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ESTIMATING THE CRITICAL RADIUS FOR RADIALLY SYMMETRIC CAVITATION

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1. Introduction. In previous work [1, 2], it was shown how the problem of radially symmetric cavitation of a hyperelastic ball can be treated by an analysis of the radial equilibrium equation. That approach has been extended to a broader class of constitutive assumptions by Meynard [3, 4]. This note continues this study in two directions. First of all, by strengthening the assumptions on the stored energy we obtain an additional property of solutions, namely, the monotonicity of the determinant of the gradient of the deformation. This has several interesting consequences. In particular, it yields an explicit upper bound for the critical radius, λ^*, beyond which cavitation occurs. These results are given in Sec. 3, after the necessary hypotheses and definitions have been recalled briefly in Sec. 2. In Sec. 4, we turn to the second main topic, which is the extension of the results concerning cavitation to a class of stored-energy functions that do not necessarily satisfy the Legendre-Hadamard condition. The study of cavitation under such hypotheses was motivated by some recent work by Marcellini [7] and by some questions raised by D. Giachetti and R. Schianchi.

The direct discussion of the equilibrium equations offers an alternative to the original and fundamental treatment of cavitation by Ball [5] based on the calculus of variations. However, it does not determine the stability of the equilibrium solutions even against radial perturbation. In this direction, the use of the field theory by Sivaloganathan [6] to discuss the stability of solutions constitutes an important complement to the present approach. The whole question of nonradially symmetric solutions and perturbations in discussed in [10].

2. Hypotheses and previous results. We confine the discussion to a brief presentation of the notions that are necessary for the subsequent new contributions. More ample information about these results is contained in [1, 2, 5].

The results concern solutions U, of the following differential equation,

$$r\frac{d}{dr}\Phi_1 = (N-1)[\Phi_2 - \Phi_1] \quad \text{for } 0 < r < 1, \tag{2.1}$$

where Φ_i denotes the ith partial derivative of a given symmetric function $\Phi:(0,\infty)^N$

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$\to \mathbb{R}$, evaluated at $(U'(r), U(r)/r, \dots, U(r)/r)$. The solution U must satisfy the constraints,

$$U'(r) > 0 \quad \text{and} \quad U(r) > 0 \quad \text{for } 0 < r < 1. \tag{2.2}$$

In elasticity, $N = 2$ or 3 and Φ is the stored-energy function for some hyperelastic material. The solution U defines a radially symmetric deformation u of the unit ball $B(0, 1)$ in $\mathbb{R}^N$ through

$$u(x) = \frac{U(r)}{r} x \quad \text{for } 0 < |x| = r < 1.$$

In a displacement boundary-value problem,

$$U(1) = \lambda \tag{2.3}$$

where $\lambda > 0$ is given. The configuration determined by U is in equilibrium provided that the radial equilibrium equation (2.1) is satisfied,

$$r^{N-1}\Phi_i \in L^1(0, 1) \quad \text{for } i = 1, 2 \tag{2.4}$$

and either

$$\lim_{r\to 0} U(r) = 0 \tag{2.5a}$$

or

$$\lim_{r\to 0} U(r) > 0 \quad \text{and} \quad \lim_{r\to 0} \widetilde{T}(r) = 0, \tag{2.5b}$$

where

$$\widetilde{T}(r) = \left[\frac{U(r)}{r}\right]^{1-N} \Phi_1 \tag{2.6}$$

and the derivatives of Φ are evaluated at $(U'(r), U(r)/r, \dots, U(r)/r)$. The expression for $\widetilde{T}(r)$ gives the radial component of the Cauchy stress on the sphere of radius $U(r)$. The condition (2.5) distinguishes the cases (a) where there is no cavitation from the cases (b) where there is a cavity of radius $R = \lim_{r\to 0} U(r) > 0$.

In [1, 2] this problem is discussed under the following hypotheses on Φ.

(A1) $\Phi: (0, \infty)^N \to \mathbb{R}$ is of class C^3 and is symmetric.

(A2) $\Phi_{11}(q, t, \dots, t) > 0$ $\forall q, t \in (0, \infty)$, and $\exists C, t_0 > 0$ such that $\Phi_{11}(q, t, \dots, t) \geq Ct^{2(N-1)}$ whenever $0 < q < t$ and $t \geq t_0$.

(A3) $\forall b > 0$ we have

$$\lim_{(q,t)\to(0,b)} t^{1-N}\Phi_1(q, t, \dots, t) = -\infty \quad \text{and} \quad \lim_{(q,t)\to(b,\infty)} t^{1-N}\Phi_1(q, t, \dots, t) = +\infty.$$

(A4) $\lim_{t\to 0+} t^{1-N}\Phi_1(t, t, \dots, t) = -\infty$ and $\lim_{t\to\infty} t^{1-N}\Phi_1(t, t, \dots, t) = +\infty$.

(A5) $\inf \Phi > -\infty$ and $(\Phi_2 - \Phi_1)/(q - t) - \Phi_{12} < 0$ $\forall q, t \in (0, \infty)$ with $q \neq t$ where the partial derivatives of Φ are evaluated at $(q, t, \dots, t)$. For $q \neq t$, set

$$R(q, t) = \frac{q\Phi_1(q, t, \dots, t) - t\Phi_2(q, t, \dots, t)}{q - t}. \tag{2.7}$$

Then R has a C^1 extension to $(0, \infty)^2$.

(A6) $\exists A, B \geq 0$ and $0 < \beta < N - 1$ such that $0 \leq R(q, t) \leq A + Bt^\beta$ for $0 < q \leq t$.

(A7) $\exists \varepsilon, t_0, K > 0$ and $0 < \gamma < 2(N-1)$ such that $|\partial R(q, t)/\partial q| \leq K t^{\gamma}$ for $0 < q \leq \varepsilon$ and $t \geq t_0$.

The following result is part of Theorem 7 of [1]. It describes the set of all radial equilibrium solutions.

THEOREM 2.1. Let the hypotheses (A1) to (A7) be satisfied.

(i) For all $\lambda > 0$, $U_t(\lambda)(r) = \lambda r$ is the unique solution of (2.1)–(2.4), (2.5a).

(ii) $\exists \lambda^* > 0$ such that, for $0 < \lambda \leq \lambda^*$, (2.1)–(2.4), (2.5b) has no solution, whereas, for $\lambda > \lambda^*$, (2.1)–(2.4), (2.5b) has a unique solution $U_c(\lambda)$.

REMARK 1. In [1, 2], the assumption (A6) is made in the stronger form,

$$0 < R(q, t) \leq A + Bt^{\beta} \quad \text{for } 0 < q \leq t. \tag{2.8}$$

However, most of the results of [1, 2] and, in particular, the conclusions that we have stated as Theorem 2.1 above, are easily seen to hold under the assumption (A6). More precisely, when (2.8) is replaced by (A6) the results in Sec. 5 of [1] are true with the following modifications:

$$\widetilde{T}'(r) \geq 0 \quad \text{instead of } \widetilde{T}'(r) > 0 \text{ in Lemma 1}, \tag{2.9}$$

$$\widetilde{T}(r) \leq \lambda^{1-N}\Phi_1(\alpha, \lambda, \ldots, \lambda) < \lambda^{1-N}\Phi_1(\lambda, \ldots, \lambda) \tag{2.10}$$

instead of (c) in Lemma 3 and,

$$-\infty < g(\lambda) \leq \lambda^{1-N}\Phi_1(\lambda, \ldots, \lambda) \tag{2.11}$$

instead of (a) in Corollary 6.

Recall that $g: (0, \infty) \to \mathbb{R}$ is the function defined by

$$g(\lambda) = \lambda^{1-N}\Phi_1(\lambda, \ldots, \lambda) - (N-1)\int_{\lambda}^{\infty} t^{-N} R(Q_{\lambda}(t), t)\, dt, \tag{2.12}$$

where Q_{λ} is the unique solution of the initial-value problem,

$$\begin{aligned} q'(t) &= (N-1)\left\{\frac{\Phi_2 - \Phi_1}{q - t} - \Phi_{12}\right\}/\Phi_{11}, \\ q(\lambda) &= \lambda \end{aligned} \tag{2.13}$$

in which the derivatives of Φ are evaluated at $(q(t), t, \ldots, t)$.

Since (2.13) cannot be integrated explicitly (except in a few special cases), the function g is not known explicitly. Hence, in Theorem 2.1, the critical radius λ^* for cavitation, which is characterised as the unique solution of the bifurcation equation

$$g(\lambda) = 0, \tag{2.14}$$

cannot be calculated explicitly. However, by (A6), $\lambda^* \geq \lambda_n$ where λ_n is any solution of the equation

$$\Phi_1(\lambda, \lambda, \ldots, \lambda) = 0. \tag{2.15}$$

If (A6) is replaced by (2.8), we have $\lambda^* > \lambda_n$. Since $\lambda^{1-N}\Phi_1(\lambda, \lambda, \ldots, \lambda)$ is the radial Cauchy stress in the homogeneous deformation $U_t(\lambda)$, λ_n is called a natural

radius for the hyperelastic ball. Under the assumptions (A1)–(A7) the ball may have more than one natural radius. However, for $\lambda > 0$,

$$\begin{aligned}\frac{d}{d\lambda}\{\lambda^{1-N}\Phi_1(\lambda, \lambda, \dots, \lambda)\} &= (1-N)\lambda^{-N}\Phi_1(\lambda, \lambda, \dots, \lambda) \\ &\quad + \lambda^{1-N}\{\Phi_{11}(\lambda, \lambda, \dots, \lambda) + (N-1)\Phi_{12}(\lambda, \lambda, \dots, \lambda)\} \\ &= N\lambda^{1-N}\Phi_{11}(\lambda, \lambda, \dots, \lambda) - (N-1)\lambda^{-N}R(\lambda, \lambda)\end{aligned}$$

by the formula (4.2) of [1] for $R(\lambda, \lambda)$. Thus, $\lambda^{1-N}\Phi_1(\lambda, \lambda, \dots, \lambda)$ is strictly increasing and λ_n is unique, provided that

$$R(\lambda, \lambda) < \frac{N}{N-1}\lambda\Phi_{11}(\lambda, \lambda, \dots, \lambda). \tag{2.16}$$

In particular, by (A2) this is so when $R(q, t) \equiv 0$, and in this case, $\lambda_n = \lambda^*$ since

$$g(\lambda) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \dots, \lambda). \tag{2.17}$$

In Sec. 3, we obtain an upper bound for λ^* under quite general conditions (Theorem 3.3) and an alternative lower bound under somewhat more restrictive hypotheses (Theorem 3.5).

REMARK 2. The proof of Theorem 2.1 and the definition of g are based on the following observations. All the solutions of (2.1)–(2.5) are to be found among the solutions $U(\lambda, \alpha)$ of the initial-value problem (2.1)–(2.3) and

$$U'(1) = \alpha, \quad \text{where } 0 < \alpha \le \lambda. \tag{2.18}$$

For $\alpha = \lambda$, we obtain the homogeneous solution $U_t(\lambda)$. For $0 < \alpha < \lambda$, the assumptions (A1)–(A7) imply that

$$\left(\frac{U(r)}{r}\right)' < 0 \quad \text{and} \quad U''(r) > 0. \tag{2.19}$$

Thus, $\lim_{r\to 0} U(\lambda, \alpha) \ge \lambda - \alpha$ and so we must find $\alpha \in (0, \lambda)$ such that

$$\tau(\lambda, \alpha) = 0. \tag{2.20}$$

Here $\tau(\lambda, \alpha) = \lim_{r\to 0}\widetilde{T}(r)$ and $\widetilde{T}(r)$ is calculated using (2.6) and the solution $U(\lambda, \alpha)$. The inequalities (2.19) lead to the change of variables,

$$t = \frac{U(r)}{r} \quad \text{and} \quad q(t) = U'(r), \tag{2.21}$$

transforming (2.1)–(2.3), (2.18) into (2.13) with the initial condition

$$q(\lambda) = \alpha. \tag{2.22}$$

To close this summary, we recall that an Ogden material has a stored energy of the form,

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{3}\phi(\lambda_i) + \psi(\lambda_1\lambda_2) + \psi(\lambda_2\lambda_3) + \psi(\lambda_3\lambda_1) + h(\lambda_1\lambda_2\lambda_3), \tag{2.23}$$

where $\phi, \psi, h: (0, \infty) \to \mathbb{R}$ are given functions. The assumptions (A1)–(A7) are satisfied provided that we impose the following conditions:

(O1) $\phi, \psi, h \in C^3((0, \infty))$,
(O2) $\phi'', \psi'' \geq 0$ and $h'' \geq \delta > 0$ on $(0, \infty)$,
(O3) $\exists A, \varepsilon_0 > 0$ such that $sh'(s) \leq -A \ \forall s \in (0, s_0)$ and $\lim_{s\to\infty} h'(s) = +\infty$,
(O4) $\phi \geq 0$ and $\psi \geq 0$.

Let $\tilde{\phi}(s) = s\phi'(s)$ and $\tilde{\psi}(s) = s\psi'(s)$.

(O5) $\exists A, B > 0$ and $0 < \beta < 2$ such that

$$0 \leq \tilde{\phi}'(s) \leq A + Bt^{\beta} \quad \text{for } 0 \leq s \leq t,$$
$$0 \leq t\tilde{\psi}'(s) \leq A + Bt^{\beta} \quad \text{for } 0 < s \leq t^2.$$

For an Ogden material satisfying these conditions,

$$\begin{aligned} t^{-2}\Phi_1(q, t, t) &= t^{-2}\phi'(q) + 2t^{-1}\psi'(qt) + h'(qt^2), \\ R(q, t) &= \{\tilde{\phi}(q) - \tilde{\phi}(t) + \tilde{\psi}(qt) - \tilde{\psi}(t^2)\}/(q - t), \\ \Phi_{11}(q, t, t) &= \phi''(q) + 2t^2\psi''(qt) + t^4 h''(qt^2), \\ \Phi_{12} - \frac{\Phi_2 - \Phi_1}{q - t} &= \frac{\phi'(q) - \phi'(t)}{(q - t)} + \frac{t\{\psi'(qt) - \psi'(t^2)\}}{(q - t)} \\ &\quad + qt\psi''(qt) + qt^3 h''(qt^2). \end{aligned} \tag{2.24}$$

By (O2), (O3), $\lim_{s\to 0} h'(s) = -\infty$, $\lim_{s\to 0} h(s) = \lim_{s\to\infty} h(s) = +\infty$ and $\exists$ a unique $p_0 > 0$ such that $h'(p_0) = 0$ and $h(s) > h(p_0)$ for $s \in (0, \infty)\backslash\{p_0\}$.

LEMMA 2.2. Under the assumptions (O1)–(O5) the Ogden material defined by (2.23) satisfies the conditions (A1)–(A7).

REMARK 3. As Meynard has shown [3, 4], the conditions (A6), (A7) can be replaced by considerably weaker assumptions without altering the conclusions of Theorem 2.1. In the context of Ogden materials, Meynard's hypotheses cover cases where $\tilde{\phi}'(s)$ and/or $\tilde{\psi}'(s)$ can tend to minus infinity as s tends to zero, and hence our condition (O5) is not necessarily satisfied. His assumptions about h are similar to (O2) and (O3). In Sec. 4, we show how cavitating solutions can be obtained under much weaker assumptions on h, but to achieve this we have to strengthen the hypotheses on ϕ and ψ.

3. Monotonicity of the determinant. In this section we impose a further restriction on the stored energy

(A8) $\partial R(q, t)/\partial q \geq 0$ for all $0 < q \leq t$.

The discussion in [9] that is used to motivate the tension-extension inequality ($\Phi_{11} > 0$) and the Baker-Eriksen inequality ($R > 0$) gives a physical interpretation of (A8). The main mathematical consequence of (A8) is that it implies the monotonicity of the quantity

$$d(r) = U'(r)\left[\frac{U(r)}{r}\right]^{N-1} \tag{3.1}$$

for all solutions of the equation (2.1). Since $U'(r)$ and $U(r)/r$ (repeated $(N-1)$ times) are the eigenvalues of the gradient of the radial deformation u associated with U, we see that d is the determinant of this gradient and as such determines the volume element corresponding to the deformation. It is convenient to derive the monotonicity of d using the variables q and t defined by (2.21). Hence, we set

$$p(t) = q(t)t^{N-1}. \tag{3.2}$$

THEOREM 3.1. Let (A1) to (A8) be satisfied. Then for $0 < \alpha \leq \lambda$, the solution of (2.13), (2.22) is defined on $[\lambda, \infty)$ and we have

$$0 < q(t) < t \quad \text{and} \quad p'(t) \leq 0 \quad \text{for } t > \lambda.$$

If $\partial R(q, t)/\partial q > 0$ for $0 < q < t$, then $p'(t) < 0$ for $0 < q < t$.

Proof. In view of Lemma 4 of [1], it is enough to prove that $p'(t) \leq 0$. From the definition of R,

$$\frac{\partial R}{\partial q}(q, t) = \frac{t}{(q-t)}\left\{\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} + qt^{-1}\Phi_{11}\right\}, \tag{3.3}$$

where the derivatives of Φ are calculated at $(q, t, \dots, t)$. By the definition of p,

$$q'(t) = p'(t)t^{1-N} - (N-1)p(t)t^{-N}$$

and substituting into the equation (2.13) we obtain

$$\begin{aligned} t^{1-N}\Phi_{11}p'(t) &= (N-1)\left\{\frac{\Phi_2 - \Phi_1}{q-t} - \Phi_{12} + t^{-N}\Phi_{11}p(t)\right\} \\ &= (N-1)t^{-1}\{q(t) - t\}\frac{\partial R}{\partial q}(q(t), t). \end{aligned} \tag{3.4}$$

From (A2), (A8), and (2.19), we see that $p'(t) \leq 0$ for $t > \lambda$.

COROLLARY 3.2. Let (A1) to (A8) be satisfied. Then for $0 < \alpha < \lambda$, $d'(r) \geq 0$ for $0 < r < 1$, with strict inequality when (A8) is strict. For $\alpha = \lambda > 0$, $d(r) \equiv \lambda^N$ for $0 < r \leq 1$.

Proof. For $0 < \alpha < \lambda$, $d(r) = p(t)$, where $t = U(r)/r$. Hence $d'(r) = p'(t)(U(r)/r)'$ and the result follows from (2.19) and Theorem 3.1.

As a first application of these observations to the problem of cavitation, we derive an upper bound for the critical radius λ^*. For this, we introduce an auxiliary function, $G: (0, \infty) \to \mathbb{R}$, defined by

$$G(\lambda) = \lambda^{1-N}\Phi_1(\lambda, \lambda, \dots, \lambda) - (N-1)\int_\lambda^\infty t^{-N}R(\lambda^N t^{1-N}, t)\,dt. \tag{3.5}$$

THEOREM 3.3. Let the conditions (A1)–(A8) be satisfied. Then

(i) $G \in C^1((0, \infty))$ and $g(\lambda) \geq G(\lambda)$ for $\lambda > 0$,
(ii) $\lim_{\lambda \to 0} G(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} G(\lambda) = +\infty$,
(iii) $\lambda_n \leq \lambda^* \leq \Lambda$, where λ_n is a natural radius and $G(\Lambda) = 0$.

Proof. For $\lambda > 0$, let $I(\lambda) = \int_\lambda^\infty t^{-N} R(\lambda^N t^{1-N}, t)\, dt$. Using (A6) and (A7), we see that $I \in C^1((0, \infty))$ with $I(\lambda) \geq 0$ and $\lim_{\lambda\to\infty} I(\lambda) = 0$. Hence, $G \in C^1((0, \infty))$ and

$$\lim_{\lambda\to\infty} G(\lambda) = \lim_{\lambda\to\infty} \lambda^{1-N} \Phi_1(\lambda, \lambda, \dots, \lambda) = +\infty.$$

By Theorem 3.1 with $\alpha = \lambda$,

$$\lambda^N = Q_\lambda(\lambda)\lambda^{N-1} \geq Q_\lambda(t) t^{N-1} \quad \text{for } t \geq \lambda$$

and so, again using (A8), we have that

$$R(\lambda^N t^{1-N}, t) \geq R(Q_\lambda(t), t) \quad \text{for } t \geq \lambda.$$

Therefore, $G(\lambda) \leq g(\lambda)$ and consequently $\lim_{\lambda\to 0} G(\lambda) = -\infty$. This completes the proof.

To give an example of how this result can be used and to show some further consequences of Theorem 3.1 we consider an Ogden material. The condition (A8) is ensured by

$$\text{(O6)} \qquad \tilde{\phi}'', \tilde{\psi}'' \geq 0 \quad \text{on } (0, \infty).$$

LEMMA 3.4. Let Φ be of the form (2.23) where (O1)–(O6) are satisfied. Then (A1)–(A8) are satisfied and if either $\tilde{\phi}'' > 0$ or $\tilde{\psi}'' > 0$ on $(0, \infty)$ we have strict inequality in (A8).

Proof. In view of Lemma 2.2, we need only verify (A8).

By a straightforward calculation based on (2.24),

$$\begin{aligned}\frac{\partial R}{\partial q}(q, t) &= \{(q-t)\tilde{\phi}'(q) - [\tilde{\phi}(q) - \tilde{\phi}(t)]\}(q-t)^{-2} \\ &\quad + \{(q-t)t\tilde{\psi}'(qt) - [\tilde{\psi}(qt) - \tilde{\psi}(t^2)]\}(q-t)^{-2} \geq 0\end{aligned}$$

and the inequality is strict if either $\tilde{\phi}'' > 0$ or $\tilde{\psi}'' > 0$ on $(0, \infty)$.

For an Ogden material, the function G can be written as

$$\begin{aligned}G(\lambda) &= \lambda^{-2}\phi'(\lambda) + 2\lambda^{-1}\psi'(\lambda^2) + h'(\lambda^3) \\ &\quad - 2\int_\lambda^\infty \frac{\tilde{\phi}(\lambda^3 t^{-2}) - \tilde{\phi}(t) + \tilde{\psi}(\lambda^3 t^{-1}) - \tilde{\psi}(t^2)}{t(\lambda^3 - t^3)}\, dt.\end{aligned} \tag{3.6}$$

In the special case where

$$\phi(s) = As^\alpha \quad \text{and} \quad \psi(s) = Cs^\gamma \tag{3.7}$$

with $A, C \geq 0$, $1 < \alpha < 3$, and $1 < \gamma < 3/2$, we set $x = (\lambda/t)^3$ and the formula for G reduces to

$$G(\lambda) = AM(\alpha)\lambda^{\alpha-3} + CN(\gamma)\lambda^{2\gamma-3} + h'(\lambda^3), \tag{3.8}$$

where

$$M(\alpha) = \alpha\left\{1 - \frac{2}{3}\int_0^1 \left(\frac{1-x^\alpha}{1-x}\right) x^{-\alpha/3}\, dx\right\}$$

and

$$N(\gamma) = 2\gamma \left\{ 1 - \frac{1}{3} \int_0^1 \left(\frac{1 - x^\gamma}{1 - x} \right) x^{-2\gamma/3} \, dx \right\}.$$

A graphical representation of the location of λ_n and Λ is given in Fig. 1 for the case where $A > 0$, $\alpha = 2$, and $C = 0$ in (3.7).

For an Ogden material we can also improve the lower bound λ_n, for the critical radius. This follows from an observation about the stress on the surface of a cavity.

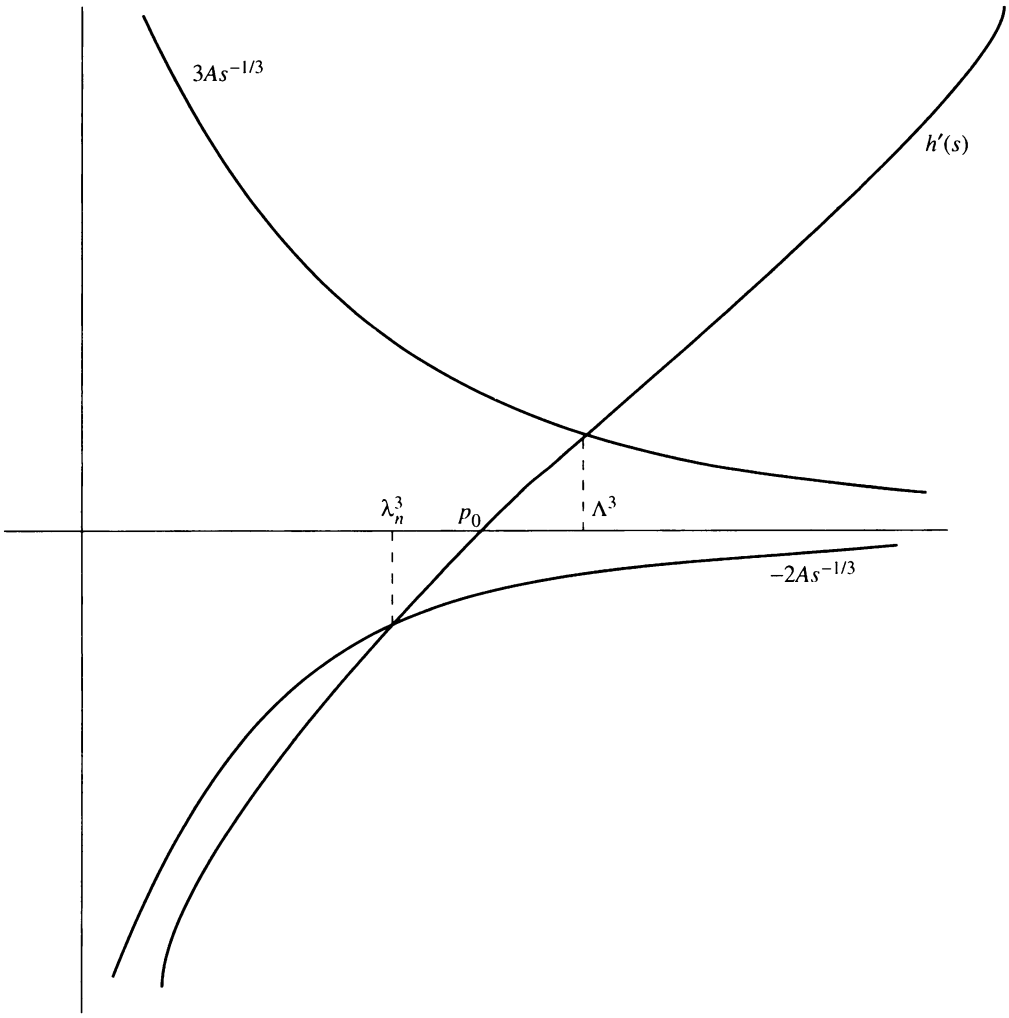

FIG. 1. Graphical representation of the determination of λ_n and Λ for the case (3.7) with $A > 0$, $\alpha = 2$, and $C = 0$, $G(\lambda) = -3A\lambda^{-1} + h'(\lambda^3)$, and $\lambda^{-2}\phi_1(\lambda, \lambda, \lambda) = 2A\lambda^{-1} + h'(\lambda^3)$.

THEOREM 3.5. Let Φ be of the form (2.23) where (O1)–(O6) are satisfied and assume, in addition, that

(O7) $\lim_{s\to 0}\tilde{\phi}'(s) > -\infty$ and $\lim_{s\to 0}\tilde{\psi}'(s) > -\infty$.

Then, for $0 < \alpha < \lambda$,

$$0 < D(\lambda, \alpha) \equiv \lim_{r\to 0} d(\lambda, \alpha)(r) \le \alpha\lambda^2 \tag{3.9}$$

and

$$\tau(\lambda, \alpha) = h'(D(\lambda, \alpha)), \tag{3.10}$$

where $d(\lambda, \alpha)$ is defined by (3.1) using the solution $U(\lambda, \alpha)$ of (2.1)–(2.3), (2.18). In particular, $\lambda^* \ge p_0^{1/3}$ and, for any $\lambda > \lambda^*$, the cavitating solution $U_c(\lambda)$ is such that $\lim_{r\to 0} d(r) = p_0$.

REMARK. For an Ogden material, the equation for a natural radius becomes

$$\phi'(\lambda) + 2\lambda\psi'(\lambda^2) + h'(\lambda^3) = 0. \tag{3.11}$$

If ϕ and ψ satisfy the condition

(O8) $\lim_{s\to 0}\phi'(s) \ge 0$ and $\lim_{s\to 0}\psi'(s) \ge 0$

as well as (O1)–(O6), we see that (O7) is verified and that $h'(\lambda_n^3) \le 0$. Hence when (O1)–(O6), (O8) are satisfied, $p_0 \ge \lambda_n^3$ and in this case $\lambda^* \ge p_0^{1/3}$ is a better estimate than $\lambda^* \ge \lambda_n$.

Proof. For an Ogden material, the definition of $\tau(\lambda, \alpha)$ reduces to

$$\tau(\lambda, \alpha) = \lim_{t\to\infty}\{t^{-2}\phi'(q(t)) + 2t^{-1}\psi'(q(t)t) + h'(q(t)t^2)\}.$$

By Theorem 3.1,

$$0 \le D(\lambda, \alpha) \le q(t)t^2 \le \alpha\lambda^2 \quad \text{for } t \ge \lambda$$

and so

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} q(t)t = 0.$$

Thus $\tau(\lambda, \alpha) = \lim_{t\to\infty} h'(q(t)t^2)$. Since $\tau(\lambda, \alpha) > -\infty$ by Lemma 6(a) of [1], it follows that $D(\lambda, \alpha) = \lim_{t\to\infty} q(t)t^2 > 0$ and that (3.10) holds.

For $\lambda > \lambda^*$ and the cavitating solution $U_c(\lambda)$ we have by (2.5b) that

$$h'(D) = 0 \quad \text{and} \quad D \le \alpha\lambda^2 < \lambda^3.$$

Thus $D = p_0$ and $\lambda^* \ge p_0^{1/3}$.

4. More general stored energies. To establish the existence of cavitating solutions for Ogden materials we have imposed the conditions (O1)–(O5). As noted in [1, 2, 5], these hypotheses imply that the stored-energy function satisfies the Legendre-Hadamard condition. Using the results of the preceding section we can now treat some stored energies of the form (2.23) which do not satisfy (O1)–(O5) and do not satisfy the L–H condition. The idea is to use the monotonicity results to weaken the assumptions on h. We replace (O2)–(O3) by the following conditions.

(O2)* ϕ'', $\psi'' \ge 0$ on $(0, \infty)$ and $\exists p_1 > p_0 > 0$ and $\delta > 0$ such that $h'(p_0) = 0$ and $h''(s) \ge \delta > 0$ for $p_0 \le s \le p_1$.

(O3)* The equation $G(\lambda) = 0$ has a solution $\Lambda \in [p_0^{1/3}, p_1^{1/3})$ where G is defined by (3.5).

THEOREM 4.1. Let Φ be of the form (2.23) where the conditions (O1), (O2)*, (O3)*, (O4), (O5), (O6), and (O7) are satisfied. Then $\exists \lambda^* \in [p_0^{1/3}, p_1^{1/3})$ such that for each $\lambda > \lambda^*$ there is a solution $U_c(\lambda)$ satisfying (2.1)–(2.4) and (2.5b).

Proof. There exists a function $h_1 \in C^3((0, \infty))$ such that the Ogden material defined by (2.23) but with h replaced by h_1 satisfies all the conditions (O1) to (O7). Hence for this material the conditions (A1) to (A8) are satisfied and by Theorem 2.1, $\exists$ a critical radius λ^* for cavitation. By Theorems 3.3 and 3.5, for this material,

$$p_0^{1/3} \le \lambda^* \le \Lambda_1,$$

where $G_1(\Lambda_1) = 0$ and G_1 is defined by (3.5) with h replaced by h_1. Since $G = G_1$ on $[p_0^{1/3}, p_1^{1/3}]$, we can choose $\Lambda_1 = \Lambda \in [p_0^{1/3}, p_1^{1/3})$. Thus $\lambda^* \in [p_0^{1/3}, p_1^{1/3})$ and for $\lambda > \lambda^*$, the unique cavitating solution $U_c(\lambda)$ for the material defined by h_1 has the following properties: $d'_\lambda(r) \geq 0$ and

$$p_0 = \lim_{r \to 0} d_\lambda(r) \le d_\lambda(r) \le d_\lambda(1) \le \lambda^3, \tag{4.1}$$

where $d_\lambda(r) = U'_c(\lambda)(r)[U_c(\lambda)(r)/r]^2$. In fact, we shall now show that for $\lambda > \lambda^*$,

$$d_\lambda(1) \le p_1. \tag{4.2}$$

From this it follows that $h(d_\lambda(r)) = h_1(d_\lambda(r))$ for $r \in (0, 1]$ and so $U_c(\lambda)$ also satisfies (2.1)–(2.4) and (2.5b) for the material defined by the given function h.

From (4.1), we see that (4.2) certainly holds for $\lambda \in (\lambda^*, p_1^{1/3}]$. To obtain the result for $\lambda > p_1^{1/3}$, we fix a value of $\mu \in (\lambda^*, p_1^{1/3}]$ and we set

$$w(r) = \frac{U_c(\mu)(sr)}{s} \quad \text{for } r, s \in (0, 1].$$

For each $s \in (0, 1]$, w satisfies (2.1), (2.2), (2.4), and (2.5b) for the function h_1 and furthermore $w(1) = U_c(\mu)(s)/s$.

Using (2.19) and the fact that $\lim_{r \to 0} U_c(\mu)(r) > 0$ we see that for each $\lambda > \mu$, $\exists$ a unique value of $s = s(\lambda) \in (0, 1)$ such that $U_c(\mu)(s)/s = \mu$.

Setting $s = s(\lambda)$, it follows that w satisfies (2.1)–(2.4) and (2.5b) for the function h_1 and consequently, by the uniqueness of the cavitating solution for this problem

$$U_c(\lambda)(r) = w(r) = \frac{U_c(\mu)(s(\lambda)r)}{s(\lambda)}$$

for $\lambda > \mu$ and $r \in (0, 1]$. In particular, for

$$0 < r \le 1, \qquad d_\lambda(1) = d_\lambda(s(\lambda)) \le d_\mu(1) = \mu^3 \le p_1$$

and (4.2) is established.

REMARK. The preceding discussion shows that as λ is increased the values of the determinant of the cavitating solution are confined to smaller and smaller intervals.

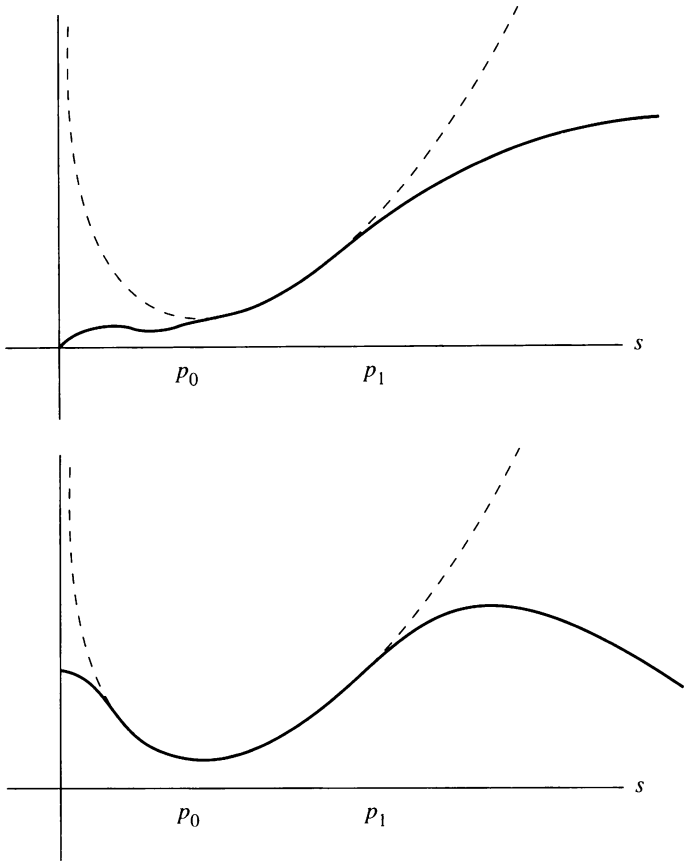

FIG. 2

In fact, in the above notation, for $\lambda > \mu$ and $0 < r \leq 1$,

$$p_0 \leq d_\lambda(r) \leq d_\lambda(1) = d_\mu(s(\lambda))$$

and $s(\lambda) \to 0$ as $\lambda \to \infty$. Furthermore, $\lim_{r \to 0} d_\mu(r) = p_0$. Hence, given any $\varepsilon > 0$, $\exists \lambda(\varepsilon) > \lambda^*$ such that $p_0 \leq d_\lambda(r) \leq p_0 + \varepsilon$ for all $\lambda > \lambda(\varepsilon)$ and $r \in (0, 1]$.

In Fig. 2, the graph of the given function h is shown by the solid curve whereas the broken curve indicates the graph of a possible extension h_1 outside the interval $[p_0, p_1]$.

For the case where Φ is defined by (2.23), (3.7) with $A > 0$, $\alpha = 2$, and $C = 0$, the conditions (O2)*, (O3)* reduce to $h \in C^3([p_0, p_1])$,

$$h'' \geq \delta > 0 \quad \text{on } [p_0, p_1],$$

$$h'(p_0) = 0 \quad \text{and} \quad h'(p_1) > 3Ap_1^{-1/3}.$$

In [8], to illustrate his approach to cavitation, Marcellini considers an example of the form (2.23), (3.7) where

$$\begin{gathered} A = 1, \ \alpha = 2, \ C = 0 \text{ and there exists } \beta \in (-1, 1) \text{ such that} \\ h(p) = 2\beta p^{-1} + (1 - \beta)(2p + p^{-2}) \text{ for } p > 0. \end{gathered} \tag{4.3}$$

As motivation for this choice, he refers to the stored energy functions introduced by Blatz and Ko. However, it should be noted that the functions proposed by Blatz and Ko are not quite of this form. In fact, they are of the form (2.23) where ϕ has a singularity at zero of the type treated by Meynard [3, 4]. Nonetheless, the example (4.3) constitutes a simple case where the function h does not satisfy the conditions (O2), (O3) but, as is easily verified, all the hypotheses of Theorem 4.1 are fulfilled. In [7], Marcellini advocates the discussion of nonconvex functions h in (2.23) and he quotes Ogden [11] in support of this. Note that for $-1 < \beta < 0$, the function h in (4.3) is not convex.

Note added in proof. In a recent paper [12], Horgan studies radially symmetric cavitation for what he calls generalised Varga materials. These are materials having a stored-energy function of the form (2.23) with

$$\phi(s) = c_1 s \quad \text{and} \quad \psi(s) = c_2 s \quad \text{for} \quad s > 0,$$

where c_1 and c_2 are constants, and with $h \in C^3((0, \infty))$.

For such materials the equilibrium equation (2.1) can be integrated explicitly and Horgan uses this to discuss cavitation. When considered as an illustration of our general results we note that a generalised Varga material satisfies the hypotheses of Theorem 4.1 provided that $c_1, c_2 \geq 0$ and that $\exists\, 0 < p_0 < p_1$ such that

$$h'(p_0) = 0 \quad \text{and} \quad h''(p) \geq \delta > 0 \quad \text{for} \quad p_0 \leq p \leq p_1.$$

Furthermore, in this case we find that

$$R(q, t) = c_1 + c_2 t,$$

and so the functions g and G defined by (2.12) and (3.5) are identical. From (3.8) with $\alpha = \gamma = 1$ we obtain

$$g(\lambda) = G(\lambda) = h'(\lambda^3) \quad \text{for all } \lambda > 0.$$

Hence for a generalised Varga material, our lower bound $p_0^{1/3}$ for the critical radius λ^* coincides with our upper bound Λ and we have

$$p_0^{1/3} = \lambda^* = \Lambda.$$

Finally, we observe that for such materials the function d that is discussed in Corollary 3.2 is constant for all solutions of (2.1).

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