily yields

$$\begin{aligned} 2^{-\alpha/2}\frac{\partial}{\partial x_2}&\left[\left(\varepsilon + \frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2} \frac{\partial v_1}{\partial x_2}\right] \\ &+ 2^{-\alpha/2}\frac{\partial}{\partial x_3}\left[\left(\varepsilon + \frac{1}{2}\left[\left(\frac{\partial v_1}{\partial x_2}\right)^2 + \left(\frac{\partial v_1}{\partial x_3}\right)^2\right]\right)^{-\alpha/2} \frac{\partial v_1}{\partial x_3}\right] \\ &= \Gamma_\varepsilon(r)^{-\alpha/2}\left[u'' + \frac{1}{r}u'(r)\right] - \alpha\Gamma_\varepsilon(r)^{-(\alpha/2-1)}u'^2(r)u''(r) \end{aligned} \tag{5.37}$$

so that for steady flow in a circular pipe (5.31) assumes, for $0 < r \le R$, the form

$$\begin{aligned} 2^{\alpha/2}\mu^0 &\left\{\Gamma_\varepsilon(r)^{-\alpha/2}\left[u''(r) + \frac{1}{r}u'(r)\right] - \alpha\Gamma_\varepsilon(r)^{-(\alpha/2+1)}u'^2(r)u''(r)\right\} \\ &- \mu_1\left\{u''''(r) + \frac{2}{r}u'''(r) - \frac{1}{r^2}u''(r) + \frac{1}{r^3}u'(r)\right\} = p_1 . \end{aligned} \tag{5.38}$$

By virtue of (3.8), (5.38) must be analyzed subject to the boundary conditions

$$u(R) = 0, \qquad u''(R) = 0. \tag{5.39}$$

A careful study of the full nonlinear boundary value problem (5.38), (5.39), which includes proofs of existence, uniqueness, and continuous dependence of $u(r)$ on both ε and μ_1 as $\varepsilon \to 0^+$, $\mu_1 \to 0^+$, is forthcoming. For the purposes of the present exposition we will content ourselves with an examination of the profiles predicted by (5.38), with $\varepsilon = \mu_1 = 0$, and the boundary condition $u(R) = 0$.

If we set $\mu_1 = 0$ in (5.38), and

$$z(r) = u'(r); \qquad c_1^\alpha = \frac{p_1}{\mu^0}\cdot 2^{\alpha/2} \tag{5.40}$$

we easily find that

$$z'(r)\left[1 - \frac{\alpha z^2(r)}{2\varepsilon + z^2(r)}\right] = -\frac{1}{r}z(r) + c_1^\alpha(2\varepsilon + z^2(r))^{\alpha/2} \tag{5.41}$$

which, for $\varepsilon = 0$, reduces to the Bernoulli equation

$$(1-\alpha)z'(r) = -\frac{1}{r}z(r) + c_1^\alpha(z^2(r))^{\alpha/2}. \tag{5.42}$$

The solutions of (5.42) are given, for arbitrary real c_2, by

$$z(r) \equiv u'(r) = \pm\left(\frac{c_1^\alpha r}{2} + \frac{c_2}{r}\right)^{1/(1-\alpha)}, \qquad 0 < r \le R. \tag{5.43}$$

For $p_1 > 0$ we must have $u'(R) < 0$, which dictates the choice of the minus sign in (5.43) on $(0, R]$; also, if $\lim_{r\to 0} u(r) < \infty$ then, clearly, we must also set $c_2 = 0$ in (5.43) so that for $0 < \alpha < 1$,

$$u'(r) = -k_\alpha r^{1/(1-\alpha)}, \qquad 0 < r \le R, \tag{5.44}$$

with

$$k_\alpha \equiv \left(\frac{c_1^\alpha}{2}\right)^{1/(1-\alpha)} = \left(\frac{p_1}{\mu^0 2^{\alpha/2+1}}\right)^{1/(1-\alpha)}.$$

For $\alpha = 0$, integration of (5.44) subject to $u(R) = 0$ yields the familiar result (5.34). For arbitrary α, $0 < \alpha < 1$, integration of (5.44) yields

$$u(r) = -k_\alpha \left(\frac{1-\alpha}{2-\alpha}\right) r^{(2-\alpha)/(1-\alpha)} + c_3 \tag{5.45}$$

and then imposition of the boundary condition at $r = R$ yields the velocity profiles

$$u(r) = k_\alpha \left(\frac{1-\alpha}{2-\alpha}\right) [R^{(2-\alpha)/(1-\alpha)} - r^{(2-\alpha)/(1-\alpha)}], \qquad 0 \le r \le R. \tag{5.46}$$

To obtain a clearer picture of the profiles (5.46) we again consider the sequence $\{\alpha_n\}$, $0 < \alpha_n < 1$, $\alpha_n \to 1^-$ as $n \to \infty$, given by $\alpha_n = (n-1)/n$. Setting $\alpha = \alpha_n$ in (5.46), and denoting the resulting profile by $u_n(r)$, we easily compute that

$$u_n(r) = K_n[R^{n+1} - r^{n+1}], \qquad 0 \le r \le R, \tag{5.47}$$

with

$$K_n = (p_1/\mu^0 2^{(3n-1)/2n})^n \cdot \frac{1}{n+1}. \tag{5.48}$$

For $n = 2$ $(\alpha = 1/2)$ we obtain

$$u_2(r) = \frac{1}{3 \cdot 2^{5/2}} \left(\frac{p_1}{\mu^0}\right)^2 [R^3 - r^3], \qquad 0 \le r \le R, \tag{5.49}$$

while for $n = 3$ $(\alpha = 2/3)$ we get

$$u_3(r) = \frac{1}{4 \cdot 2^4} \left(\frac{p_2}{\mu^0}\right)^3 [R^4 - r^4], \qquad 0 \le r \le R, \tag{5.50}$$

and it is clear that the profiles given explicitly by (5.47) exhibit the "flattening out" one sees in Poiseuille flow, for small values of the standard kinematic viscosity, prior to the breakdown of laminar flow and the onset of turbulence. The persistence of the profiles (5.46) for $\varepsilon \ne 0$, $\mu_1 \ne 0$ (but small) can be demonstrated by means of appropriate continuous dependence theorems for the behavior of the solutions of (5.38), (5.39) as ε and $\mu_1 \to 0^+$; an example of just such a continuous dependence theorem is proven in Sec. 6 for the problem of plane Poiseuille flow between parallel plates.

(iii) *Plane Couette flow.* As our last example we consider the problem of steady flow between parallel plates at $x_2 = \pm a$, but now assume that the bottom plate at

$x_2 = a$ is moving with constant velocity $\overline{u}$. The flow still has the form (5.3) but the boundary conditions are now

$$v_1(-a) = 0; \qquad v_1(a) = \overline{u}. \tag{5.51}$$

The solution of this problem within the framework of the steady Navier-Stokes equations is well known and is given by

$$v_1(x_2) = \frac{p_1}{2\mu_0} x_2^2 + \frac{\overline{u}}{2a} x_2 + \frac{1}{2}\left(\overline{u} - \frac{p_1}{\mu_0} a^2\right) \tag{5.52}$$

which, for $\overline{u} = 0$, reduces to the solution (5.5) of the plane Poiseuille flow problem between fixed parallel plates. In order to examine an example of this flow within the framework of the nonlinear theory for the dipolar fluid we consider the differential equation (5.13_1) for $n = 2$ $(\alpha = 1/2)$, i.e.,

$$\sqrt{2}u_2'(x_2) = \pm\left(\frac{p_1}{\mu^0} x_2 + \gamma\right)^2 \tag{5.53}$$

subject to the boundary conditions $u_2(-a) = 0$, $u_2(a) = \overline{u}$. Integration of (5.53) again yields (5.15), and as we must still have $u_2'(-a) > 0$, $u_2'(a) < 0$, imposition of the boundary conditions and some elementary algebraic manipulations yield

$$\gamma = -\frac{\mu^0 \overline{u}}{\sqrt{2} p_1 a^2}; \qquad \tilde{\gamma} = \frac{1}{3}\left(\frac{p_1}{\mu^0}\right)^2 a^3 + \frac{\overline{u}}{\sqrt{2}} + \frac{1}{2}\left(\frac{\mu^0}{p_1}\right)^2 \frac{\overline{u}^2}{a^3}. \tag{5.54}$$

Inserting the constants γ, $\tilde{\gamma}$ into (5.15), and simplifying, we find for the profile $u_2(x_2)$ the following explicit form:

$$\begin{aligned} u_2(x_2) = 2^{-1/2}\Bigg\{&\frac{1}{3}\left(\frac{p_1}{\mu^0}\right)^2 (x_2^3 + a^3) + \frac{\overline{u}}{\sqrt{2}}\left(1 - \frac{x_2^2}{a^2}\right) \\ &+ \frac{1}{2}\left(\frac{\mu^0}{p_1}\right)^2 \frac{\overline{u}^2}{a^4}(x_2 + a)\Bigg\}, \quad \text{for } x_2 \geq \frac{1}{\sqrt{2}}\left(\frac{\mu^0}{p_1}\right)^2 \frac{\overline{u}}{a^2}, \end{aligned} \tag{5.55a}$$

$$\begin{aligned} u_2(x_2) = 2^{-1/2}\Bigg\{&\frac{1}{3}\left(\frac{p_1}{\mu^0}\right)^2 (-x_2^3 + a^3) + \frac{\overline{u}}{\sqrt{2}}\left(1 + \frac{x_2^2}{a^2}\right) \\ &+ \frac{1}{2}\left(\frac{\mu^0}{p_1}\right)^2 \frac{\overline{u}^2}{a^4}(-x_2 + a)\Bigg\}, \quad \text{for } x_2 \leq \frac{1}{\sqrt{2}}\left(\frac{\mu^0}{p_1}\right)^2 \frac{\overline{u}}{a^2}. \end{aligned} \tag{5.55b}$$

A simple comparison shows that (5.55a, b) reduce to (5.18) if $\overline{u} = 0$.

6. Existence, uniqueness, and continuous dependence for the problem of plane Poiseuille flow. In this section we consider the problems of existence, uniqueness, and continuous dependence for the nonlinear boundary value problem

$$-\left[\frac{u'(x)}{(\varepsilon + u'^2(x))^{\alpha/2}}\right]' + \mu u''''(x) = f(x), \qquad -a \leq x \leq a, \tag{6.1}$$

$$u(\pm a) = u''(\pm a) = 0 \tag{6.2}$$

of which (5.4), subject to the boundary conditions $v_1(\pm a) = v_1''(\pm a) = 0$, is but a special case.[6] In (6.1), $\mu > 0$ and we will be interested in the behavior of solutions of (6.1), (6.2) as $\mu \to 0^+$; our continuous dependence result cannot, of course, hold in the C^2 sense (boundary-layer theory comes into effect at that level of smoothness) but does hold in the norm of $C^{1+\delta}$ for $0 < \delta < 1/2$. We will require that $f \in L^2(-a, a)$.

(i) *Existence of solutions.* Let $V = H_0^{3/2+\delta}(-a, a)$ with $0 < \delta < 1/2$ and let $B_M(0)$ be the ball of radius M in V where $M > 0$. We set

$$W_M = B_M(0) \cap H^2(-a, a). \tag{6.3}$$

By virtue of standard embedding results, W_M is compact in V for any $\delta < 1/2$. For $v \in V$ we define

$$L_v u = -\left[\frac{u'}{(\varepsilon + v'^2)^{\alpha/2}}\right]' + \mu u''''. \tag{6.4}$$

Then for fixed $v \in V$, the linear boundary value problem

$$\begin{cases} L_v u = f, & -a \le x \le a, \\ u(\pm a) = u''(\pm a) = 0 \end{cases} \tag{6.5}$$

has, as a consequence of the Lax-Milgram lemma, a unique solution $u \in H^2(-a, a)$ for which

$$\|u\|_{H^2(-a,a)} \le c\|f\|_{L^2(-a,a)} \tag{6.6}$$

with $c > 0$ independent of u. Let $T: v \to u$ where u is the unique solution of (6.5). For any given $f \in L^2(-a, a)$ it is a direct consequence of (6.6) that $\exists M > 0$ sufficiently large such that $T: W_M \to W_M$. We want to show that T is a continuous map. For $v, w \in W_M$, let $u_1 = Tv$, $u_2 = Tw$; then

$$-\left[\frac{u_1'}{(\varepsilon + v'^2)^{\alpha/2}}\right]' + \left[\frac{u_2'}{(\varepsilon + w'^2)^{\alpha/2}}\right]' + \mu[u_1 - u_2]'''' = 0. \tag{6.7}$$

Multiplying (6.7) by $u_1 - u_2$, integrating over $(-a, a)$, and then integrating by parts we obtain, in view of (6.5_2)

$$\begin{aligned} \mu \int_{-a}^{a} [(u_1 - u_2)''(x)]^2\,dx + \int_{-a}^{a} \frac{u_1'(x)(u_1 - u_2)'(x)}{(\varepsilon + v'^2(x))^{\alpha/2}}\,dx \\ - \int_{-a}^{a} \frac{u_2'(x)(u_1 - u_2)'(x)}{(\varepsilon + w'^2(x))^{\alpha/2}}\,dx = 0 \end{aligned} \tag{6.8}$$

or

$$\begin{aligned} \mu\|u_1 - u_2\|^2_{H^2(-a,a)} + \int_{-a}^{a} \frac{(u_1 - u_2)'(x)u_1'(x)[(\varepsilon + w'^2(x))^{\alpha/2} - (\varepsilon + v'^2(x))^{\alpha/2}]}{(\varepsilon + v'^2(x))^{\alpha/2}(\varepsilon + w'^2(x))^{\alpha/2}}\,dx \\ + \int_{-a}^{a} \frac{[(u_1 - u_2)'(x)]^2}{(\varepsilon + w'^2(x))^{\alpha/2}}\,dx = 0. \end{aligned} \tag{6.9}$$

[6] Continuous dependence on μ will be established, below, only for the case where $f(x) = \text{const}$.

As $u_1, u_2 \in H^2(-a, a)$, $u_1', u_2' \in L^\infty(-a, a)$, we may estimate the first integral in (6.9) from above, and drop the (nonnegative) second integral, so as to obtain an estimate of the form

$$\|u_1 - u_2\|_{H^2(-a,a)} \leq c_1 \left[\int_{-a}^{a} (|w'(x)|^\alpha - |v'(x)|^\alpha)^2 \, dx\right]^{1/2} \tag{6.10}$$

for some $c_1 > 0$; in obtaining (6.10) we have also employed the mean value theorem in the integrand of the first integral in (6.9). The continuity of T follows directly from the estimate (6.10). By the Schauder fixed point theorem it now follows that there exists, for $M > 0$ sufficiently large, a unique $u \in W_M$ such that $u = Tu$; we have established the existence of a unique solution of (6.1), (6.2) for arbitrary $\mu > 0$.

(ii) *Continuous dependence on* μ. From the results in (i), we infer the existence of a unique solution $u \in W_M$ $(M > 0$ sufficiently large) of the equation

$$-\left[\frac{u'(x)}{(\varepsilon + u'^2(x))^{\alpha/2}}\right]' + \mu\mu''''(x) = K, \qquad -a \leq x \leq a, \tag{6.11}$$

$K \in R^1$, subject to the boundary conditions (6.2). We set $v = u'$ in (6.11), and integrate the resulting equation over $(-a, x)$, $x \leq a$, so as to obtain

$$-\frac{v(x)}{(\varepsilon + v^2(x))^{\alpha/2}} + \mu v''(x) = K(x + a) - A_\mu. \tag{6.12}$$

For fixed $\varepsilon > 0$, it will be understood that $u(x)$ in (6.11) and $v(x)$ in (6.12) depend on μ, but we will refrain for now from writing u_μ or v_μ. Also, in (6.12)

$$A_\mu = \frac{v(-a)}{(\varepsilon + v^2(-a))^{\alpha/2}} - \mu v''(-a). \tag{6.13}$$

In view of the boundary conditions (6.2), $v'(-a) = v'(a) = 0$ and

$$\int_{-a}^{a} v(x)\, dx = u(a) - u(-a) = 0. \tag{6.14}$$

We now multiply (6.12) by $v(x)$, integrate over $(-a, a)$, and then integrate by parts to obtain

$$\int_{-a}^{a} \frac{v^2(x)}{(\varepsilon + v^2(x))^{\alpha/2}}\, dx + \mu \int_{-a}^{a} v'^2(x)\, dx = -K \int_{-a}^{a} xv(x)\, dx, \tag{6.15}$$

where we have used (6.14). Now, let

$$E_\varepsilon = \{x | v^2(x) > \varepsilon\}. \tag{6.16}$$

Then $\forall x \in E_\varepsilon$, $v^2/(\varepsilon + v^2)^{\alpha/2} > \beta_1 v^{2-\alpha}$, $\beta_1 = 2^{-\alpha/2} > 0$; similarly, as $u' \in$

$L^\infty(-a, a)$, on $E_\varepsilon^c = [-a, a]/E_\varepsilon$, $\exists \beta_2, \rho > 0$ such that $v^{2-\alpha} \le \beta_2 \varepsilon^\rho$. Therefore

$$\begin{aligned}\int_{-a}^{a} v^{2-\alpha}(x)\,dx &= \int_{E_\varepsilon} v^{2-\alpha}\,dx + \int_{E_\varepsilon^c} v^{2-\alpha}\,dx \\ &\le \frac{1}{\beta_1}\int_{E_\varepsilon} \frac{v_2}{(\varepsilon + v^2)^{\alpha/2}}\,dx + \beta_2\varepsilon^\rho \operatorname{meas}(E_\varepsilon^c) \\ &\le \frac{1}{\beta_1}\int_{-a}^{a} \frac{v^2}{(\varepsilon + v^2)^{\alpha/2}}\,dx + \beta_3 .\end{aligned} \tag{6.17}$$

Using the last estimate in (6.17) in (6.15) we have

$$\int_{-a}^{a} v^{2-\alpha}(x)\,dx + \frac{\mu}{\beta_1}\int_{-a}^{a} v'^2(x)\,dx \le \frac{K}{\beta_1}\int_{-a}^{a} |x|\,|v(x)|\,dx + \beta_3 . \tag{6.18}$$

By virtue of the Hölder inequality, (6.18) yields

$$\int_{-a}^{a} |v|^{2-\alpha}\,dx \le \frac{K}{\beta_1}\left[\int_{-a}^{a} |x|^{(2-\alpha)/(1-\alpha)}\,dx\right]^{(1-\alpha)/(2-\alpha)} \left[\int_{-a}^{a} |v|^{2-\alpha}\,dx\right]^{1/(2-\alpha)} + \beta_3 . \tag{6.19}$$

For arbitrary $\delta > 0$, we now use the Young inequality

$$|a|\cdot|b| \le \delta|a|^p + \delta^{-1/(p-1)}|b|^{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

on the right-hand side of (6.19) with $p = 2-\alpha$; for δ chosen sufficiently small we obtain from (6.19) an estimate of the form

$$\int_{-a}^{a} v^{2-\alpha}(x)\,dx \le a_1 \int_{-a}^{a} |x|^{(2-\alpha)/(1-\alpha)}\,dx + a_2 \tag{6.20}$$

with $a_1, a_2 > 0$. To obtain our next set of estimates we multiply (6.12) by $v''(x)$, integrate over $(-a, a)$, and integrate by parts; inasmuch as $v'(-a) = v'(a) = 0$ we easily find that

$$\begin{aligned}\int_{-a}^{a} \left[\frac{v(x)}{(\varepsilon + v^2(x))^{\alpha/2}}\right]' v'(x)\,dx + \mu\int_{-a}^{a} (v''(x))^2\,dx &= -K\int_{-a}^{a} v'(x)\,dx \\ &= K[v(-a) - v(a)]\end{aligned} \tag{6.21}$$

or

$$\int_{-a}^{a} \frac{v'^2(x)}{(\varepsilon + v^2(x))^{\alpha/2}} \cdot \left[\frac{\varepsilon + (1-\alpha)v^2(x)}{\varepsilon + v^2(x)}\right] dx + \mu\int_{-a}^{a} (v''(x))^2\,dx = K[v(-a) - v(a)]. \tag{6.22}$$

Now, $\forall\alpha$ with $0 < \alpha < 1$, $\exists \bar{c}_1, \bar{c}_2 > 0$ such that

$$\bar{c}_1 \le \frac{\varepsilon + (1-\alpha)\eta}{\varepsilon + \eta} \le \bar{c}_2, \qquad \forall \eta \ge 0, \tag{6.23}$$

where the $\bar{c}_i$, $i = 1, 2$, depend on α but not on η. Applying (6.23) to (6.22) with $\eta = v^2$ we get the estimate

$$\begin{aligned}\int_{-a}^{a} \frac{v'^2(x)}{(\varepsilon + v^2(x))^{\alpha/2}}\,dx + \frac{\mu}{\bar{c}_1}\int_{-a}^{a} (v''(x))^2\,dx &\le \frac{K}{\bar{c}_1}[v(-a) - v(a)] \\ &\le \bar{c}_3 \max_{[-a, a]} |v(x)| .\end{aligned} \tag{6.24}$$

We now set

$$\Psi(v) = \int_0^v \frac{ds}{(\varepsilon + s^2)^{\alpha/4}}. \tag{6.25}$$

Then it follows directly from (6.24) that

$$\int_{-a}^{a} \left(\frac{d}{dx}\Psi(v(x))\right)^2 dx \le \bar{c}_3 \max_{[-a,a]} |v(x)|. \tag{6.26}$$

As $\Psi(v)$ is an even function, and $1/(\varepsilon + s^2)^{\alpha/4} \le 1/(s^2)^{\alpha/4}$, we have that

$$|\Psi(v)| = \Psi(|v|) \le \left(1 - \frac{\alpha}{2}\right) |v|^{1-\alpha/2} \le |v|^{1-\alpha/2}. \tag{6.27}$$

Therefore, by virtue of our previous estimate (6.20), $\exists \Psi_0 > 0$ (const.) such that

$$\int_{-a}^{a} \Psi^2(v(x))\, dx \le \Psi_0. \tag{6.28}$$

Now, $\forall w \in H^1[-a, a]$, and $\forall \delta > 0, \exists c_\delta > 0$ such that

$$\max_{[-a,a]} |w| \le \delta \left(\int_{-a}^{a} w'^2(x)\, dx\right)^{1/2} + c_\delta \left(\int_{-a}^{a} w^2(x)\, dx\right)^{1/2} \tag{6.29}$$

(see, e.g., [11, Lemma 5.1]); applying (6.29) with $w = \Psi(v)$ and making use of both (6.26) and (6.28) we find that for some $d_\delta > 0$

$$\max_{[-a,a]} |\Psi(v)| \le \delta \left[\max_{[-a,a]} |v(x)|\right]^{1/2} + d_\delta. \tag{6.30}$$

Our goal now is to show that for some $c > 0$

$$\max_{[-a,a]} |v(x)| \le c\left[\max\left\{1, \max_{[-a,a]} |\Psi(v)|^\beta\right\}\right] \tag{6.31}$$

with c independent of v, and $\beta = 1/(1 - \alpha/2)$. To this end we define, for $s \in R^1$,

$$F(s) = \Psi(s) - ks^{1-\alpha/2}, \tag{6.32}$$

where k is chosen so that $\Psi(1) > k$. Thus, $F(1) \ge 0$ while

$$F'(s) = \frac{1}{(\varepsilon + s)^{\alpha/4}} - k\left(1 - \frac{\alpha}{2}\right) s^{-\alpha/2}$$

so that, for k chosen sufficiently small, $F'(s) \ge 0$, $\forall s \in R^1$. Consequently, $F(s) \ge 0$, $\forall s \ge 1$ so that

$$ks^{1/\beta} \le \Psi(s), \quad \forall s \ge 1 \tag{6.33}$$

with $\beta = 1/(1 - \alpha/2)$. Employing (6.33) in (6.30) we obtain

$$\max_{[-a,a]} |\Psi(v)| \le \delta'\left[\max\left\{1, \max_{[-a,a]} |\Psi(v)|^{\beta/2}\right\}\right] + d_\delta \tag{6.34}$$

with $\delta' = \delta k^{-\beta/2}$. If $\max_{[-a,a]} |\Psi(v)|^{\beta/2} > 1$, then

$$\max_{[-a,a]} |\Psi(v)| - \delta' \max_{[-a,a]} |\Psi(v)|^{\beta/2} \le d_\delta$$

or, as $\beta/2 = 1/(2-\alpha) < 1$, for $0 < \alpha < 1$,

$$(1-\delta') \max_{[-a,a]} |\Psi(v)| \le d_\delta .$$

Therefore, for δ chosen sufficiently small, it follows that $\exists C > 0$ such that (recall that $|\Psi(v)| = \Psi(|v|)$):

$$\max_{[-a,a]} [\Psi(|v(x)|)] \le C . \tag{6.35}$$

Clearly, an estimate of the form (6.35) also follows from (6.34) if $\max_{[-a,a]} |\Psi(v)|^{\beta/2} \ge 1$. Now, the estimate (6.31) is a direct consequence of (6.33), and the use of (6.35) in (6.33) then produces a bound of the form

$$\max_{[-a,a]} |v(x)| \le C'$$

for some $C' > 0$. Then, by virtue of (6.24), we have, for some $C > 0$,

$$\int_{-a}^{a} \left[\frac{d}{dx}\Psi(v(x))\right]^2 dx + \mu \int_{-a}^{a} (v''(x))^2\, dx \le C . \tag{6.36}$$

Combining (6.36) with (6.28), and no longer suppressing the dependence of v on μ, it follows that $\exists \tilde{C}$, independent of μ, such that

$$\|\Psi(v_\mu)\|_{H^1(-a,a)} \le \tilde{C} . \tag{6.37}$$

Therefore, $\exists \Psi^0 \in H^1(-a, a)$ such that

$$\Psi(v_\mu) \to \Psi^0 \quad \text{in } H^1(-a, a) \text{ as } \mu \to 0^+ \tag{6.38}$$

and, by virtue of (6.36), we also note that

$$\mu v''_\mu \to 0 \quad \text{in } L^2(-a, a) \text{ as } \mu \to 0^+ . \tag{6.39}$$

In view of (6.38), and our existence theorem, for some $\overline{\Psi}$ we have

$$\Psi(v_\mu) \to \overline{\Psi} \quad \text{in } C^{0,\delta}, \qquad 0 < \delta < 1/2 \text{ as } \mu \to 0^+ . \tag{6.40}$$

But Ψ, being monotone, is invertible, and as $\Psi^{-1} \in C^1(R^1)$ we find that

$$v_\mu \to \overline{v} \quad \text{in } C^{0,\delta}, \qquad 0 < \delta < 1/2 \text{ as } \mu \to 0^+ . \tag{6.41}$$

Finally, as $u'_\mu = v_\mu$ we have, for the unique solution $u_\mu(x)$ of (6.11), (6.2), that

$$u_\mu \to \overline{u} \quad \text{in } C^{1+\delta}, \qquad 0 < \delta < 1/2 \text{ as } \mu \to 0^+ \tag{6.42}$$

with $\overline{u}$ the unique solution of (6.2), with $\mu = 0$, subject to the boundary conditions $u(\pm a) = 0$; this concludes the demonstration of the continuous dependence of u_μ on μ, as $\mu \to 0^+$, in the $C^{1+\delta}$ norm, $0 < \delta < 1/2$. Continuous dependence of the solutions of the nonlinear boundary-value problem (6.11), (6.2) on ε, as $\varepsilon \to 0^+$, follows directly from the algebraic equation (5.8).

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