VARIATIONAL FORMULATIONS FOR THE VIBRATION
OF A PIEZOELECTRIC BODY

BY

J. S. YANG

Princeton University, Princeton, New Jersey

Abstract. This paper presents a systematic discussion on the variational principles for the vibration of a piezoelectric body. It is shown that there exist four types of variational formulations depending on the internal energy, electric enthalpy, mechanical enthalpy, and enthalpy, respectively. The one depending on the internal energy is in a positive-definite form which immediately leads to a few important properties of the lowest resonant frequency.

1. Introduction. In classical elasticity, there are two types of variational principles for the free vibration of an elastic body. One is associated with potential energy, the other with complementary energy [1–3]. A variational formulation for the free vibration of a piezoelectric body is given in [4], which is related to the electric enthalpy. In this paper, three other variational formulations are given for the free vibration of a piezoelectric body which are related to the internal energy, mechanical enthalpy, and enthalpy, respectively. The one depending on the electric enthalpy in [4] is presented in a more general form in this paper. These variational principles can be considered as generalizations of the corresponding variational formulations in classical elasticity. Because of the presence of the electric fields, there can be four generalizations for the two formulations in classical elasticity. They each have a different set of independent arguments, which allow different but equivalent formulations of the same eigenvalue problem. The variational principles are given without constraints. They can be reduced to various constraint variational principles. The constraint internal energy formulation is in a positive-definite form, which can be used to show a few properties of the lowest resonant frequency.

2. The eigenvalue problem. Let the region occupied by the piezoelectric body be $V$, the boundary surface of $V$ be $S$, the unit outward normal of $S$ be $n_i$, and $S$ be partitioned in the following way:

$$S_u \cup S_T = S_\phi \cup S_D = S,$$
$$S_u \cap S_T = S_\phi \cap S_D = \emptyset.$$  

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Then the eigenvalue problem for the resonance of a linear piezoelectric body is [5]

\[- T_{ij,j} = \rho \omega^2 u_i \quad \text{in} \ V, \]
\[- D_{i,i} = 0 \quad \text{in} \ V, \]
\[- S_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in} \ V, \]
\[E_i + \phi_{,i} = 0 \quad \text{in} \ V, \]
\[- T_{ij} + \frac{\partial H}{\partial S_{ij}} = 0 \quad \text{in} \ V, \]
\[D_i + \frac{\partial H}{\partial E_i} = 0 \quad \text{in} \ V, \]
\[- u_i = 0 \quad \text{on} \ S_u, \]
\[T_{ji} n_j = 0 \quad \text{on} \ S_T, \]
\[- \phi = 0 \quad \text{on} \ S_\phi, \]
\[D_i n_i = 0 \quad \text{on} \ S_D, \]  

(2)

where \( u_i \) is displacement, \( S_{ij} \) strain, \( T_{ji} \) stress, \( \phi \) electric potential, \( E_i \) electric field, \( D_i \) electric displacement, \( \rho \) mass density, and \( \omega \) resonant frequency. \( H = H(S_{ij}, E_i) \) is the electric enthalpy function for the piezoelectric material. Since we are only considering linear materials, \( H \) and the corresponding linear constitutive relations assume the following form:

\[H = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - \frac{1}{2} \varepsilon_{ij} E_i E_j - \varepsilon_{ijk} E_i E_j S_{jk},\]  

(3)

\[T_{ij} = c_{ijkl} S_{kl} - \varepsilon_{kij} E_k, \]
\[D_i = \varepsilon_{ij} E_j + \varepsilon_{ijk} S_{jk},\]  

(4)

where the elastic moduli \( c_{ijkl} \), electric permittivity \( \varepsilon_{ij} \), and piezoelectric constants \( \varepsilon_{ijk} \) are material constants with the following symmetry properties:

\[c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij},\]
\[\varepsilon_{ijk} = \varepsilon_{ikj}, \quad \varepsilon_{ij} = \varepsilon_{ji},\]  

(5)

and for any nonzero symmetric tensor \( a_{ij} \) and vector \( b_i \)

\[c_{ijkl} a_{ij} a_{kj} > 0, \quad \varepsilon_{ij} b_i b_j > 0.\]  

(6)

Given \( \rho \) and \( H \), values of \( \omega^2 \) are sought corresponding to which nontrivial solutions of \( u_i, S_{ij}, T_{ij}, \phi, E_i \), and \( D_i \) exist.
3. The electric enthalpy $H(S_{ij}, E_i)$ formulation. We first give the formulation involving the electric enthalpy $H(S_{ij}, E_i)$ in a form that generalizes what is in [4]. Let

$$\Lambda_1(u_i, S_{ij}, T_{ij}, \phi, E_i, D_i) = \int_V \left[ T_{ij} u_{i,j} + D_i \phi_{,i} + H(S_{ij}, E_i) + E_i D_i - T_{ij} S_{ij} \right] dV$$

$$- \int_{S_u} T_{ji} n_j u_j dS - \int_{S_s} D_i n_i \phi dS,$$

$$\Gamma_1(u_i) = \int_V \frac{1}{2} \rho u_i u_i dV,$$

$$\Pi_1(u_i, S_{ij}, T_{ij}, \phi, E_i, D_i) = \frac{\Lambda_1(u_i, S_{ij}, T_{ij}, \phi, E_i, D_i)}{\Gamma_1(u_i)}.$$  (7)

To obtain the stationary conditions of $\Pi_1$ with all its arguments as independent variables, we begin with

$$\delta \Pi_1 = \frac{1}{\Gamma_1^2} (\Gamma_1 \delta \Lambda_1 - \Lambda_1 \delta \Gamma_1) = \frac{1}{\Gamma_1} \left( \delta \Lambda_1 - \frac{\Lambda_1}{\Gamma_1} \delta \Gamma_1 \right).$$  (8)

Therefore $\delta \Pi_1 = 0$ implies

$$\delta \Lambda_1 - \frac{\Lambda_1}{\Gamma_1} \delta \Gamma_1 = 0.$$  (9)

With integration by parts, we can obtain

$$\delta \Lambda_1 = \int_V \left\{ -T_{ji,j} \delta u_i - D_{i,i} \delta \phi + \left( \frac{\partial H}{\partial S_{ij}} - T_{ij} \right) \delta S_{ij} + \left( \frac{\partial H}{\partial E_i} + D_i \right) \delta E_i \right. \right.$$

$$\left. + \left[ \frac{1}{2} (u_{i,j} + u_{j,i}) - S_{ij} \right] \delta T_{ij} + (E_i + \phi_{,i}) \delta D_i \right\} dV$$

$$- \int_{S_u} u_i \delta T_{ji} n_j dS + \int_{S_F} T_{ji} n_j \delta u_i dS - \int_{S_S} \phi \delta D_i n_i dS + \int_{S_D} D_i n_i \delta \phi dS,$$

$$\delta \Gamma_1 = \int_V \rho u_i \delta u_i dV.$$  (10)

Since all the variations of $\delta u_i, \delta S_{ij}, \delta T_{ij}, \delta \phi, \delta E_i,$ and $\delta D_i$ are independent, $\delta \Pi_1 = 0$ implies
Comparing (11) to (2), we have the following variational principle: the stationary condition of \( \Pi_1 \) gives the eigenvalue problem (2), with the stationary value of \( \Pi_1 \) as \( \omega^2 \).

The above variational principle has no constraints. If we choose our admissible functions to satisfy

\[
-S_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } V, \\
E_i + \phi_{,i} = 0, \\
-T_{ij} + \frac{\partial H}{\partial S_{ij}} = 0 \quad \text{in } V, \\
D_i + \frac{\partial H}{\partial E_i} = 0 \quad \text{in } V, \\
-u_i = 0 \quad \text{on } S_u, \\
T_{ij}n_j = 0 \quad \text{on } S_T, \\
-\phi = 0 \quad \text{on } S_\phi, \\
D_i n_i = 0 \quad \text{on } S_D.
\]  

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The above variational principle has no constraints. If we choose our admissible functions to satisfy

\[
-S_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } V, \\
E_i + \phi_{,i} = 0, \\
-T_{ij} + \frac{\partial H}{\partial S_{ij}} = 0 \quad \text{in } V, \\
D_i + \frac{\partial H}{\partial E_i} = 0 \quad \text{in } V, \\
-u_i = 0 \quad \text{on } S_u, \\
T_{ij}n_j = 0 \quad \text{on } S_T, \\
-\phi = 0 \quad \text{on } S_\phi, \\
D_i n_i = 0 \quad \text{on } S_D.
\]  

then \( \Pi_1 \) reduces to

\[
\Pi_1 = \frac{\int_V H dV}{\int_V \frac{1}{2} \rho u_i u_i dV},
\]  

and the stationary condition of \( \Pi_1 \) becomes

\[
-T_{ji,j} = \rho \frac{\Lambda_1}{\Gamma_1} u_i \quad \text{in } V, \\
-D_{i,i} = 0 \quad \text{in } V, \\
T_{ji}n_j = 0 \quad \text{on } S_T, \\
D_i n_i = 0 \quad \text{on } S_D.
\]

This constraint version of \( \Lambda_1 \) is equivalent to what is given in [4]. The above \( H \) formulation can be considered as a generalization of the potential energy formulation for the vibration problem in classical elasticity.
4. The mechanical enthalpy $M(T_{ij}, D_i)$ formulation. The mechanical enthalpy $M$ can be introduced through Legendre transform from $H$ as

$$M = M(T_{ij}, D_i) = H + E_iD_i - T_{ij}S_{ij},$$

which generates the following constitutive relations:

$$S_{ij} = -\frac{\partial M}{\partial T_{ij}}, \quad E_i = \frac{\partial M}{\partial D_i}.$$  \hspace{1cm} (16)

For this formulation, we need to introduce (when $\omega \neq 0$)

$$a_i = -\omega^2 u_i, \quad \psi = -\omega^2 \phi.$$  \hspace{1cm} (17)

We note that the physical meaning of $a_i$ is acceleration. Let

$$\Lambda_2(a_i, T_{ij}, \psi, D_i) = \int_V (-T_{ij,j}a_i + \frac{1}{2}\rho a_i a_i - D_{i,i}\psi)dV + \int_{S_T} T_{ji,n_j}a_i dS$$

$$+ \int_{S_D} D_{i,n_i}\psi dS,$$

$$\Gamma_2(T_{ij}, D_i) = \int_V M(T_{ij}, D_i)dV,$$

$$\Pi_2(a_i, T_{ij}, \psi, D_i) = \frac{\Lambda_2(a_i, T_{ij}, \psi, D_i)}{\Gamma_2(T_{ij}, D_i)},$$

where independent arguments are now $a_i$, $T_{ij}$, $\psi$, and $D_i$. Since

$$\delta \Lambda_2 = \int_V \left[ (-T_{ij,j} + \rho a_i)\delta a_i - D_{i,i}\delta \psi + \frac{1}{2}(a_{i,j} + a_{j,i})\delta T_{ij} + \psi, \delta D_i \right]dV$$

$$- \int_{S_u} a_i \delta T_{ij,n_j} dS + \int_{S_T} T_{ji,n_j} \delta a_i dS - \int_{S_\phi} \psi \delta D_{i,n_i} dS + \int_{S_D} D_{i,n_i}\delta \psi dS,$$

$$\delta \Gamma_2 = \int_V \left( \frac{\partial M}{\partial T_{ij}} \delta T_{ij} + \frac{\partial M}{\partial D_i} \delta D_i \right) dV,$$

$$\delta \Pi_2 = 0 \text{ implies}$$

$$- T_{ji,j} + \rho a_i = 0 \quad \text{in } V,$$

$$- D_{i,i} = 0 \quad \text{in } V,$$

$$\frac{1}{2}(a_{i,j} + a_{j,i}) = \frac{\Lambda_2}{\Gamma_2} \frac{\partial M}{\partial T_{ij}} \quad \text{in } V,$$

$$\psi, = \frac{\Lambda_2}{\Gamma_2} \frac{\partial M}{\partial D_i} \quad \text{in } V,$$

$$a_i = 0 \quad \text{on } S_u,$$

$$T_{ji,n_j} = 0 \quad \text{on } S_T,$$

$$- \psi = 0 \quad \text{on } S_\phi,$$

$$D_{i,n_i} = 0 \quad \text{on } S_D,$$

which is an equivalent system of the original eigenvalue problem (2) (when $\omega \neq 0$).

Equations (20)$_{3,4}$ can be obtained by multiplying both sides of (16) by $\omega^2$ and by
substituting (23) and (17). Hence, the stationary condition of the functional $\Pi_2$ gives the eigenvalue problem (20) with the stationary value of $\Pi_2$ as $\omega^2$. If we choose the admissible functions to satisfy

$$
-T_{ji,j} + \rho a_i = 0 \quad \text{in} \ V,
-D_{i,i} = 0 \quad \text{in} \ V,
T_{ji} n_j = 0 \quad \text{on} \ S_T,
D_i n_i = 0 \quad \text{on} \ S_D,
$$

then $\Pi_2$ reduces to

$$
\Pi_2 = \frac{\int_V -\frac{1}{2} \rho a_i a_i \, dV}{\int_V M \, dV},
$$

and the stationary condition of $\Pi_2$ under constraints (21) gives

$$
\frac{1}{2}(a_{i,j} + a_{j,i}) = \frac{\Lambda_2}{\Gamma_2} \frac{\partial M}{\partial T_{ij}} \quad \text{in} \ V,
\psi_{,i} = \frac{\Lambda_2}{\Gamma_2} \frac{\partial M}{\partial D_i} \quad \text{in} \ V,
-a_i = 0 \quad \text{on} \ S_u,
-\psi = 0 \quad \text{on} \ S_{\phi},
$$

with the stationary value of $\Pi_2$ as $\omega^2$.

This $M$ formulation can be considered as a generalization of the complementary energy formulation for the vibration problem in classical elasticity.

5. The enthalpy $G(T_{ij}, E_i)$ formulation. The enthalpy $G$ can be obtained from $H$ as

$$
G = G(T_{ij}, E_i) = H - T_{ij} S_{ij},
$$

which generates the following constitutive relations:

$$
S_{ij} = -\frac{\partial G}{\partial T_{ij}}, \quad d_i = -\frac{\partial G}{\partial E_i}.
$$

We introduce

$$
a_i = -\omega^2 u_i, \quad D_i = -\omega^2 D_i,
$$

and let

$$
\Lambda_3(a_i, T_{ij}, \phi, E_i, \mathcal{D}_i) = \int_V \left[-T_{ji,j} a_i + \frac{1}{2} \rho a_i a_i + \mathcal{D}_i \phi_{,i} + \mathcal{D}_i E_i \right] dV
+ \int_{S_T} T_{ji} n_j a_i dS - \int_{S_{\phi}} \mathcal{D}_i n_i \phi dS,
\Gamma_3(T_{ij}, E_i) = \int_V G(T_{ij}, E_i) dV,
\Pi_3(a_i, T_{ij}, \phi, E_i, \mathcal{D}_i) = \frac{\Lambda_3(a_i, T_{ij}, \phi, E_i, \mathcal{D}_i)}{\Gamma_3(T_{ij}, E_i)}.
$$
We have
\[ \delta \Lambda_3 = \int_V \left( (-T_{ij,j} + \rho a_i) \delta a_i - \mathcal{D}_{i,i} \delta \phi + \frac{1}{2} (a_{i,j} + a_{j,i}) \delta T_{ij} + (E_i + \phi_{,i}) \delta E_i + \mathcal{D}_i \delta E_i \right) dV \]
\[ - \int_{S_u} a_i \delta T_{ij} n_j dS + \int_{S_T} T_{ji} n_j \delta a_i dS - \int_{S_\phi} \phi \delta \mathcal{D}_i n_i dS + \int_{S_D} \mathcal{D}_i n_i \delta \phi dS , \]
\[ \delta \Gamma_3 = \int_V \left( \frac{\partial G}{\partial T_{ij}} \delta T_{ij} + \frac{\partial G}{\partial E_i} \delta E_i \right) dV . \] (28)

Hence \( \delta \Pi_3 = 0 \) implies
\[ - T_{ji,j} + \rho a_i = 0 \text{ in } V , \]
\[ - \mathcal{D}_{i,i} = 0 \text{ in } V , \]
\[ E_i + \phi_{,i} = 0 \text{ in } V , \]
\[ \frac{1}{2} (a_{i,j} + a_{j,i}) = \frac{\Lambda_3}{\Gamma_3} \frac{\partial G}{\partial T_{ij}} \text{ in } V , \]
\[ \mathcal{D}_i = \frac{\Lambda_3}{\Gamma_3} \frac{\partial G}{\partial E_i} \text{ in } V , \]
\[ - a_i = 0 \text{ on } S_u , \]
\[ T_{ji} n_j = 0 \text{ on } S_T , \]
\[ - \phi = 0 \text{ on } S_\phi , \]
\[ \mathcal{D}_i n_i = 0 \text{ on } S_D , \]

which is another equivalent system of the original eigenvalue problem (2). Hence the stationary condition of \( \Pi_3 \) gives the eigenvalue problem (29), with the stationary value of \( \Pi_3 \) as \( \omega^2 \). If we choose our admissible functions to satisfy
\[ - T_{ji,j} + \rho a_i = 0 \text{ in } V , \]
\[ E_i + \phi_{,i} = 0 , \]
\[ T_{ji} n_j = 0 \text{ on } S_T , \]
\[ - \phi = 0 \text{ on } S_\phi , \] (30)
then \( \Pi_3 \) reduces to
\[ \Pi_3 = \frac{\int_V - \frac{1}{2} \rho a_i a_i dV}{\int_V G dV} , \]
and the stationary condition of \( \Pi_3 \) becomes
\[ - \mathcal{D}_{i,i} = 0 \text{ in } V , \]
\[ \frac{1}{2} (a_{i,j} + a_{j,i}) = \frac{\Lambda_3}{\Gamma_3} \frac{\partial G}{\partial T_{ij}} \text{ in } V , \]
\[ \mathcal{D}_i = \frac{\Lambda_3}{\Gamma_3} \frac{\partial G}{\partial E_i} \text{ in } V , \]
\[ - a_i = 0 \text{ on } S_u , \]
\[ \mathcal{D}_i n_i = 0 \text{ on } S_D . \] (32)
This $G$ formulation can be considered as another generalization of the complementary energy formulation for the vibration problem in classical elasticity.

6. The internal energy $U(S_{ij}, D_i)$ formulation. The internal energy $U$ can be obtained from $H$ as

$$U = U(S_{ij}, D_i) = H + E_i D_i,$$

which generates the following constitutive relations:

$$T_{ij} = \frac{\partial U}{\partial S_{ij}}, \quad E_i = \frac{\partial U}{\partial D_i}. \quad (34)$$

Let

$$\Lambda_4(u_i, S_{ij}, T_{ij}, \phi, D_i) = \int_V \{T_{ij} u_{i,j} - D_{i,i} \phi + U(S_{ij}, D_i) - T_{ij} S_{ij}\} dV$$

$$- \int_{S_u} T_{ji} n_j u_i dS + \int_{S_D} D_i n_i \phi dS, \quad (35)$$

$$\Pi_4(u_i, S_{ij}, T_{ij}, \phi, D_i) = \frac{\Lambda_4(u_i, S_{ij}, T_{ij}, \phi, D_i)}{\Gamma_4(u_i)}.$$  

Then

$$\delta \Lambda_4 = \int_V \{ -T_{ji,j} \delta u_{i,j} - D_{i,i} \delta \phi + \left( \frac{\partial U}{\partial S_{ij}} - T_{ij} \right) \delta S_{ij} + \left( \frac{\partial U}{\partial D_i} + \phi_{,i} \right) \delta D_i$$

$$+ \left[ \frac{1}{2} (u_{i,j,j} + u_{j,i,j}) - S_{ij} \right] \delta T_{ij} \} dV$$

$$- \int_{S_u} u_i \delta T_{ji} n_j dS + \int_{S_T} T_{ji} n_j \delta u_i dS - \int_{S_D} \phi \delta D_i n_i dS + \int_{S_D} D_i n_i \delta \phi dS, \quad (36)$$

$$\delta \Gamma_4 = \int_V \rho u_t \delta u_i dV.$$

Hence $\delta \Pi_4 = 0$ implies

$$-T_{ji,j} = \rho \frac{\Lambda_4}{\Gamma_4} u_i \quad \text{in } V,$$

$$-D_{i,i} = 0 \quad \text{in } V,$$

$$-S_{ij} + \frac{1}{2} (u_{i,j,j} + u_{j,i,j}) = 0 \quad \text{in } V,$$

$$-T_{ij} + \frac{\partial U}{\partial S_{ij}} = 0 \quad \text{in } V,$$

$$\phi_{,i} + \frac{\partial U}{\partial D_i} = 0 \quad \text{in } V,$$

$$-u_i = 0 \quad \text{on } S_u,$$

$$T_{ji} n_j = 0 \quad \text{on } S_T,$$

$$-\phi = 0 \quad \text{on } S_{\phi},$$

$$D_i n_i = 0 \quad \text{on } S_D,$$

$$(37)$$
which is also an equivalent form of the original eigenvalue problem (2). The stationary condition of \( \Pi_4 \) gives the eigenvalue problem (37), with the stationary value of \( \Pi_4 \) as \( \omega^2 \). If we choose admissible functions to satisfy

\[
-D_{i,j} = 0 \quad \text{in } V,
\]

\[
-S_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } V,
\]

\[
-u_i = 0 \quad \text{on } S_u,
\]

\[
-D_i n_i = 0 \quad \text{on } S_D,
\]

then \( \Pi_4 \) reduces to

\[
\Pi_4 = \frac{\int_V U dV}{\int_V \frac{1}{2} \rho u_i u_i dV},
\]

and the stationary condition of \( \Pi_4 \) becomes

\[
-T_{i,j} = \rho \frac{\Lambda_4}{\Gamma_4} u_i \quad \text{in } V,
\]

\[
-T_{ij} + \frac{\partial U}{\partial S_{ij}} = 0 \quad \text{in } V,
\]

\[
\phi, i + \frac{\partial U}{\partial D_i} = 0 \quad \text{in } V,
\]

\[
T_{ji} n_j = 0 \quad \text{on } S_T,
\]

\[
-\phi = 0 \quad \text{on } S_\phi.
\]

This \( U \) formulation can be considered as another generalization of the potential energy formulation for the vibration problem in classical elasticity.

Finally, we note that from (33), (3), and (4) the internal energy can be expressed in a positive-definite form in terms of \( S_{ij} \) and \( E_i \) as follows:

\[
U = H + E_i D_i
\]

\[
= \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - \frac{1}{2} \epsilon_{ij} E_i E_j - \epsilon_{ijk} E_i S_{jk} + E_{ij}(\epsilon_{ij} E_j + \epsilon_{ijk} S_{jk})
\]

\[
= \frac{1}{2} c'_{ijkl} S_{ij} S_{kl} + \frac{1}{2} \epsilon_{ij} E_i E_j.
\]

Since the internal energy function \( U \) is positive definite, the constraint \( \Pi_4 \) is bounded from below. Therefore, the lowest resonant frequency must be a minimum. Following some standard arguments in variational analysis [6], we have the following immediate properties.

The lowest resonant frequency will increase if any of the following happens:

(i) \( S_u \) increases;
(ii) \( S_D \) increases;
(iii) \( \rho \) decreases;
(iv) \( c_{ijkl} \) increases to \( c'_{ijkl} \) such that \( (c'_{ijkl} - c_{ijkl})a_{ijkl} > 0 \) for any nonzero symmetric \( a_{ijkl} \);
(v) \( \epsilon_{ij} \) increases to \( \epsilon'_{ij} \) such that \( (\epsilon'_{ij} - \epsilon_{ij})b_i b_j > 0 \) for any nonzero \( b_i \).
Some of the above properties may be considered as generalizations of the corresponding properties in classical elasticity.

REFERENCES


