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AN INVERSE PROBLEM FOR THE LINEAR VISCOELASTIC KIRCHHOFF PLATE

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Abstract. We consider the linear Kirchhoff plate model subject to a viscoelastic damping of integral type. The damping term contains a time-dependent convolution kernel accounting for the long-range memory effects. The mechanical behavior of the plate is influenced by this memory function which is a priori unknown. That leads us to consider the question of identifying it. We study this kind of inverse problem by using the integrodifferential evolution equation governing the vertical deflection of the plate and coupling it with a set of overposed initial and boundary conditions. The problem obtained is then reduced to a nonlinear Volterra integral equation of second kind for the unknown memory kernel. Then, via the Contraction Principle, we prove local (in time) existence and uniqueness results. In addition, we show the Lipschitz continuous dependence upon the data. These results also apply to a viscoelastic beam model.

1. Introduction. The mechanical behavior of some materials like, e.g., polymers, wood, concretes, and metal at high temperature, is well described by stress-strain relationships in which the actual stress depends both on the actual strain and on the ones exerted in the past. Such a description is usually called *viscoelasticity* of *Boltzmann* (or *integral*) type (cf., e.g., [7, 8, 13, 14, 26, 27]).

The quoted stress-strain laws contain some time-dependent functions, named *relaxation* functions, accounting for the memory effects. They act as convolution kernels of time integrals involving the past strains, influencing the dissipative properties of the medium (cf., e.g., [2, 3, 4, 7, 27] and their references).

A practical problem arising in the applications of such models (e.g., in Engineering and in Geophysics) consists in the fact that the relaxation functions are *a priori* unknown or scarcely known. Thus we have to deal with the inverse problem of identifying such functions.

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This problem has been analyzed by different approaches. Among them we recall the *creep* and *relaxation* experimental tests (see, e.g., [2, 4, 8, 27]) and the optimization techniques (see, e.g., [1, 24, 29, 30]). Recently, however, a new method has been proposed and applied in particular to one-dimensional and three-dimensional linear viscoelasticity (cf. [9, 10, 11, 12]; see also [28] for a similar technique and [23] for a nonlinear model).

It can be described as follows. First, the integrodifferential equation governing the evolution of the displacement field is derived from the constitutive assumptions and the balance of linear momentum. Then an *overdetermined* initial and boundary value problem for this equation is formulated and studied, the information being chosen according to the physical context.

The main aim of our paper is to show that the method just described applies to a two-dimensional mechanical model studied in recent years mainly from the point of view of *controllability*, i.e., the linear viscoelastic Kirchhoff plate (cf. [15, 16, 17, 19, 20]; see also [21] for the viscoelastic beam).

This model essentially consists of a linear hyperbolic Volterra integrodifferential equation governing the evolution of the *vertical deflection* of the plate (cf. [15, Chap. 1, Sec. 7] for its derivation from the constitutive assumptions). More precisely, let us assume that the viscoelastic Kirchhoff plate of uniform thickness h $(h > 0)$ is homogeneous (with unitary density) and isotropic. We assume besides that it lies in a region $\Omega \times (-\frac{h}{2}, \frac{h}{2})$ of the space $\mathbf{R}^3$, for each $t \in [0, T]$, $T > 0$, $\Omega \subset \mathbf{R}^2$ being an open, bounded, and connected set having a smooth boundary Γ (for instance of class $\mathscr{C}^4$).

Denoting by u the vertical deflection of the plate from its equilibrium position $u \equiv 0$, the governing equation is (cf. [15, Chapter 1, Section 7]; see also [16, 17, 19])

$$hu'' - \frac{h^3}{12}\Delta u'' + D(0)\Delta^2 u + D' * \Delta^2 u = f \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

where the symbol "$'$" denotes the time derivative, Δ is the Laplace operator, Δ^2 is the biharmonic operator, and the symbol "$*$" represents the convolution product with respect to time over $(0, t)$, that is $(a*b)(t) := \int_0^t a(t-s)b(s)\,ds$. Here $D\colon (0, +\infty) \to \mathbf{R}$ is the so-called *viscoelastic flexural rigidity*, $D(0) > 0$, and $f\colon \Omega \times (0, T) \to \mathbf{R}$ is the *body force* acting along the vertical direction (i.e., orthogonal to Ω).

Our inverse problem is to find D.

In order to solve it we introduce a set of initial and boundary conditions that can be coupled with Eq. (1.1) (cf., e.g., [15, Chap. 1, Sec. 7], [16, 17]). We assume that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 and Γ_1 are nonempty, relatively open, disjoint subsets of Γ, and that $|\Gamma_0| > 0$, $|\Gamma_0|$ being the Lebesgue measure of Γ_0. Moreover, we denote by $\nu = (\nu_1, \nu_2)$ and $\tau = (-\nu_2, \nu_1)$ the unit normal vector to Γ pointing outward and the unit positively-oriented tangent vector to Γ, respectively.

Let us consider the conditions

$$u(\cdot, 0) = u_0, \qquad u'(\cdot, 0) = u_1 \qquad \text{in } \Omega, \tag{1.2}$$

$$u = u_\nu = 0 \qquad \text{on } \Gamma_0 \times [0, T], \tag{1.3}$$

$$D(0)\mathscr{B}_1 u + D' * \mathscr{B}_1 u = g^1 \qquad \text{on } \Gamma_1 \times [0, T], \tag{1.4}$$

$$D(0)\mathscr{B}_2 u + D' * \mathscr{B}_2 u - \frac{h^3}{12} u''_\nu = g^2 \qquad \text{on } \Gamma_1 \times [0, T], \tag{1.5}$$

where u_0, $u_1 \colon \Omega \to \mathbf{R}$ and g^1, $g^2 \colon [0, T] \to \mathbf{R}$ are given functions. Here $\mu \in (0, 1/2)$ is the *viscoelastic Poisson ratio*, $(\cdot)_\nu$ represents the derivative in the ν-direction, and $\mathscr{B}_1$, $\mathscr{B}_2$ are the linear operators defined by

$$\mathscr{B}_1 z := \Delta z + (1-\mu) B_1 z, \qquad \mathscr{B}_2 z := (\Delta z)_\nu + (1-\mu)(B_2 z)_\tau, \tag{1.6}$$

$(\cdot)_\tau$ being the derivative in the τ-direction and

$$B_1 z := 2\nu_1\nu_2 z_{xy} - \nu_1^2 z_{yy} - \nu_2^2 z_{xx}, \qquad B_2 z := (\nu_1^2 - \nu_2^2) z_{xy} + \nu_1\nu_2 (z_{yy} - z_{xx}). \tag{1.7}$$

Note that

$$g^1 = -M^\tau, \qquad g^2 = -M^\nu_\tau - g^3, \tag{1.8}$$

where M^τ represents the *bending moment* about the tangent vector τ (per unit length), M^ν is the *twisting moment* about the normal vector ν (per unit length), and g^3 is the *shear force* acting along the vertical direction.

The *direct* problem, consisting in finding u satisfying Eq. (1.1) and the initial and boundary conditions (1.2–1.5), is well posed (see [16, Theorem 2.1] and also Theorem 3.1 in this paper), but this information does not suffice to determine D as well. To this aim we need another datum involving the time only. From the physical viewpoint, it is reasonable to measure, for each time $t \in [0, T]$, e.g., the bending moment M^τ exerted on a portion $\widetilde{\Gamma} \subseteq \Gamma_0$ having positive measure, that is (cf. (1.5), (1.8))

$$\int_{\widetilde{\Gamma}} \{D(0)\mathscr{B}_1 u + D' * \mathscr{B}_1 u\}\, d\Gamma = g \quad \text{on } [0, T], \tag{1.9}$$

where $d\Gamma$ denotes the Lebesgue measure on Γ and $g \colon [0, T] \to \mathbf{R}$ is a known function.

Summing up, a possible formulation of our inverse problem as an overdetermined problem for Eq. (1.1) is the following

PROBLEM (P1). Find u and D satisfying (1.1–1.5), (1.9)

This paper is mainly devoted to the study of Problem (P1).

The plan goes as follows. The next section contains a more precise formulation of Problem (P1), the statements of our main results, and some remarks about their applicability. In Sec. 3 we show a result related to the direct problem (1.1–1.5) which will be useful in the following sections. Problem (P1) is reduced, in Sec. 4, to a nonlinear Volterra integral equation of the second kind for the unknown D''. Taking advantage of this equivalent formulation, the main results are proved in Secs. 5 and 6. Finally, Sec. 7 is concerned with some final remarks on the hypotheses we have used and the treatment of additional information different from (1.9). Furthermore, an application to a viscoelastic beam model is also discussed.

2. Main reuslts. First, we introduce some notation. The symbol $C^r([0, T]; X)$ ($r \in \mathbf{N}$, X being a real Banach space) will denote the space consisting of all the functions $z: [0, T] \to X$ continuous along with their first r time-derivatives ($C([0, T]; X) := C^0([0, T]; X)$). Endowing it with the norm

$$\|z\|_{r,X} := \sum_{i=0}^{r} \operatorname{ess\,sup}_{t\in[0,T]} \|z^{(i)}(t)\|_X$$

we obtain a Banach space. If $X \equiv \mathbf{R}$ then we simply set $\|\cdot\|_r := \|\cdot\|_{r,X}$.

Recalling now that $H^k(\Omega)$ represents the usual Sobolev space of order $k \in \mathbf{N}$, we set $H := H^0(\Omega) = L^2(\Omega)$ and

$$V := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\}, \tag{2.1}$$

$$W := \{w \in H^2(\Omega) \mid v = v_\nu = 0 \text{ on } \Gamma_0\}. \tag{2.2}$$

V and W become Hilbert spaces thanks to the following inner products, respectively:

$$c(v, \tilde{v}) := h\int_\Omega v\tilde{v} + \gamma\int_\Omega \nabla v \cdot \nabla\tilde{v} \quad \forall v, \tilde{v} \in V, \tag{2.3}$$

$$\begin{aligned} a(w, \tilde{w}) := \int_\Omega \{&w_{xx}\tilde{w}_{xx} + w_{yy}\tilde{w}_{yy} + \mu(w_{xx}\tilde{w}_{yy} + w_{yy}\tilde{w}_{xx}) \\ &+2(1-\mu)w_{xy}\tilde{w}_{xy}\}\,dx\,dy \quad \forall w, \tilde{w} \in W, \end{aligned} \tag{2.4}$$

where $\gamma := h^3/12$ and the dot denotes the scalar product in $\mathbf{R}^2$. Note that, since $|\Gamma_0| > 0$, the bilinear form $a(\cdot, \cdot)$ is coercive owing to Korn's Lemma (cf., e.g., [5, Chap. III, Sec. 3.3]).

We observe that

$$W \hookrightarrow V \hookrightarrow H \hookrightarrow V' \hookrightarrow W' \tag{2.5}$$

with dense and compact embeddings, V' and W' being the dual spaces of V and W. Then we introduce the linear operators $C: V \to V'$, $A: W \to W'$ defined by

$$\langle Cv, \tilde{v}\rangle := c(v, \tilde{v}) \quad \forall v, \tilde{v} \in V, \tag{2.6}$$

$$\langle Aw, \tilde{w}\rangle := a(w, \tilde{w}) \quad \forall w, \tilde{w} \in W, \tag{2.7}$$

$\langle\cdot, \cdot\rangle$ being the duality pairing between V' and V or between W' and W. We remark that C and A are the canonical isomorphisms of V onto V' and of W onto W'.

We can now reformulate Problem (P1) in a more rigorous way using the functional framework just introduced.

By Green's formulas one easily gets the identities (cf. (2.3–2.4), (2.6–2.7))

$$(hz'' - \gamma\Delta z'', v) = \langle Cz'', v\rangle - \gamma\int_{\Gamma_1} z''_\nu v\,d\Gamma \quad \forall v, z'' \in H^2(\Omega) \cap V, \tag{2.8}$$

$$(\Delta^2 z, w) = \langle Az, w\rangle + \int_{\Gamma_1} \{(\mathscr{B}_2 z)w - (\mathscr{B}_1 z)w_\nu\}\,d\Gamma \quad \forall w, z \in H^4(\Omega) \cap W, \tag{2.9}$$

where $(\cdot, \cdot)$ denotes the usual inner product in H (for (2.9) see, e.g., [5, Chap. IV, Sec. 2.3]).

Assuming, for the sake of simplicity, that (cf. (1.4–1.5))

$$g^1 = g^2 \equiv 0 \tag{2.10}$$

and recalling (1.1–1.5), we restate Problem (P1) as follows:

PROBLEM (P1). Find $u\colon [0, T] \to W$ and $D\colon [0, T] \to \mathbf{R}$ such that

$$Cu''(t) + A(D(0)u + D' * u)(t) = f(t) \quad \text{in } W', \ \forall t \in (0, T), \tag{2.11}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{2.12}$$

$$\int_{\widetilde{\Gamma}} \{D(0)\mathscr{B}_1 u + D' * \mathscr{B}_1 u\}\, d\Gamma = g \quad \text{on } [0, T], \tag{2.13}$$

where $f\colon [0, T] \to V'$ and $u_0, u_1 \in W$, $g\colon [0, T] \to \mathbf{R}$ are given.

As far as the data are concerned, we assume

$$f = f^1 + f^2 \in C^3([0, T]; V') + C^2([0, T]; H), \tag{2.14}$$

$$C^{-1} f^2 \in C^2([0, T]; W), \tag{2.15}$$

$$u_0, u_1 \in W, \tag{2.16}$$

$$C^{-1} A u_0, C^{-1} A u_1 \in W, \tag{2.17}$$

$$C^{-1} f(0), C^{-1} f'(0) \in W, \tag{2.18}$$

$$AC^{-1} A u_0, AC^{-1} f(0) \in V', \tag{2.19}$$

$$g \in C^2([0, T]), \tag{2.20}$$

where $C^{-1}\colon V' \to V$ is the inverse of the operator C (cf. (2.3), (2.6)).

In addition we assume the consistency conditions

$$D_0 := g(0) \left(\int_{\widetilde{\Gamma}} \mathscr{B}_1 u_0 \, d\Gamma \right)^{-1} > 0, \tag{2.21}$$

$$D(0) \int_{\widetilde{\Gamma}} \mathscr{B}_1 u_1 d\Gamma + D'(0) \int_{\widetilde{\Gamma}} \mathscr{B}_1 u_0 d\Gamma = g'(0), \tag{2.22}$$

along with the further hypotheses

$$\mathscr{B}_1 u_i = \mathscr{B}_2 u_i = 0 \quad \text{on } \Gamma_1, \ i = 0, 1, \tag{2.23}$$

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing. \tag{2.24}$$

Note that the boundary integrals appearing in (2.21–2.22) make sense owing to (2.2), (2.16) (cf. also (1.6–1.7)). The geometric condition (2.24) is rather restrictive but it is technically necessary if one needs, as we do, to apply regularity results for elliptic equations (see (3.21)).

The first result is concerned with local (in time) existence and uniqueness.

THEOREM 2.1. Let the assumptions (2.10), (2.14–2.24) hold. Then there exists $T_0 \in (0, T]$ such that Problem (P1) admits a unique local solution (u, D) having the

following properties:

$$u \in C^2([0, T_0]; H^3(\Omega) \cap W) \cap C^3([0, T_0]; W) \cap C^4([0, T_0]; V), \tag{2.25}$$

$$D \in C^2([0, T_0]), \qquad D(0) > 0. \tag{2.26}$$

REMARK 2.1. As we shall see in Sec. 4, the nonvanishing condition (2.21) allows us to transform Problem (P1) into a nonlinear Volterra integral equation of the second kind for D''. If this condition fails and our problem is generally equivalent to a Volterra integral equation of the *first kind*, then it is essentially *ill posed*. Thus, the solvability of Problem (P1) strongly depends on a condition like (2.21) (see also Sec. 7, 7.1, (7.12)). Concerning the difficulties related to the existence *in the large*, deriving from the nonlinearity of the problem, the reader is referred to [11, Introduction] and the reference quoted therein.

The second result consists in a continuous dependence estimate, which shows that the differences of possible solutions to Problem (P1) depend upon the differences of the data in a Lipschitz way, that is

THEOREM 2.2. Let $(f_j, u_{0j}, u_{1j}, g_j)$, $j = 1, 2$, be two sets of data satisfying the hypotheses (2.14–2.23) and let (2.24) hold. Besides, assume (cf. (2.10)) $g_j^1 = g_j^2 \equiv 0$, for $j = 1, 2$, and (cf. (2.21)) $D_{01} = D_{02}$. Denoting by (u_j, D_j), $j = 1, 2$, the corresponding solutions of Problem (P1) and letting K, L be constants such that

$$\begin{aligned} \max\{&\|f_j^1\|_{3,V'}, \|C^{-1}f_j^2\|_{2,W}, \|u_{0j}\|_W, \|u_{1j}\|_W, \|C^{-1}f_j(0)\|_W, \\ &\|AC^{-1}f_j(0)\|_{V'}, \|C^{-1}f_j'(0)\|_W, \|C^{-1}Au_{0j}\|_W, \\ &\|C^{-1}Au_{1j}\|_W, \|AC^{-1}Au_{0j}\|_{V'}, (D_{0j})^{-1}\} \le K, \end{aligned} \tag{2.27}$$

$$\|D_j\|_2 \le L, \tag{2.28}$$

for $j = 1, 2$, there exists a positive function $\Lambda \in C^0((0, +\infty)^3)$ such that

$$\begin{aligned} &\|u_1 - u_2\|_{2,H^3(\Omega)} + \|u_1 - u_2\|_{3,W} + \|u_1 - u_2\|_{4,V} + \|D_1 - D_2\|_2 \\ &\le \Lambda(K, L, T)\Big\{\|f_1^1 - f_2^1\|_{3,V'} + \|C^{-1}(f_1^2 - f_2^2)\|_{2,W} + \|u_{01} - u_{02}\|_W \\ &\quad + \|u_{11} - u_{12}\|_W + \|C^{-1}(f_1 - f_2)(0)\|_W \\ &\quad + \|C^{-1}(f_1' - f_2')(0)\|_W + \|AC^{-1}(f_1 - f_2)(0)\|_{V'} \\ &\quad + \|C^{-1}A(u_{01} - u_{02})\|_W + \|C^{-1}A(u_{11} - u_{12})\|_W \\ &\quad + \|AC^{-1}A(u_{01} - u_{02})\|_{V'} + \|g_1 - g_2\|_2\Big\}. \end{aligned} \tag{2.29}$$

Moreover, the function Λ (also depending on Ω, Γ_0, Γ_1, $\tilde{\Gamma}$, ν_1, ν_2, h, μ) is increasing with respect to each of its arguments.

REMARK 2.2. Recalling the definitions of the operators A, C (cf. (2.3–2.4), (2.6–2.7)) and using well-known regularity results on linear elliptic boundary value problems (cf., e.g., [22, Chap. 2, Secs. 5, 9]), we can deduce from the estimate (2.29) a

more readable one, that is,

$$\begin{aligned}\|u_1 - u_2\|_{2,H^3(\Omega)} + \|u_1 - u_2\|_{3,W} + \|u_1 - u_2\|_{4,V} + \|D_1 - D_2\|_2 \\ \leq \widetilde{\Lambda}(K, L, T)\Big\{\|f_1^1 - f_2^1\|_{3,V'} + \|(f_1^2 - f_2^2)\|_{2,H} + \|(f_1 - f_2)(0)\|_V \\ + \|(f_1' - f_2')(0)\|_H + \|u_{01} - u_{02}\|_{H^5(\Omega)} \\ + \|u_{11} - u_{12}\|_{H^4(\Omega)} + \|g_1 - g_2\|_2\Big\}.\end{aligned}$$

Here $\widetilde{\Lambda}$ is a function quite similar to Λ.

REMARK 2.3. From the practical viewpoint, it is not easy to construct initial data u_0 and u_1 satisfying the assumptions (2.16–2.17), (2.19), (2.21–2.22). Thus, the above results are a bit unrealistic. Despite this fact, we have chosen to present the main results in this form for the sake of generality. Nevertheless, if the initial data vanish identically, which is quite common in the applications, it is possible to get analogous results, where $C^{-1}f(0)$ and $C^{-1}f'(0)$ play the role of the initial data. In this case, especially if f does not depend on time, the assumptions seems to be more reasonable from the point of view of applicability (see Sec. 7, 7.1, (7.5–7.14)). What exactly happens if the initial data have compact support in Ω, so that (2.21) fails, is still an open question (cf. Remark 2.1, however).

REMARK 2.4. Here we are looking for time-continuous solutions. Nonetheless, it is possible to show that Theorems 2.1–2.2 and the related Remarks 7.1–7.4 keep their validity even if the time regularity of f and g is slightly relaxed. More precisely, considering Problem (P1), the assumptions (2.14–2.15) and (2.20) can be replaced by, respectively,

$$f = f^1 + f^2 \in W^{3,p}(0, T; V') + W^{2,p}(0, T; H), \tag{2.30}$$

$$C^{-1}f^2 \in W^{2,p}(0, T; W), \tag{2.31}$$

$$g \in W^{2,p}(0, T) \tag{2.32}$$

for some $p \in [1, +\infty]$. Here $W^{r,p}$ denotes, as usual, the Sobolev space of order $r \in \mathbf{N}$ and index p.

Then the counterpart of Theorem 2.1 will yield the local existence and uniqueness of a pair (u, D) solving Problem (P1) such that (cf. (2.25–2.26))

$$u \in W^{2,p}(0, T_0; H^3(\Omega) \cap W) \cap W^{3,p}(0, T_0; W) \cap W^{4,p}(0, T_0; V), \tag{2.33}$$

$$D \in W^{2,p}(0, T_0), \qquad D(0) > 0. \tag{2.34}$$

Concerning Theorem 2.2, the bounds (2.27–2.28) and the continuous dependence estimate (2.29) should be modified according to (2.30–2.34).

For an analysis developed in this functional framework and applied to the three-dimensional case the reader is referred to [11].

The regularity stated in (2.34) is a bit stronger than the one usually required by the Principle of *fading memory*. That is essentially due to the treatment of the integro-differential term as a perturbation (see, e.g., [14, Chap. III, Theorem III.11]). In order to further relax the regularity of D, some monotonicity or convexity conditions on

D (cf., e.g., [14, Chap. III., Theorem III.22]) are needed, but our technique does not seem suitable to deal with this kind of assumptions. More precisely, it is not clear how a monotonicity (or convexity) property of D can be preserved in our fixed-point formulation (see Sec. 4).

REMARK 2.5. The term $(h^3/12)\Delta u''$ in Eq. (1.1) is sometimes neglected (cf., e.g., [20]), due to the smallnesss of h^3. In this case the operator C (cf. (2.3), (2.6)) reduces to the identity and the problem can be treated by the same technique developed in this paper.

3. The direct problem. In this section we will suppose that the relaxation function D is given. We will prove that the unique solution u to the direct problem (2.11–2.12) satisfies an estimate that shows how u depends on D. This estimate will be very useful in the proofs of our main results (see Secs. 4–6).

Let us assume

$$D \in C^2([0, T]), \tag{3.1}$$

$$D(0) > 0, \tag{3.2}$$

$$f^1 \in C^1([0, T]; V'), \tag{3.3}$$

$$C^{-1} f^2 \in C([0, T]; W), \tag{3.4}$$

$$u_0, u_1 \in W, \tag{3.5}$$

$$Au_0 \in V'. \tag{3.6}$$

Then we have

THEOREM 3.1. Under the assumptions (2.10), (2.24), (3.1–3.6), there exists a unique function $u \in C([0, T]; H^3(\Omega) \cap W) \cap C^1([0, T]; W) \cap C^2([0, T]; V)$ solving equation (2.11) with $f = f^1 + f^2$ and satisfying the initial conditions (2.12). Moreover, there exist a positive constant Λ_1 and a pair of positive functions Λ_2, $\Lambda_3 \in C^0((0, +\infty)^2)$ such that

$$\begin{aligned} &\|u\|_{C([0,T]); H^3(\Omega)\cap W)} + \|u\|_{C^1([0,t]; W)} \\ &\quad \le \Lambda_1 \Big\{ \|u_0\|_{H^3(\Omega)} + \|u_1\|_W + \|f^1\|_{C^1([0,t]; V')} \\ &\qquad + \int_0^t \|(f^1)'\|_{C([0,s]; V')}\, ds + \int_0^t \|C^{-1} f^2\|_{C([0,s]; W)}\, ds \\ &\qquad\qquad + t\|u_0\|_W \Lambda_2(\|D'\|_{C^1([0,t])}, t) \Big\} \\ &\quad \times \{1 + t\Lambda_3(\|D'\|_{C^1([0,t])}, t) \exp[t\Lambda_3(\|D'\|_{C^1([0,T])}, t)]\} \end{aligned} \tag{3.7}$$

for any $t \in [0, T]$. Here the constant Λ_1 depends on Ω, h, μ, $(D(0))^{-1}$, $D'(0)$, T and the functions Λ_2, Λ_3 are increasing in each of their arguments and also depend on $(D(0))^{-1}$.

Proof. Consider the equation

$$z = D(0)u + D' * u \quad \text{on } [0, T], \tag{3.8}$$

where $u \in C([0, T]; H^3(\Omega) \cap W) \cap C^1([0, T]; W) \cap C^2([0, T]; V)$ is supposed to be given. Then, thanks to (3.1–3.2), z belongs to the same space as u and besides

$$D(0)u = z + R * z \quad \text{on } [0, T], \tag{3.9}$$

where R is the so-called *resolvent* of $(D(0))^{-1}D'$. Moreover, R solves the integral equation (see, e.g., [14, Chap. IV, Sec. 4])

$$D(0)R + R * D' = -D' \quad \text{on } [0, T]. \tag{3.10}$$

Then

$$D(0)R' = -D'' - D'(0)R - R * D'' \quad \text{on } [0, T], \tag{3.11}$$

and $R \in C^1([0, T])$.

From Eqs. (3.10–3.11) we can also infer, via Gronwall's Lemma, the following inequality

$$\begin{aligned} \|R'\|_{C([0,t])} \le{}& (D(0))^{-1}\|D''\|_{C([0,t])} + (D(0))^{-2}(|D'(0)| + t\|D''\|_{C([0,t])}) \\ &\times (\|D'\|_{C([0,t])} + t(D(0))^{-1}\|D'\|^2_{C([0,t])} \exp[t(D(0))^{-1}\|D'\|_{C([0,t])}]) \end{aligned} \tag{3.12}$$

for any $t \in [0, T]$.

Eqs. (3.8–3.9) allow us to transform the Cauchy problem (2.11–2.12) into the following equivalent one:

$$C(z + R * z)''(t) + D(0)Az(t) = D(0)f(t) \quad \text{in } W', \ \forall t \in (0, T), \tag{3.13}$$

$$z(0) = z_0 := D(0)u_0, \qquad z'(0) = z_1 := D(0)u_1 + D'(0)u_0. \tag{3.14}$$

To prove the theorem it is useful to transform the Cauchy problem (3.13–3.14) into a Cauchy problem for a first-order equation.

Let us introduce the linear operators $\mathscr{C}$, $\mathscr{A}$ defined by

$$\mathscr{C} := \begin{pmatrix} D(0)A & 0 \\ 0 & C \end{pmatrix}, \qquad \mathscr{A} := \begin{pmatrix} 0 & -D(0)A \\ D(0)A & 0 \end{pmatrix}.$$

Note that $\mathscr{C}$ and $\mathscr{A}$ are isomorphisms of $W \times V$ onto $W' \times V'$ and of $W \times W$ onto $W' \times W'$.

Then we consider the linear operator $\mathbf{A} := \mathscr{C}^{-1}\mathscr{A} : \mathscr{D}(\mathbf{A}) \to W \times V$, where

$$\mathscr{D}(\mathbf{A}) := \{v \in W \times W \mid Av \in W' \times V'\} \hookrightarrow W \times V.$$

We remark that, owing to the regularity of Γ and to the geometric assumption (2.24), it can be proved that (cf. [18, Theorem 2.3 and Remark 2.4])

$$\mathscr{D}(\mathbf{A}) \equiv \{v = (v^1, v^2) \in W \times W \mid v^1 \in H^3(\Omega) \cap W, \ \mathscr{B}_1 v^1 = 0 \quad \text{on } \Gamma_1\}. \tag{3.15}$$

Thus $\mathscr{D}(\mathbf{A})$ can be endowed with the norm $\|v\|_{\mathscr{D}(\mathbf{A})} := \|v^1\|_{H^3(\Omega)} + \|v^2\|_W$, which is equivalent to the graph norm.

Setting now

$$v := (z, z')^{\mathrm{tr}}, \qquad v_0 := (z_0, z_1)^{\mathrm{tr}}, \tag{3.16}$$

where tr denotes the transposition operation, the Cauchy problem (3.13–3.14) turns out to be

$$v'(t) + \mathbf{A}v(t) = \mathscr{R}(v)(t) + \mathscr{F}(t) \quad \text{in } W \times V, \ \forall t \in (0, T), \tag{3.17}$$

$$v(0) = v_0, \tag{3.18}$$

where $\mathscr{R}$ and $\mathscr{F}$ are defined by

$$\mathscr{R} := \begin{pmatrix} 0 & 0 \\ 0 & -(\delta\mathscr{I} + R'*) \end{pmatrix}, \tag{3.19}$$

$$\mathscr{F} := (0, -R'z_0 + D(0)C^{-1}f)^{\mathrm{tr}}. \tag{3.20}$$

Here we have used the identity

$$(R * z)''(t) = (\delta\mathscr{I} + R'*)z'(t) + R'(t)z_0 \quad \forall t \in [0, T],$$

where $\mathscr{I}$ denotes the identity operator and $\delta := R(0) = -(D(0))^{-1}D'(0)$ (see (3.10)).

We now show that the Cauchy problem (3.17–3.18) has a unique solution $v \in C([0, T]; \mathscr{D}(\mathbf{A})) \cap C^1([0, T]; W \times V)$. Indeed, thanks to the geometric assumption (2.24), regularity results for linear elliptic operators (cf., e.g., [22, Chap. 2, Sec. 7]) allow us to deduce from (3.6) that $u_0 \in H^3(\Omega) \cap W$; then, owing to (3.5–3.6) and (3.14–3.16), we get

$$v_0 \in \mathscr{D}(\mathbf{A}). \tag{3.21}$$

On the other hand, we have (cf. (3.19))

$$\|\mathscr{R}(v_1) - \mathscr{R}(v_2)\|_{C([0,t];\mathscr{D}(\mathbf{A}))} \le (|\delta| + \|R'\|_{C([0,t])})\|v_1 - v_2\|_{C([0,t];\mathscr{D}(\mathbf{A}))}, \tag{3.22}$$

for any $t \in [0, T]$ and any $v_1, v_2 \in C([0, T]; \mathscr{D}(\mathbf{A}))$. Besides (cf. (3.3–3.5), (3.14), (3.20)),

$$\mathscr{F} = \mathscr{F}^1 + \mathscr{F}^2 \in C^1([0, T]; W \times V) + C([0, T]; \mathscr{D}(\mathbf{A})), \tag{3.23}$$

where

$$\mathscr{F}^1 := (0, D(0)C^{-1}f^1)^{\mathrm{tr}}, \qquad \mathscr{F}^2 := (0, -R'z_0 + D(0)C^{-1}f^2)^{\mathrm{tr}}. \tag{3.24}$$

Since the operator $-\mathbf{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $W \times V$ (cf. [18, Theorem 2.1]), taking (3.21–3.23) into account, we infer that there exists a unique solution $v \in C([0, T]; \mathscr{D}(\mathbf{A})) \cap C^1([0, T]; W \times V)$ solving the Cauchy problem (3.17–3.18) (see, e.g., [25, Chap. 4, Theorem 2.4, Corollaries 2.5–2.6 and Chap. 6, Theorem 1.7]).

Recalling (3.8–3.9), (3.15–3.16), this result implies that the direct problem (2.11–2.12) admits a unique solution u with the regularity properties stated in Theorem 3.1.

It remains to prove the estimate (3.7).

We observe that the solution v to the problem (3.17–3.18) also solves the integral equation (cf., e.g., [25, Chap. 6, Theorem 1.2 and Corollary 1.3])

$$v(t) = S(t)v_0 + \int_0^t S(t-s)[\mathscr{R}(v)(s) + \mathscr{F}(s)]\,ds \quad \forall t \in [0, T], \tag{3.25}$$

S being the strongly continuous semigroup generated by $-\mathbf{A}$.

Let us recall that S has the following properties (cf., e.g., [25, Chapter 1, Secs. 1.1–1.2]):

$$S(0) = \mathscr{I}, \tag{3.26}$$

$$S\colon [0, T] \to \mathscr{L}(W \times V) \cap \mathscr{L}(\mathscr{D}(\mathbf{A})) \text{ is strongly continuous}, \tag{3.27}$$

$$S'(t)y = -\mathbf{A}S(t)y \quad \forall y \in \mathscr{D}(\mathbf{A}), \ \forall t \in [0, T], \tag{3.28}$$

$$\|S(t)\|_{\mathscr{L}(\mathscr{D}(\mathbf{A}))} \le \Lambda_4, \tag{3.29}$$

where Λ_4 is a constant depending on $\mathbf{A}$, Ω, and T. Here $\mathscr{L}(X)$ (X being a Banach space) denotes the Banach space consisting of all linear and bounded operators acting from X into itself.

Then, owing to (3.3), (3.26), (3.28), integrating by parts in time we obtain

$$\begin{aligned}\int_0^t \mathbf{A}S(t-s)\mathscr{F}^1(s)\,ds &= -\int_0^t S'(t-s)\mathscr{F}^1(s)\,ds \\ &= \mathscr{F}^1(t) - S(t)\mathscr{F}^1(0) \\ &\quad - \int_0^t S(t-s)(\mathscr{F}^1)'(s)\,ds \quad \forall t \in [0, T].\end{aligned} \tag{3.30}$$

Taking (3.22–3.24), (3.29–3.30) into account, from Eq. (3.25) we derive the inequality

$$\begin{aligned}\|v\|_{C([0,t];\mathscr{D}(\mathbf{A}))} \le \Lambda_5 \Big\{ &\|v_0\|_{\mathscr{D}(\mathbf{A})} + \int_0^t (1 + \|R'\|_{C[(0,s])})\|v\|_{C([0,s];\mathscr{D}(\mathbf{A}))}\,ds \\ &+ \|f^1\|_{C([0,t];V')} + \int_0^t \|(f^1)'\|_{C([0,s];V')}\,ds \\ &+ \int_0^t \|C^{-1}f^2\|_{C([0,s];W)}\,ds + \|z_0\|_W \int_0^t \|R'\|_{C([0,s])}\,ds \Big\}\end{aligned} \tag{3.31}$$

for any $t \in [0, T]$, where Λ_5 is a positive constant analogous to Λ_4 and depending on $D(0)$ as well.

Applying now Gronwall's Lemma to (3.31) we get

$$\begin{aligned}&\|v\|_{C([0,t];\mathscr{D}(\mathbf{A}))} \\ &\quad \le \Lambda_5 \Big\{ \|v_0\|_{\mathscr{D}(\mathbf{A})} + \|f^1\|_{C([0,t];V')} + \int_0^t \|(f^1)'\|_{C([0,s];V')}\,ds \\ &\qquad + \int_0^t \|C^{-1}f^2\|_{C([0,s];W)}\,ds + \|z_0\|_W \int_0^t \|R'\|_{C([0,s])}\,ds \Big\} \\ &\qquad \times \{1 + t\Lambda_5(1 + \|R'\|_{C([0,t])}) \exp[t\Lambda_5(1 + \|R'\|_{C([0,t])})]\}\end{aligned} \tag{3.32}$$

for any $t \in [0, T]$.

Finally, recalling Eqs. (3.8–3.9), the estimate (3.12), and the relationships (3.14–3.16), from the estimate (3.32) we derive (3.7).

REMARK 3.1. As we shall see in the next sections, the estimate (3.7) is crucial in order to prove our results. It is worthwhile to note that this estimate seems to be difficult to get by using energy identity arguments as in [10–11] (cf. also [18, p. 231, observation at the end of 2.1]).

4. An equivalent problem. Here we will show that, under the assumptions of Theorem 2.1, Problem (P1) has a solution (u, D) fulfilling the properties (2.25–2.26) if and only if $D'' \in C([0, T])$ solves a Volterra nonlinear integral equation of the second kind.

Let us assume (2.10), (2.14–2.24) hold and let (u, D) be the solution to Problem (P1) satisfying (2.25–2.26).

First, observe that, taking the limit as $t \to 0^+$ in Eq. (2.13) and recalling (1.2), (2.21), we have

$$D(0) = D_0 > 0. \tag{4.1}$$

Moreover, using (4.1), from the consistency condition (2.22) we get

$$D'(0) = D^1 = \left(\int_{\tilde{\Gamma}} \mathscr{B}_1 u_0 \, d\Gamma\right)^{-1} \left(g'(0) - D_0 \int_{\tilde{\Gamma}} \mathscr{B}_1 u_1 \, d\Gamma\right). \tag{4.2}$$

Setting now

$$w := u'', \tag{4.3}$$

we prove, by differentiation in time, that (w, D) solves an inverse problem quite analogous to Problem (P1).

Indeed, differentiating in time both sides of Eq. (2.11), we obtain (cf. (2.12), (4.1))

$$Cu^{(3)}(t) + A(D_0 u' + D' * u')(t) = -D'(t)Au_0 + f'(t) \quad \text{in } W', \tag{4.4}$$

for any $t \in (0, T)$.

A further differentiation in time yields (cf. (2.12), (4.3))

$$Cw''(t) + A(D_0 w + D' * w)(t) = -D''(t)Au_0 - D'(t)Au_1 + f''(t) \quad \text{in } W', \tag{4.5}$$

for any $t \in (0, T)$.

We remark that, owing to the conditions (2.23), the function $w \equiv u''$ satisfies the same boundary conditions as u does (cf. (2.8–2.10)).

On the other hand, taking the limit as $t \to 0^+$ in Eqs. (2.11), (4.4), recalling the initial conditions (2.12) and Eqs. (4.1–4.2), we get

$$w(0) = w_0 := C^{-1}(-D_0 Au_0 + f(0)), \tag{4.6}$$

$$w'(0) = w_1 := C^{-1}(-D_0 Au_1 - D^1 Au_0 + f'(0)). \tag{4.7}$$

Then, differentiating twice with respect to time both the members of Eq. (2.13) and taking (2.20), (4.1–4.2) into account, we have

$$\begin{aligned} &\int_{\tilde{\Gamma}} \{D_0 \mathscr{B}_1 w + D' * \mathscr{B}_1 w\} \, d\Gamma \\ &\quad = g'' - \left(\int_{\tilde{\Gamma}} \mathscr{B}_1 u_0 \, d\Gamma\right) D'' - \left(\int_{\tilde{\Gamma}} \mathscr{B}_1 u_1 \, d\Gamma\right) D' \quad \text{on } [0, T]. \end{aligned} \tag{4.8}$$

Consider the direct problem consisting in finding w solving Eq. (4.5) and satisfying the initial conditions (4.6–4.7). The assumptions of Theorem 3.1 are satisfied owing to (2.14–2.19); hence this problem has a unique solution $w \in C([0, T]; H^3(\Omega) \cap$

$W) \cap C^1([0, T]; W) \cap C^2([0, T]; V)$. As a consequence, observing that (cf. (4.1–4.2))

$$D(t) := D_0 + tD^1 + \int_0^t (t-s)G(s)\,ds \quad \forall t \in [0, T], \tag{4.9}$$

where $G := D''$, we can define a mapping $\mathscr{K}: C([0, T]) \to C([0, T]; H^3(\Omega) \cap W) \cap C^1([0, T]; W) \cap C^2([0, T]; V)$ by setting

$$\mathscr{K}(G) := w, \tag{4.10}$$

where w solves the direct problem (4.5–4.7).

Thanks to these considerations and to (2.21), from Eq. (4.8) we derive that G solves the following functional equation in the fixed-point form:

$$G = \mathscr{T}(G) \quad \text{on } [0, T], \tag{4.11}$$

where $\mathscr{T}: C([0, T]) \to C([0, T])$ is the nonlinear Volterra operator defined by (cf. (4.9))

$$\mathscr{T}(G) := \left(\int_{\widetilde{\Gamma}} \mathscr{B}_1 u_0\, d\Gamma\right)^{-1} \left\{ g'' - \int_{\widetilde{\Gamma}} \{D_0 \mathscr{B}_1 \mathscr{K}(G) + (D^1 + 1 * G) * \mathscr{B}_1 \mathscr{K}(G)\} d\Gamma \right.$$
$$\left. - \left(\int_{\widetilde{\Gamma}} \mathscr{B}_1 u_1\, d\Gamma\right)(D^1 + 1 * G) \right\} \quad \text{on } [0, T]. \tag{4.12}$$

Conversely, assume that $G \in C([0, T])$ is a solution to the functional equation (4.11). Then the pair (w, D), defined by (4.9–4.10), solves the inverse problem (4.5–4.8). Hence, owing to the consistency conditions (2.21–2.22), the pair (u, D), where u is defined by (cf. (4.3))

$$u(t) := u_0 + tu_1 + \int_0^t (t-s)w(s)\,ds \quad \forall t \in [0, T], \tag{4.13}$$

solves Problem (P1), fulfilling (2.25–2.26).

Summing up we have proved the following equivalence result:

THEOREM 4.1. Let the same assumptions of Theorem 2.1 hold. Problem (P1) has a unique solution (u, D) satisfying (2.25–2.26) if and only if the functional equation (4.11) admits a unique solution $G \in C([0, T])$.

5. Proof of Theorem 2.1. Let us set

$$E(\rho, T) := \{G \in C([0, T]) \mid \|G\|_{C([0,T])} \le \rho\},$$

for any $(\rho, T) \in (0, +\infty)^2$.

Thanks to Theorem 4.1, it suffices to prove that there is a pair $(\rho_0, T_0) \in (0, +\infty)^2$ such that $\mathscr{T}$ (cf. (4.12)) takes $E(\rho_0, T_0)$ into itself and $\mathscr{T}^l$ is a contraction for some $l \in \mathbf{N}$. Then the generalized Contraction Principle will yield the proof. To prove that, we need to consider first the mapping $\mathscr{K}$ (cf. (4.10)).

Recalling (1.6), Hölder's inequality, and a well-known trace theorem (cf., e.g., [22, Chap. 1, Sec. 8, Theorem 8.3]) give

$$\left|\int_{\widetilde{\Gamma}} \mathscr{B}_1 z\, d\Gamma\right| \le \Lambda_6 |\widetilde{\Gamma}|^{1/2} \|z\|_{H^3(\Omega)} \quad \forall z \in H^3(\Omega), \tag{5.1}$$

where Λ_6 is a positive constant depending on Γ and μ.

On the other hand, since $\mathscr{K}(D'')$ solves the Cauchy problem (4.5–4.7) (cf. (4.9–4.10)), the estimate (3.7) gives

$$\begin{aligned}
&\|\mathscr{K}(D'')\|_{C([0,t];H^3(\Omega))} \\
&\quad \le \Lambda_1 \Big\{ \|w_0\|_{H^3(\Omega)} + \|w_1\|_W + \|D'Au_1 + (f^1)''\|_{C([0,t];V')} \\
&\qquad + \int_0^t \|D''Au_1 + (f^1)^{(3)}\|_{C([0,s];V')}\,ds + \int_0^t \|C^{-1}(f^2)''\|_{C([0,s];W)}\,ds \\
&\qquad + \int_0^t \|D''C^{-1}Au_0\|_{C([0,s];W)}\,ds + t\|w_0\|_W \Lambda_2(\|D'\|_{C^1([0,t])}, t)\Big\} \\
&\quad \times \{1 + t\Lambda_3(\|D'\|_{C'([0,t])}, t)\exp[t\Lambda_3(\|D'\|_{C^1([0,t])}, t)]\} \quad \forall t \in [0, T].
\end{aligned} \tag{5.2}$$

Then, taking (4.9) into account and using assumptions (2.14–2.19), we get from (5.2)

$$\|\mathscr{K}(G)\|_{C([0,t];H^3(\Omega))} \le \Lambda_7(t) + t\Lambda_8(\rho, t)\exp[t\Lambda_9(\rho, t)] \tag{5.3}$$

for any $t \in [0, T]$ and any $G \in E(\rho, T)$. Here $\Lambda_7 \in C^0((0, +\infty))$ and $\Lambda_8, \Lambda_9 \in C^0((0, +\infty)^2)$ are positive functions, increasing in their variables. Moreover, Λ_7, λ_8, and Λ_9 also depend upon the data.

Thanks to the estimates (5.1), (5.3), and to the hypotheses (2.20–2.21), we derive from (4.12) the following inequality

$$\|\mathscr{T}(G)\|_{C([0,t])} \le \Lambda_{10}(t) + t\Lambda_{11}(\rho, t)\exp[t\Lambda_9(\rho, t)] + \rho t\Lambda_{12} \tag{5.4}$$

for any $t \in [0, T]$ and for any $G \in E(\rho, T)$, where $\Lambda_{10} \in C^0((0, +\infty))$ and $\Lambda_{11} \in C^0((0, +\infty)^2)$ are positive functions analogous to Λ_7 and Λ_8, respectively. Moreover, Λ_{12} is a positive constant depending only on the data.

Looking at (5.4) it is easy to realize that one can pick $(\rho_0, T_0) \in (0, +\infty) \times [0, T]$ such that

$$\|\mathscr{T}(G)\|_{C([0,T_0])} \le \rho_0 \tag{5.5}$$

for any $G \in E(\rho_0, T_0)$. That is to say, $\mathscr{T}$ takes $E(\rho_0, T_0)$ into itself. We will prove that $\mathscr{T}$ has a unique fixed point in $E(\rho_0, T_0)$.

Consider G, $\widehat{G} \in E(\rho_0, T_0)$ and set (cf. (4.10))

$$\mathscr{W} := w - \hat{w} = \mathscr{K}(G) - \mathscr{K}(\widehat{G}). \tag{5.6}$$

It is easy to prove that $\mathscr{W}$ solves the following Cauchy problem (cf. (4.5–4.7), (4.9))

$$\begin{aligned}
&C\mathscr{W}''(t) + A[D_0\mathscr{W} + (D^1 + 1 * G) * \mathscr{W}](t) \\
&\quad = -(G - \widehat{G})(t)Au_0 - [1 * (G - \widehat{G})](t)Au_1 \\
&\qquad - A[1 * (G - \widehat{G}) * \mathscr{K}(\widehat{G})](t) \quad \text{in } W', \forall t \in (0, T_0),
\end{aligned} \tag{5.7}$$

$$\mathscr{W}(0) = \mathscr{W}'(0) = 0. \tag{5.8}$$

Applying the estimate (3.7) to the problem (5.7–5.8) and recalling (5.6) we obtain, for any $t \in [0, T_0]$,

$$\begin{aligned}
&\|\mathscr{K}(G) - \mathscr{K}(\widehat{G})\|_{C([0,t];H^3(\Omega))} \\
&\quad \le \Lambda_1 \Big\{ \|[1 * (G - \widehat{G})]Au_1\|_{C([0,t];V')} \\
&\qquad + \|1 * (G - \widehat{G}) * A\mathscr{K}(\widehat{G})\|_{C([0,t];V')} + \int_0^t \|(G - \widehat{G})Au_1\|_{C([0,s];V')}\, ds \\
&\qquad + \int_0^t \|(G - \widehat{G})A\mathscr{K}(\widehat{G})\|_{C([0,s];V')}\, ds + \int_0^t \|(G - \widehat{G})C^{-1}Au_0\|_{C([0,s];W)}\, ds \Big\} \\
&\qquad \times \{1 + t\Lambda_3(\|D^1 + 1 * G\|_{C^1([0,t])}, t) \exp[t\Lambda_3(\|D^1 + 1 * G\|_{C^1([0,t])}, t)]\}.
\end{aligned} \tag{5.9}$$

The assumptions (2.16–2.17), the estimate (5.3), and the fact that G, $\widehat{G}$ belong to $E(\rho_0, T_0)$ allow us to get from (5.9) the inequality

$$\|\mathscr{K}(G) - \mathscr{K}(\widehat{G})\|_{C([0,t];H^3(\Omega))} \le \Lambda_{13}(\rho_0, T_0) \int_0^t \|G - \widehat{G}\|_{C([0,s])}\, ds \tag{5.10}$$

for any $t \in [0, T_0]$, where $\Lambda_{13}(\rho_0, T_0)$ is a positive constant also depending upon the data.

Using now (4.12) and (5.10), one can find a positive constant $\Lambda_{14}(\rho_0, T_0)$, also depending on the data, such that

$$\|\mathscr{T}(G) - \mathscr{T}(\widehat{G})\|_{C([0,t])} \le \Lambda_{14}(\rho_0, T_0) \int_0^t \|G - \widehat{G}\|_{C([0,s])}\, ds \quad \forall t \in [0, T_0], \tag{5.11}$$

for any G, $\widehat{G} \in E(\rho_0, T_0)$.

Finally, (5.11) implies that there is some $l \in \mathbf{N}$ such that $\mathscr{T}^l$ is a contraction from $E(\rho_0, T_0)$ into itself and this concludes the proof.

6. Proof of Theorem 2.2. We start to observe that, owing to the bounds (2.27–2.28) and the estimate (5.2), there exists a positive function $\Lambda \in C^0((0, +\infty)^3)$ such that

$$\|\mathscr{K}(D_j'')\|_{C([0,T];H^3(\Omega))} \le \Lambda(K, L, T) \tag{6.1}$$

for $j = 1, 2$. The function Λ is nondecreasing in each of its variables and it also depends on Ω, Γ_0, Γ_1, ν_1, ν_2, h, and μ. From now on Λ will denote a function with the previous properties, which could also depend on $\widetilde{\Gamma}$.

Consider now (cf. (4.9–4.10))

$$w := w_1 - w_2 = \mathscr{K}(D_1'') - \mathscr{K}(D_2'') \tag{6.2}$$

and set

$$D := D_1 - D_2, \quad u^0 := u_{01} - u_{02}, \quad u^1 := u_{11} - u_{12}, \quad f := f_1 - f_2. \tag{6.3}$$

Recalling that w_j solves a Cauchy problem like (4.5–4.7), we can easily prove that w (cf. (6.2)) solves the following:

$$\begin{aligned} Cw''(t) + A(D_0 w + D_1' * w)(t) = & - D''(t)Au_{01} - D_2''(t)Au^0 - D'(t)Au_{11} \\ & - D_2'(t)Au^1 - A(D' * w_2)(t) + f''(t) \\ & \text{in } W', \forall t \in (0, T), \end{aligned} \tag{6.4}$$

$$w(0) = w^0 := w_{01} - w_{02}, \tag{6.5}$$

$$w'(0) = w^1 := w_{11} - w_{12}, \tag{6.6}$$

where $D_0 := D_{01} = D_{02}$ (cf. hypotheses of Theorem 2.2) and w_{0j}, w_{1j} are defined by Eqs. (4.6–4.7) in an obvious way.

Applying the estimate (3.7) to the Cauchy problem (6.4–6.6) we get

$$\begin{aligned} & \|w\|_{C([0,t];H^3(\Omega))} + \|w\|_{C^1([0,t];W)} \\ & \quad \le \Lambda_1 \Big\{ \|w^0\|_{H^3(\Omega)} + \|w^1\|_W + \|D' Au_{11}\|_{C([0,t];V')} \\ & \qquad + \|D_2' Au^1\|_{C([0,t];V')} + \|(f^1)''\|_{C([0,t];V')} \\ & \qquad + \|A(D' * w_2)\|_{C([0,t];V')} + \int_0^t \|D'' Au_{11}\|_{C([0,s];V')}\, ds \\ & \qquad + \int_0^t \|D_2'' Au^1\|_{C([0,s];V')}\, ds + \int_0^t \|(f^1)^{(3)}\|_{C([0,s];V')}\, ds \\ & \qquad + \int_0^t \|(D' * Aw_2)'\|_{C([0,s];V')}\, ds + \int_0^t \|C^{-1}(f^2)''\|_{C([0,s];W)}\, ds \\ & \qquad + \int_0^t \|D'' C^{-1} Au_{01}\|_{C([0,s];W)}\, ds + \int_0^t \|D_2'' C^{-1} Au^0\|_{C([0,s];W)}\, ds \\ & \qquad\qquad + t\|w_0\|_W \Lambda_2(\|D_1'\|_{C^1([0,t])}, t) \Big\} \\ & \quad \times \{1 + t\Lambda_3(\|D_1'\|_{C^1([0,t])}, t) \exp[t\Lambda_3(\|D_1'\|_{C^1([0,t])}, t)]\} \end{aligned} \tag{6.7}$$

for any $t \in [0, T]$. Here $f^1 := f_1^1 - f_2^1$ and $f^2 := f_1^2 - f_2^2$, where $f_1 := f_1^1 + f_1^2$, $f_2 := f_2^1 + f_2^2$.

Observe that

$$D_j(t) = D_0 + tD_{1j} + \int_0^t (t-s) D_j''(s)\, ds \quad \text{on } [0, T], \tag{6.8}$$

where (cf. (4.2))

$$D_{1j} = \left(\int_{\tilde{\Gamma}} \mathscr{B}_1 u_{0j}\, d\Gamma \right)^{-1} \left(g_j'(0) - D_0 \int_{\tilde{\Gamma}} \mathscr{B}_1 u_{1j}\, d\Gamma \right) \tag{6.9}$$

for $j = 1, 2$.

Then, thanks to (2.21), (2.27), and (5.1), we deduce from (6.8–6.9) the estimate (cf. (6.3))

$$\begin{aligned}\|D\|_{C^1([0,t])} \le \Lambda(K, L, T) \Big\{ &\|u^0\|_{H^3(\Omega)} + \|u^1\|_{H^3(\Omega)} \\ &+ \|g\|_{C^1([0,t])} + \int_0^t \|D''\|_{C([0,s])}\, ds \Big\} \\ &\forall t \in [0, T],\end{aligned} \tag{6.10}$$

where $g := g_1 - g_2$.

Recalling hypotheses (2.16–2.19) and using elliptic estimates (cf., e.g., [22, Chap. 2, Secs. 5, 7, 9]), from Eq. (4.6), written for w^0 (cf. (6.5)), we infer

$$\begin{aligned}\|w^0\|_{H^3(\Omega)} \le \Lambda(K, L) \Big\{ &\|AC^{-1}Au^0\|_{V'} + \|C^{-1}Au^0\|_W \\ &+ \|AC^{-1}f(0)\|_{V'} + \|C^{-1}f(0)\|_W \Big\}.\end{aligned} \tag{6.11}$$

Moreover, owing to hypotheses (2.16–2.18), (2.20) and to Eq. (6.9), from Eq. (4.7), written for w^1 (cf. (6.6)), we get

$$\begin{aligned}\|w^1\|_W \le \Lambda(K, L) \Big\{ &\|C^{-1}Au^1\|_W + \|C^{-1}Au^0\|_W \\ &+ \|u^0\|_{H^3(\Omega)} + \|u^1\|_{H^3(\Omega)} + \|g\|_{C^1([0,T])} + \|C^{-1}f'(0)\|_W \Big\}.\end{aligned} \tag{6.12}$$

Note that the assumptions (2.16–2.17) combining with elliptic estimates (cf., e.g., [22, Chap. 2, Secs. 5, 9] imply

$$\begin{aligned}&\|u^0\|_{H^3(\Omega)} + \|u^1\|_{H^3(\Omega)} \\ &\quad \le \Lambda(K, L)\{\|C^{-1}Au^0\|_W + \|u^0\|_W + \|C^{-1}Au^1\|_W + \|u^1\|_W\}.\end{aligned} \tag{6.13}$$

The bound (6.1) and the estimates (6.10–6.13) allow us to derive from (6.7) the following:

$$\begin{aligned}&\|w\|_{C([0,t];H^3(\Omega))} + \|w\|_{C^1([0,t];W)} \\ &\quad \le \Lambda(K, L, T) \Big\{ \|u^0\|_W + \|u^1\|_W + \|C^{-1}Au^0\|_W \\ &\qquad + \|C^{-1}Au^1\|_W + \|C^{-1}f(0)\|_W \\ &\qquad + \|C^{-1}f'(0)\|_W + \|AC^{-1}Au^0\|_{V'} + \|AC^{-1}f(0)\|_{V'} \\ &\qquad + \|(f^1)''\|_{C^1([0,t];V')} + \|C^{-1}(f^2)''\|_{C([0,t];W)} \\ &\qquad + \|g\|_{C^1([0,t])} + \int_0^t \|D''\|_{C([0,s])}\, ds \Big\}\end{aligned} \tag{6.14}$$

for any $t \in [0, T]$. Observe now that $G_j = D_j''$ solves an equation like (4.11); then

D'' (cf. (6.3)) solves the following (cf. also (4.12), (6.3)):

$$\begin{aligned} D'' = \left(\int_{\tilde\Gamma} \mathscr{B}_1 u_{01}\, d\Gamma\right)^{-1} \Big\{ g'' - \int_{\tilde\Gamma} [D_0 \mathscr{B}_1 w + D' * \mathscr{B}_1 w_1 + D_2' * \mathscr{B}_1 w]\, d\Gamma \\ - \left(\int_{\tilde\Gamma} \mathscr{B}_1 u^1\, d\Gamma\right) D_1' - \left(\int_{\tilde\Gamma} \mathscr{B}_1 u_{12}\, d\Gamma\right) D' \Big\} \\ + \left[\left(\int_{\tilde\Gamma} \mathscr{B}_1 u_{01}\, d\Gamma\right)^{-1} - \left(\int_{\tilde\Gamma} \mathscr{B}_1 u_{02}\, d\Gamma\right)^{-1} \right] \\ \times \left(g_2'' - \int_{\tilde\Gamma} [D_0 \mathscr{B}_1 w_2 + D_2' * \mathscr{B}_1 w_2] d\Gamma - \left(\int_{\tilde\Gamma} \mathscr{B}_1 u_{12}\, d\Gamma\right) D_2' \right) \quad \text{on } [0, T]. \end{aligned} \tag{6.15}$$

Thanks to the hypotheses (2.21), (2.27–2.28) and to the estimates (5.1), (6.1), (6.10), and (6.14), one easily gets from (6.15) the following inequality:

$$\begin{aligned} \|D''\|_{C([0,t])} \le \Lambda(K, L, T) \Big\{ & \|u^0\|_W + \|u^1\|_W + \|C^{-1} A u^0\|_W \\ & + \|C^{-1} A u^1\|_W + \|C^{-1} f(0)\|_W \\ & + \|C^{-1} f'(0)\|_W + \|A C^{-1} A u^0\|_{V'} \\ & + \|A C^{-1} f(0)\|_{V'} + \|(f^1)''\|_{C^1([0,t]; V')} \\ & + \|C^{-1} (f^2)''\|_{C([0,t]; W)} + \|g\|_{C^2([0,t])} \\ & + \int_0^t \|D''\|_{C([0,s])}\, ds \Big\} \end{aligned} \tag{6.16}$$

for any $t \in [0, T]$. Applying Gronwall's lemma to (6.16) we obtain

$$\begin{aligned} \|D''\|_{C([0,T])} \le \Lambda(K, L, T) \Big\{ & \|u^0\|_W + \|u^1\|_W + \|C^{-1} A u^0\|_W \\ & + \|C^{-1} A u^1\|_W + \|C^{-1} f(0)\|_W \\ & + \|C^{-1} f'(0)\|_W + \|A C^{-1} A u^0\|_{V'} \\ & + \|A C^{-1} f(0)\|_{V'} + \|(f^1)''\|_{C^1([0,T]; V')} \\ & + \|C^{-1} (f^2)''\|_{C([0,T]; W)} + \|g\|_{C^2([0,T])} \Big\}. \end{aligned} \tag{6.17}$$

Combining (6.10), (6.14), and (6.17) we get

$$\begin{aligned} & \|w\|_{C([0,t]; H^3(\Omega))} + \|w\|_{C^1([0,t]; W)} + \|D\|_{C^2([0,T])} \\ & \le \Lambda(K, L, T) \Big\{ \|u^0\|_W + \|u^1\|_W + \|C^{-1} A u^0\|_W + \|C^{-1} A u^1\|_W \\ & \quad + \|C^{-1} f(0)\|_W + \|C^{-1} f'(0)\|_W \\ & \quad + \|A C^{-1} A u^0\|_{V'} + \|A C^{-1} f(0)\|_{V'} \\ & \quad + \|f^1\|_{C^3([0,T]; V')} + \|C^{-1} f^2\|_{C^2([0,T]; W)} + \|g\|_{C^2([0,T])} \Big\}. \end{aligned} \tag{6.18}$$

Recalling that $C\colon V \to V'$ is an isomorphism and taking advantage of (6.10) and (6.18), from Eq. (6.4) one can easily deduce

$$\begin{aligned}\|w''\|_{C([0,T];V)} \le \Lambda(K, L, T) \Big\{ &\|u^0\|_W + \|u^1\|_W + \|C^{-1}Au^0\|_W \\ &+ \|C^{-1}Au^1\|_W + \|C^{-1}f(0)\|_W + \|C^{-1}f'(0)\|_W \\ &+ \|AC^{-1}Au^0\|_{V'} + \|AC^{-1}f(0)\|_{V'} + \|f^1\|_{C^3([0,T];V')} \\ &+ \|C^{-1}f^2\|_{C^2([0,T];W)} + \|g\|_{C^2([0,T])} \Big\}. \end{aligned} \tag{6.19}$$

We now observe that (cf. ((4.3), (6.2–6.3))

$$(u_1 - u_2)(t) = u_{01} - u_{02} + t(u_{11} - u_{12}) + \int_0^t (t-s)w(s)\,ds \quad \forall t \in [0, T]. \tag{6.20}$$

Then (6.18–6.20) yield the wanted estimate (2.29).

7. Concluding remarks.

7.1. In the applications the plate is usually considered at rest at the initial time $t = 0$; thus the initial data u_0, u_1 identically vanish. This implies that the basic condition (2.21) fails. Nevertheless, reinforcing a little the assumptions on the data f and g, one can prove results quite similar to Theorems 2.1 and 2.2. Indeed, observe that, if $u_0 = u_1 \equiv 0$ then the problem (4.5–4.8), which is actually equivalent to Problem (P1), turns out to be

$$Cw''(t) + A(D_0 w + D' * w)(t) = \tilde{f}(t) \quad \text{in } W', \ \forall t \in (0, T), \tag{7.1}$$

$$w(0) = w_0 := C^{-1}f(0), \tag{7.2}$$

$$w'(0) = w_1 := C^{-1}f'(0), \tag{7.3}$$

$$\int_{\tilde{\Gamma}} \{D_0 \mathscr{B}_1 w + D' * \mathscr{B}_1 w\}\, d\Gamma = \tilde{g} \quad \text{on } [0, T], \tag{7.4}$$

where $\tilde{f} := f''$ and $\tilde{g} := g''$. The identification problem consisting in finding w and D satisfying (7.1–7.4) is quite identical to Problem (P1). Hence, assuming

$$f = f^1 + f^2 \in C^5([0, T]; V') + C^4([0, T]; H), \tag{7.5}$$

$$C^{-1}f^2(0) \in C^4([0, T]; W), \tag{7.6}$$

$$C^{-1}f(0),\ C^{-1}f'(0) \in W, \tag{7.7}$$

$$C^{-1}AC^{-1}f(0),\ C^{-1}AC^{-1}f'(0) \in W, \tag{7.8}$$

$$C^{-1}f''(0),\ C^{-1}f^{(3)}(0) \in W, \tag{7.9}$$

$$AC^{-1}AC^{-1}f(0),\ AC^{-1}f''(0) \in V', \tag{7.10}$$

$$g \in C^4([0, T]), \tag{7.11}$$

$$D_0 := g''(0)\left(\int_{\tilde{\Gamma}} \mathscr{B}_1 C^{-1} f(0)\, d\Gamma\right)^{-1} > 0, \tag{7.12}$$

$$D(0)\int_{\tilde{\Gamma}} \mathscr{B}_1 C^{-1} f'(0)\, d\Gamma + D'(0)\int_{\tilde{\Gamma}} \mathscr{B}_1 C^{-1} f(0)\, d\Gamma = g^{(3)}(0), \tag{7.13}$$

$$\mathscr{B}_j C^{-1} f(0) = \mathscr{B}_j C^{-1} f'(0) = 0 \quad \text{on } \Gamma_1,\ j = 0, 1, \tag{7.14}$$

it is easy to check that $\tilde{f}$, w_0, w_1, and $\tilde{g}$ satisfy the hypotheses (2.14–2.23). Then, assuming in addition that (2.10) and (2.24) hold, we can apply Theorem 2.1 to the identification problem (7.1–7.4). Hence, we can find a $T_0 \in (0, T]$ such that the problem (7.1–7.4) has a unique local solution (w, D) fulfilling the properties (2.25–2.26). Recalling now (4.13) we get the following

THEOREM 7.1. Let the assumptions (2.10), (2.24), (7.5–7.14) be satisfied. Then there exists $T_0 \in (0, T]$ such that Problem (P1) admits a unique local solution (u, D) having the following properties:

$$u \in C^4([0, T_0]; H^3(\Omega) \cap W) \cap C^5([0, T_0]; W) \cap C^6([0, T_0]; V),$$
$$D \in C^2([0, T_0]), \qquad D(0) > 0.$$

Clearly we also can establish an analogous result to Theorem 2.2.

Finally, we note that we can obtain similar results even if $u_0 \equiv 0$ only. It suffices to consider the identification problem for $(v, D) := (u', D)$, namely

$$Cv''(t) + A(D_0 v + D' * v)(t) = f'(t) \quad \text{in } W',\ \forall t \in (0, T), \tag{7.15}$$

$$v(0) = u_1, \qquad v'(0) = C^{-1}f(0), \tag{7.16}$$

$$\int_{\tilde{\Gamma}} \{D_0 \mathscr{B}_1 v + D' * \mathscr{B}_1 v\} d\Gamma = g' \quad \text{on } [0, T]. \tag{7.17}$$

Here Eq. (7.15) derives from (4.4) setting $u_0 \equiv 0$, the initial condition for v' derives from (4.6), and the additional information (7.17) comes out by differentiating in time Eq. (2.13). We point out that the initial velocity u_1 will replace u_0 in the basic condition (2.21).

7.2. Throughout this paper we have implicitly assumed that the memory mechanism is *casual*, i.e., it acts only over the time interval $(0, t)$ (cf., e.g., [2, Appendix K] and [26, Chap. I, Sec. 5]. However, other assumptions may be made (see, e.g.,

[6] and its references). For instance, if the memory is considered as *infinite* then the convolution integrals in time must be taken over $(0, +\infty)$ (or $(-\infty, t)$ according to the choice of the integration variables). In this case, assuming that the material is free of stresses and strains in the past, that is for $t \leq 0$, we can take again the mentioned integrals over $(0, t)$ (cf. [15, Chap. 1, Sec. 7]) and our results still hold. More generally, they hold even if the material is free of stresses and strains only in a given time interval $(-\tilde{T}, 0]$, $\tilde{T} > 0$, and the residual stress involving the past history of u from $-\infty$ to $-\tilde{T}$ is known (see [11, Remark 4.3.] for details in the three-dimensional case). Note that the previous assumptions on the past histories of stress and strain imply that $u_0 = u_1 \equiv 0$. Thus, in order to get our results, Remark 7.1 has to be taken into account.

7.3. Let us consider the direct problem (1.1–1.5) and its variational formulation (2.11–2.12) (cf. Sec. 2). Reinforcing the regularity assumptions on the data f, u_0, u_1 (cf. (3.3–3.6)) and on Γ it is possible to show the existence and uniqueness of a *classical* solution (i.e., Eq. (1.1) is satisfied a.e. in $\Omega \times (0, T)$).

More precisely, replacing (3.3–3.6) by

$$C^{-1} f^1 \in C^1([0, T]; W), \tag{7.18}$$

$$C^{-1} f^2 \in C([0, T]; H^3(\Omega) \cap W), \tag{7.19}$$

$$u_0, u_1 \in W, \tag{7.20}$$

$$Au_0 \in H, \qquad Au_1 \in V', \tag{7.21}$$

and assuming Γ of class $\mathscr{C}^5$, we prove

THEOREM 7.2. Under the assumptions (2.10), (2.24), (3.1–3.2), (7.18–7.21), there exists a unique function $u \in C([0, T]; H^4(\Omega) \cap W) \cap C^1([0, T]; H^3(\Omega) \cap W) \cap C^2([0, T]; W)$ solving Eq. (1.1) a.e. in $\Omega \times (0, T)$ with $f = f^1 + f^2$ and satisfying the conditions (1.2–1.5). Moreover, there are a positive constant Λ_{15} and a pair of positive functions Λ_{16}, $\Lambda_{17} \in C^0((0, +\infty)^2)$ such that

$$\begin{aligned} &\|u\|_{C([0,t]; H^4(\Omega)\cap W)} + \|u\|_{C^1([0,t]; H^3(\Omega)\cap W)} \\ &\quad \leq \Lambda_{15} \Big\{ \|u_0\|_{H^4(\Omega)} + \|u_1\|_{H^3(\Omega)} + \|f^1\|_{C([0,t]; H)} \\ &\qquad + \int_0^t \|(f^1)'\|_{C([0,s]; H)}\, ds + \int_0^t \|C^{-1} f^2\|_{C([0,s]; H^3(\Omega))}\, ds \\ &\qquad\qquad + t\|u_0\|_{H^3(\Omega)} \Lambda_{16}(\|D'\|_{C^1([0,t])}, t) \Big\} \\ &\quad \times \{1 + t\Lambda_{17}(\|D'\|_{C^1([0,t]}, t) \exp[t\Lambda_{17}(\|D'\|_{C^1([0,t])}, t)]\} \end{aligned} \tag{7.22}$$

for any $t \in [0, T]$. The constant Λ_{15} depends on Ω, h, μ, $(D(0))^{-1}$, $D'(0)$, T and the functions Λ_{16}, Λ_{17} are increasing in each of their arguments and also depend on $(D(0))^{-1}$, Ω, h, μ.

Proof. Recalling the proof of Theorem 3.1 (cf. Sec. 3) we apply the operator $\mathbf{A}$ to both sides of Eqs. (3.17–3.18). This implies that $\tilde{v} := \mathbf{A}v$ formally solves the

Cauchy problem:

$$\tilde{v}' + \mathbf{A}\tilde{v}(t) = \mathbf{A}\mathscr{R}(\mathbf{A}^{-1}\tilde{v})(t) + \tilde{F}(t) \quad \text{in } W \times V, \ \forall t \in (0, T), \tag{7.23}$$

$$\tilde{v}(0) = \tilde{v}_0 := \mathbf{A}v_0, \qquad \tilde{v}'(0) = \tilde{v}_1 := \mathbf{A}v_1, \tag{7.24}$$

where $\widetilde{\mathscr{F}} := \mathbf{A}\mathscr{F}$ (cf. (3.20)).

Thanks to the assumptions (7.18–7.21) we have (cf. also (3.15))

$$\tilde{v}_0 \in D(\mathbf{A}), \tag{7.25}$$

$$\widetilde{\mathscr{F}} \in C^1([0, T]; W \times V) + C([0, T]; \mathscr{D}(\mathbf{A})). \tag{7.26}$$

Moreover, we can easily prove the estimate (cf. (3.22))

$$\|\mathbf{A}[\mathscr{R}(\mathbf{A}^{-1}\tilde{v}_1) - \mathscr{R}(\mathbf{A}^{-1}\tilde{v}_2)]\|_{C([0,t];\mathscr{D}(\mathbf{A}))} \le (|\delta| + \|R'\|_{C([0,t])})\|\tilde{v}_1 - \tilde{v}_2\|_{C([0,t];\mathscr{D}(\mathbf{A}))} \tag{7.27}$$

for any $t \in [0, T]$ and any $\tilde{v}_1, \tilde{v}_2 \in C([0, T]; \mathscr{D}(\mathbf{A}))$.

Taking advantages of (7.25–7.27) and using the same arguments used in the proof of Theorem 3.1 (see from (3.24) on) we get that there exists a unique solution $\tilde{v} \in C([0, T]; \mathscr{D}(\mathbf{A})) \cap C^1([0, T]; W \times V)$ to problem (7.23–7.24) and $\tilde{v}$ satisfies an estimate quite similar to (3.32). Observe now that $\mathbf{A}$ is invertible and $\mathbf{A}^{-1}: W \times V \to \mathscr{D}(\mathbf{A})$. Then, setting $v := \mathbf{A}^{-1}\tilde{v}$, using Eqs. (3.9), (3.16), and elliptic regularity results, we prove that the function u defined by

$$u := (D(0))^{-1}[(\mathbf{A}^{-1}\tilde{v})^1 + R * (\mathbf{A}^{-1}\tilde{v})^1] \quad \text{on } [0, T] \tag{7.28}$$

belongs to $C([0, T]; H^4(\Omega) \cap W) \cap C^1([0, T]; H^3(\Omega) \cap W) \cap C^2([0, T]; W)$ and is the unique solution to Problem (1.1–1.5).

Finally, the analogue of estimate (3.7) holding for $\tilde{v}$ and Eq. (3.10) allow us to infer from (7.28) the wanted inequality (7.22). □

Owing to Theorem 7.2 we can change the additional information (1.9) by assuming knowledge of the force $-M_\tau^\nu - g^3$ (cf. (1.5), (1.8)) acting on a portion $\tilde{\Gamma} \subseteq \Gamma_0$, $|\tilde{\Gamma}| > 0$, for each time $t \in [0, T]$, that is,

$$\int_{\tilde{\Gamma}} \{D(0)\mathscr{B}_2 u + D' * \mathscr{B}_2 u\} d\Gamma = g \quad \text{on } [0, T], \tag{7.29}$$

where $g: [0, T] \to \mathbf{R}$ is a given function. Here we recall that $u''_\nu = 0$ on $\tilde{\Gamma} \times [0, T]$ since $u''(\cdot, t) \in W$, for any $t \in [0, T]$ (cf. (2.2) and Theorem 7.2).

Since the integral on the left-hand side of (7.29) contains a third-order linear differential operator (cf. (1.6)), in place of the inequality (5.1), we must consider the following inequality (cf., e.g., [22, Chap. 1, Sec. 8, Theorem 8.3]):

$$\left| \int_{\tilde{\Gamma}} \mathscr{B}_2 z d\Gamma \right| \le \Lambda_{16} |\tilde{\Gamma}|^{1/2} \|z\|_{H^4(\Omega)} \tag{7.30}$$

for any $z \in H^4(\Omega)$, where Λ_{16} is a positive constant depending on Γ, μ.

Inequality (7.30) tells us that estimate (7.22) has to be used in place of (3.7). Then arguments quite identical to the ones developed in Secs. 5 and 6 yield results similar to Theorems 2.1–2.2.

7.4. All the results proved or outlined in this paper also apply to the one-dimensional counterpart of the Kirchhoff viscoelastic plate model, namely the *viscoelastic beam* (cf., e.g., [21]).

In this case, assuming $\Omega = (0, \omega) \subset \mathbf{R}$, $\omega > 0$ being the length of the beam (supposed to be homogeneous with unitary density), Eq. (1.1) becomes

$$hu_{tt} - \frac{h^3}{12}u_{xxtt} + D(0)u_{xxxx} + D' * u_{xxxx} = f \quad \text{in } (0, \omega) \times (0, T), \tag{7.31}$$

where $h > 0$ denotes the thickness of the beam.

Moreover, the initial and boundary conditions (1.2–1.5) can be translated as follows:

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \quad \forall x \in (0, \omega), \tag{7.32}$$

$$u(0, t) = u_x(0, t) = 0 \quad \forall t \in [0, T], \tag{7.33}$$

$$D(0)u_{xx}(\omega, t) + (D' * u_{xx})(\omega, t) = g_1(t) \quad \forall t \in [0, T], \tag{7.34}$$

$$D(0)u_{xxx}(\omega, t) + (D' * u_{xxx})(\omega, t) - \frac{h^3}{12}u_{xtt}(\omega, t) = g_2(t) \quad \forall t \in [0, T]. \tag{7.35}$$

As far as the additional information is concerned, we can choose for instance (cf. (1.9))

$$D(0)u_{xx}(0, t) + (D' * u_{xx})(0, t) = g(t) \quad \forall t \in [0, T] \tag{7.36}$$

or (cf. Remark 7.3, (7.28))

$$D(0)u_{xxx}(0, t) + (D' * u_{xxx})(0, t) - \frac{h^3}{12}u_{xtt}(0, t) = g(t) \quad \forall t \in [0, T], \tag{7.37}$$

$g\colon [0, T] \to \mathbf{R}$ being a given function. Either assuming (7.36) or (7.37) we can prove results like Theorems 2.1–2.2.

Let us consider for example the additional information (7.36), namely the identification problem consisting in finding u and D satisfying Eq. (7.31) and the conditions (7.32–7.36).

We define (cf. (2.1–2.4))

$$V := \{v \in H^1(0, \omega) \mid v(0) = 0\}, \tag{7.38}$$

$$W := \{w \in H^2(0, \omega) \mid w(0) = w_x(0) = 0\}, \tag{7.39}$$

and we endow them with the following inner products:

$$c(v, \tilde{v}) := h\int_0^\omega v\tilde{v} + \frac{h^3}{12}\int_0^\omega v_x\tilde{v}_x \quad \forall v, \tilde{v} \in V, \tag{7.40}$$

$$a(w, \tilde{w}) := \int_0^\omega w_{xx}\tilde{w}_{xx} \quad \forall w, \tilde{w} \in W. \tag{7.41}$$

Moreover, by using the coercive bilinear forms defined by (7.40–7.41), we introduce the canonical isomorphisms $C\colon V \to V'$ and $A\colon W \to W'$.

Assume now that (2.14–2.20) hold according to (7.38–7.41) and to the new definitions of C and A. In addition, assume that (cf. 2.21–2.22))

$$D_0 := g(0)\left(\frac{d^2u_0}{dx^2}(0)\right)^{-1} > 0, \tag{7.42}$$

$$D(0)\frac{d^2u_1}{dx^2}(0) + D'(0)\frac{d^2u_0}{dx^2}(0) = g'(0), \tag{7.43}$$

$$\frac{d^2u_j}{dx^2}(\omega) = \frac{d^3u_j}{dx^3}(\omega) = 0, \qquad j = 0, 1. \tag{7.44}$$

Clearly condition (2.24) is trivially satisfied in the one-dimensional case.

Thanks to the previous positions the same technique used in the two-dimensional case gives

THEOREM 7.3. Let the assumptions (2.10), (2.14–2.20), (7.42–7.44) hold. Then there is $T_0 \in (0, T]$ such that the problem (7.30–7.35) admits a unique local solution (u, D) having the properties (2.25–2.26).

We point out that Theorem 2.2 can be restated as well. Similar considerations hold when the additional information (7.37) is considered, taking Remark 7.3 into account.

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