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ON THE SECOND SOLUTION OF FLOW OF VISCOELASTIC FLUID OVER A STRETCHING SHEET

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Abstract. For the boundary layer flow of a viscoelastic fluid caused by the stretching of a sheet, it is demonstrated that, besides the well-known solution, a second solution exists for all nonzero values of k, the viscoelastic fluid parameter. This solution is obtained in closed form. It exhibits an oscillatory behavior with the oscillations increasing as k approaches the value zero.

1. Introduction. The Navier-Stokes equations characterizing the flow of a viscous fluid admit very few closed-form solutions. For viscoelastic fluids such solutions are even more rare because of the increased complexity of the constitutive equations of the fluids. Not unnaturally, Rajagopal et. al. [1], without suspecting the existence of an exact closed-form solution, employed the perturbation method to study the flow of viscoelastic fluid over a stretching sheet. The boundary value problem (BVP) governing the dimensionless stream function f, for this problem, is given by

$$f''' + ff'' - f'^2 + k(ff^{\mathrm{iv}} - 2f'f''' + f''^2) = 0, \tag{1}$$

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0, \tag{2}$$

where k is the nondimensional measure of the viscoelastic fluid parameter and a prime denotes the derivative with respect to η, the similarity variable.

For $k = 0$, it is well known that the BVP (1) and (2) admits a closed-form solution

$$f = 1 - e^{-\eta}, \tag{3}$$

which was first given by Crane [2].

That the above solution is unique was independently shown by McLeod and Rajagopal [3] and Troy et al [4]. The latter were also able to spot a closed-form solution of the full BVP (1) and (2) for $k \neq 0$, namely

$$f = \sqrt{1-k}\left(1 - e^{-\eta/\sqrt{1-k}}\right). \tag{4}$$

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The present author [5] also obtained this solution numerically by integrating BVP (1) and (2). For the numerical solution it can be shown that if $f''(0)$ is known then it is possible to integrate Eq. (1) notwithstanding the fact that the order of the differential equation is four and there are only three boundary conditions. This is because one can obtain a Taylor series expansion for f in a neighborhood of $\eta = 0$ in terms of $f''(0)$. We shall be returning to this point presently.

It was thought that the solution (4) is also unique like its counterpart (solution (3)) for Newtonian fluids. However, recently Chang [6] demonstrated that a dual solution exists for $k = \frac{1}{2}$. The value $\frac{1}{2}$ for k was determined in the process of finding the solution because of the form of trial solution chosen by Chang. It would be of interest to know if another, perhaps similar, solution also exists for other values of k, and if it does, would it be possible to obtain it in closed form. Also, since the solution (4) is valid only for $k < 1$, one would like to know if this second solution is applicable for $k \geq 1$.

In the present note we investigate these queries by choosing a more general trial function than the one chosen by Chang.

2. The second solution. Chang [6] has shown that a second solution is possible of the form

$$f(\eta) = a(1 - e^{-b\eta} \cos \omega\eta), \tag{5}$$

where a, b, and ω are constants.

By substituting for $f(\eta)$ in Eq. (1) and equating the coefficients of various functions of η he was able to obtain

$$a = \sqrt{2}, \qquad b = \frac{1}{\sqrt{2}}, \qquad \omega = \sqrt{\frac{3}{2}}. \tag{6}$$

However, in order to derive this solution he was forced to take $k = \frac{1}{2}$.

It may be instructive to look at the procedure of numerical solution of BVP (1) and (2) to find if another solution, besides the solution (4), is possible for other values of k. For this we develop a Taylor series expansion for f.

Let

$$f''(0) = s. \tag{7}$$

Setting $\eta = 0$ in Eq. (1) and using the boundary conditions (2), we obtain

$$f'''(0) = \frac{1 - ks^2}{1 - 2k}. \tag{8}$$

For a solution of BVP (1) and (2) to exist at $k = \frac{1}{2}$, clearly

$$1 - \tfrac{1}{2}s^2 = 0 \tag{9}$$

for $k = \frac{1}{2}$, giving $s = \pm 2^{1/2}$. The value with the negative sign, of course, corresponds to the solution (4). Since the value with the positive sign is also a possibility, a second solution may be plausible for this value. Indeed, this is the value corresponding to the solution obtained by Chang [6].

We further differentiate Eq. (1) and set $\eta = 0$ to obtain

$$f^{\text{iv}}(0) = \frac{s}{1-k}. \tag{10}$$

Equation (10) shows that a solution may exist at $k = 1$, provided $s = 0$. This suggests that it may be possible to extend the solution of Chang so that it holds for $k = 1$, in which case $f''(0)$ must vanish.

The BVP (1) and (2) accordingly was integrated numerically starting with Chang's solution for $k = 0.5$ using the technique of parameter differentiation (Ariel [7]), in which k was incremented by 0.01. It was confirmed that Chang's solution could be extended to $k = 1$ where $f''(0)$ became zero. In the process of numerical integration it was discovered that the oscillations persisted even beyond $k = 1$. While the amplitude of the oscillations decreased with increasing k, the corresponding wavelength became larger with an increase in k. We, therefore, decided to attempt an exact analytical solution of the form

$$f(\eta) = a[1 - e^{-b\eta}(\cos\omega\eta + c\sin\omega\eta)], \tag{11}$$

where c is an additional constant. Note that Eq. (11) still satisfies the boundary conditions (2), except $f'(0) = 1$. This is the condition that will eventually determine the value of c.

We can rewrite (11) as

$$f(\eta) = a[1 - \rho e^{-b\eta}\cos(\omega\eta + \alpha)], \tag{12}$$

where

$$\rho\cos\alpha = 1, \qquad \rho\sin\alpha = -c. \tag{13}$$

Successive differentiation of (12) gives

$$f^{(n)}(\eta) = -a\rho r^n e^{-b\eta}\cos(\omega\eta + \alpha + n\beta), \tag{14}$$

where

$$r\cos\beta = -b, \qquad r\sin\beta = \omega. \tag{15}$$

Substituting for f and its derivatives in Eq. (1) and equating the coefficients of $\exp(-b\eta)$ and $\exp(-2b\eta)$, we obtain

$$r^3\cos(\omega\eta + \alpha + 3\beta) + ar^2\cos(\omega\eta + \alpha + 2\beta) + akr^4\cos(\omega\eta + \alpha + 4\beta) = 0, \tag{16}$$

and

$$1 + 2kr^2\cos 2\beta = 0. \tag{17}$$

Further expanding (16) and equating the coefficients of $\cos(\omega\eta+\alpha)$ and $\sin(\omega\eta+\alpha)$, we get

$$r^3\cos 3\beta + ar^2\cos 2\beta + akr^4\cos 4\beta = 0, \tag{18}$$

$$r^3\sin 3\beta + ar^2\sin 2\beta + akr^4\sin 4\beta = 0. \tag{19}$$

Note that we have three equations, (17), (18), and (19), for three unknowns a, r, and β, or equivalently a, b, and ω. The introduction of the additional term

containing the constant c has not affected our equations for determining a, b, and ω.

The unknown a can be readily eliminated from Eqs. (18) and (19) and this at once yields

$$kr^2 = 1 \quad \text{or} \quad r = k^{-1/2}, \tag{20}$$

which when combined with (17) and (15) gives

$$\beta = \frac{2\pi}{3}. \tag{21}$$

If r and β are substituted in Eq. (18), one obtains

$$a = r = k^{-1/2}. \tag{22}$$

Finally, c is determined using the initial condition $f'(0) = 1$. We have

$$c = (1 - 2k)/\sqrt{3}. \tag{23}$$

The second solution thus can be written as

$$f(\eta) = \frac{1}{\sqrt{k}}\left[1 - e^{-\eta/(2\sqrt{k})}\left(\cos\frac{\sqrt{3}\eta}{2\sqrt{k}} + \frac{1-2k}{\sqrt{3}}\sin\frac{\sqrt{3}\eta}{2\sqrt{k}}\right)\right]. \tag{24}$$

For this solution it can easily be verified that

$$f''(0) = \frac{1-k}{k\sqrt{k}}, \tag{25}$$

which vanishes when $k = 1$.

The solution (24), it may be remarked, holds for all values of $k > 0$, including $k \geq 1$ unlike the solution (4). Nevertheless, it is open to doubt if this solution can be realized in practice. One reason is, of course, that $f(\eta)$ becomes unbounded as $k \to 0$, a most unlikely circumstance.

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