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EXPONENTIAL STABILITY OF THE KIRCHHOFF PLATE WITH THERMAL OR VISCOELASTIC DAMPING

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Abstract. The exponential stability of the semigroup associated with the Kirchhoff plate with thermal or viscoelastic damping and various boundary conditions is proved. This improves the corresponding results by Lagnese by showing that the semigroup is still exponentially stable even without feedback control on the boundary. The proof is essentially based on PDE techniques and the method is remarkable in the sense that it also throws light on applications to other higher-dimensional problems.

1. Introduction. The purpose of this paper is to show that the semigroups associated with linear thermoelastic plates and linear viscoelastic plates of the Kirchhoff type with various boundary conditions are exponentially stable, which further leads to the exponential decay of energy of these plates.

Suppose a thin plate of the Kirchhoff type occupies a bounded region $\Omega \in \mathbf{R}^2$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. The plate is rigidly clamped along Γ_0, simply supported along Γ_1, and free along Γ_2. In addition, we assume that

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \varnothing. \tag{1.1}$$

If thermal damping is considered, then the vertical deflection w of the plate and the temperature θ satisfy the following partial differential equations (see [L]):

$$w'' - \gamma\Delta w'' + \Delta^2 w + \alpha\Delta\theta = 0, \tag{1.2}$$

$$\beta\theta' - \eta\Delta\theta + \sigma\theta - \alpha\Delta w' = 0, \tag{1.3}$$

with $\alpha, \beta, \eta, \sigma > 0, \gamma \geq 0$ being constants, and the prime being time derivative. Various boundary conditions could be imposed on θ depending on what is assumed about the

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temperature dynamics at the edge of the plate. We assume that the temperature is subject to the Newton law of cooling. Therefore, we have

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_0, \quad t > 0; \tag{1.4}$$

$$w = \Delta w + (1-\mu)B_1 w + \alpha\theta = 0 \quad \text{on } \Gamma_1, \quad t > 0; \tag{1.5}$$

$$\begin{cases} \Delta w + (1-\mu)B_1 w + \alpha\theta = 0, \\ \dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)\dfrac{\partial B_2 w}{\partial \tau} - \gamma\dfrac{\partial w''}{\partial \nu} + \alpha\dfrac{\partial \theta}{\partial \nu} = 0 \end{cases} \quad \text{on } \Gamma_2, \quad t > 0; \tag{1.6}$$

$$\frac{\partial \theta}{\partial \nu} = -\lambda\theta \quad \text{on } \Gamma, \quad t > 0, \tag{1.7}$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward normal to Γ, $\tau = \{-\nu_2, \nu_1\}$ is the unit tangent vector, λ is a constant ($\lambda = 0$ corresponds to the insulated temperature boundary condition), μ ($\frac{1}{2} > \mu > 0$) is the Poisson ratio and

$$\begin{cases} B_1 w = 2\nu_1\nu_2 \dfrac{\partial^2 w}{\partial x \partial y} - \nu_1^2 \dfrac{\partial^2 w}{\partial y^2} - \nu_2^2 \dfrac{\partial^2 w}{\partial x^2}, \\ B_2 w = (\nu_1^2 - \nu_2^2)\dfrac{\partial^2 w}{\partial x \partial y} + \nu_1\nu_2 \left(\dfrac{\partial^2 w}{\partial y^2} - \dfrac{\partial^2 w}{\partial x^2} \right). \end{cases} \tag{1.8}$$

The initial state of the plate is

$$w(0) = w^0(x,y), \qquad w'(0) = w^1(x,y), \qquad \theta(0) = \theta^0(x,y) \tag{1.9}$$

with w^0, w^1, θ^0 being given functions.

The energy of the thermoelastic plate is defined by

$$E(t) = \frac{1}{2}\{a(w(t)) + \|w'(t)\|^2 + \gamma\|\nabla w'(t)\|^2 + \beta\|\theta\|^2\} \tag{1.10}$$

where

$$a(w(t)) = \int_\Omega \left\{ \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\mu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} d\Omega. \tag{1.11}$$

When viscoelastic damping is considered, the vertical deflection of the plate satisfies

$$w''(t) - \gamma\Delta w''(t) + \Delta^2 w(t) + \Delta^2 \int_0^\infty g'(s) w(t-s)\, ds = 0 \tag{1.12}$$

with $\gamma \geq 0$, $g(0) = 1$ (see [L]). The corresponding boundary conditions are

$$w = \frac{\partial w}{\partial \nu} = 0, \quad \text{on } \Gamma_0, \quad t > 0, \tag{1.13}$$

$$w = \mathcal{B}_1 \left(w(t) + \int_0^\infty g'(s) w(t-s)\, ds \right) = 0, \quad \text{on } \Gamma_1, \quad t > 0, \tag{1.14}$$

and

$$\begin{cases} \mathcal{B}_1\left(w(t)+\displaystyle\int_0^\infty g'(s)w(t-s)\,ds\right)=0 \\ \mathcal{B}_2\left(w(t)+\displaystyle\int_0^\infty g'(s)w(t-s)\,ds\right)-\gamma\dfrac{\partial w''}{\partial\nu}=0 \end{cases} \quad \text{on } \Gamma_2, \quad t>0, \tag{1.15}$$

where

$$\mathcal{B}_1 w=\Delta w+(1-\mu)B_1 w, \qquad \mathcal{B}_2 w=\frac{\partial\Delta w}{\partial\nu}+(1-\mu)\frac{\partial B_2 w}{\partial\tau} \tag{1.16}$$

with the operators B_1, B_2 being the same as above and $\tau=\{-\nu_2,\nu_1\}$.

The initial state of the plate is

$$w(0+)=w^0, \qquad w'(0+)=w^1, \qquad w(-s)=w_h(s) \quad \text{for } 0<s<\infty \tag{1.17}$$

with w^0, w^1, w_h being given functions.

We shall assume that the relaxation function $g(s)$ satisfies the following conditions:
(g1) $g\in C^2(0,\infty)\cap C[0,\infty)$,
(g2) $g(t)>0, g'(t)<0, g''(t)>0$, for $t>0$,
(g3) $g(\infty)=g_\infty>0$,
(g4) $g''(t)+kg'(t)\geq 0$ for some $k>0$.
Condition (g4) implies that $g'(t)$ decays exponentially, and (g3) means that the material behaves like an elastic solid at $t=\infty$.

The total energy corresponding to (1.12) is defined by

$$\begin{aligned} E(t)=&\frac{1}{2}\int_\Omega (g_\infty a(w(t))+[w'(t)]^2+\gamma[\nabla w'(t)]^2)\,d\Omega \\ &-\frac{1}{2}\int_0^\infty\int_\Omega g'(s)a(w(t)-w(t-s))\,d\Omega\,ds. \end{aligned} \tag{1.18}$$

To study the exponential stability of a linear system is of great importance from both the theoretical and practical point of view. First, needless to say, to show the exponential stability of a semigroup has its own merit. Second, it is also important for the study of the global existence of the solution to the corresponding nonlinear system with small initial data (e.g., see [S]). Moreover, as an application, it is known from optimal control theory (see [GRT] and the references cited there) that the exponential stability of a linear system is a sufficient condition for the existence of optimal control for the Linear Quadratic Regulator problem governed by such a system.

In his monograph [L], Lagnese systematically studied thin plate models, including the above thermoelastic and viscoelastic plates, and proved exponential stability when $\Gamma_1=\varnothing$ and an appropriate feedback mechanism is implemented along the free edge Γ_2. Since systems (1.2)–(1.3) and (1.12) already possess dissipative mechanisms, an open question is whether they are still exponentially stable without feedback control.

The first result in this direction was obtained by Kim [K] in 1990. He proved the exponential decay of energy of a linear thermoelastic plate ($\gamma=0$ in (1.2)–(1.3)) for rigidly clamped and constant temperature boundary conditions, i.e., on the whole boundary Γ,

$w = \frac{\partial w}{\partial \nu} = \theta = 0$. More recently, Rivera and Racke [RR] obtained the same result for the case of $w = \Delta w = \theta = 0$ on the whole boundary Γ. It seems that their methods cannot be applied to the case of more general boundary conditions (1.4)–(1.7).

Our main purpose in this paper is to give a positive answer to the above open problem by showing the exponential stability of semigroups associated with the linear thermoelastic plate equation (1.2)–(1.7) and the linear viscoelastic plate equation (1.12)–(1.16) through a unified treatment. These results are presented in Theorem 2.2 and Theorem 3.2, respectively. We point out that the results by Kim and Racke can easily be proved by our method. Our method of proof is still a combination of a theorem about the necessary and sufficient conditions for a semigroup to be exponentially stable (see [H]) and a contradiction argument. This has been successfully applied to the one-dimensional system with thermal and viscoelastic dampings (see [LZ1], [BLZ], [LZ2]). However, we would like to emphasize that we are now dealing with two-dimensional problems. The information about a positive gap of the eigenvalues of the system, which we heavily used in the proof in our previous papers, is not available. This information is also crucial for the proof of the exponential stability of one-dimensional elastic systems with local damping (see [CFNS]). To overcome this mathematical difficulty, we adopt a different approach without using such information. Therefore, it also throws light on the applications to other higher-dimensional problems.

2. Linear thermoelastic plate. In this section, we first prove the exponential stability of the semigroup associated with the linear thermoelastic plate (1.2)–(1.7), then give a brief discussion to generalize our result to other boundary conditions including the ones treated by Kim and Racke. We set $\gamma = 0$ for simplicity. The case of $\gamma \neq 0$ is more delicate and will be treated elsewhere. In what follows, we always assume that $\Gamma_2 \neq \Gamma$.

Let

$$Z = H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Omega) \tag{2.1}$$

be equipped with the energy-related norm which is induced by the inner product

$$\langle z, \tilde{z} \rangle_Z = a(w, \tilde{w}) + \langle v, \tilde{v} \rangle + \beta \langle \theta, \tilde{\theta} \rangle \tag{2.2}$$

for any $z = \{w, v, \theta\}, \tilde{z} = \{\tilde{w}, \tilde{v}, \tilde{\theta}\} \in Z$. Hereafter, we denote by $H^k_{\Gamma_j}$ $(k = 1, 2,\ j = 0, 1)$ the subspace of H^k whose elements up to $k-1$ order derivatives vanish on Γ_j in the trace sense. If we denote w' by v, then the initial boundary problem (1.2)–(1.7) can be reduced to a first-order evolution equation of the form

$$z'(t) = \mathcal{A} z(t), \qquad z(0) = z_0 \tag{2.3}$$

where $z_0 = \{w^0, w^1, \theta^0\}$ and

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\alpha\Delta \\ 0 & \frac{\alpha}{\beta}\Delta & \frac{1}{\beta}(-\sigma I + \eta\Delta) \end{bmatrix}, \tag{2.4}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in Z : \begin{array}{l} w \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega), v \in H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega), \\ \theta \in H^2(\Omega), \quad w, \theta \text{ satisfy (1.4)–(1.7)} \end{array} \right\}. \tag{2.5}$$

We recall the following Green's formula (see [L]) since it will be used extensively in the rest of this paper:

$$\int_\Omega (\Delta^2 w)\tilde{w}\,d\Omega = a(w;\tilde{w}) + \int_\Gamma \left\{ \left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau} \right] \tilde{w} - [\Delta w + (1-\mu)B_1 w]\frac{\partial \tilde{w}}{\partial \nu} \right\} d\Gamma. \tag{2.6}$$

THEOREM 2.1. The operator $\mathcal{A}$ defined by (2.4)–(2.5) is the infinitesimal generator of a C_0-semigroup, $T(t)$, of contractions on $H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

Proof. It is clear that $\mathcal{D}(\mathcal{A})$ is dense in Z. For any $z = \{w, v, \theta\} \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}
\mathrm{Re}\langle \mathcal{A}z, z\rangle_Z &= \mathrm{Re}\{a(v,w) + \langle -\Delta^2 w - \alpha\Delta\theta, v\rangle + \langle \alpha\Delta v - \sigma\theta + \eta\Delta\theta, \theta\rangle\} \\
&= \mathrm{Re}\left\{ a(v,w) - a(w,v) - \int_\Gamma \left\{ \left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau} - \alpha\frac{\partial \theta}{\partial \nu} \right] v \right.\right. \\
&\qquad \left.\left. -[\Delta w + (1-\mu)B_1 w - \alpha\theta]\frac{\partial v}{\partial \nu} \right\} d\Gamma - \alpha\langle \theta, \Delta v\rangle + \langle \alpha\Delta v - \sigma\theta + \eta\Delta\theta, \theta\rangle \right\} \\
&= -\sigma\|\theta\|^2 + \eta\langle \Delta\theta, \theta\rangle \\
&= -\sigma\|\theta\|^2 - \eta\|\nabla\theta\|^2 - \int_\Gamma \theta^2\,d\Gamma \le 0
\end{aligned} \tag{2.7}$$

where we have used Green's formula (2.6) and the boundary conditions (1.4)–(1.7). Hereafter, we denote by $\|\cdot\|$ the usual $L^2(\Omega)$ norm. Thus $\mathcal{A}$ is dissipative. It remains to show that $\mathrm{Range}(I - \mathcal{A}) = Z$, i.e., for any $F = \{f_1, f_2, f_3\} \in Z$, we want to show that the equation

$$(I - \mathcal{A})z = F \tag{2.8}$$

has a unique solution z. Instead of (2.8), we look for $w \in H^4$ and $\theta \in H^2$ satisfying

$$w + \Delta^2 w + \alpha\Delta\theta = f_1 + f_2, \tag{2.9}$$

$$\beta\theta - \alpha\Delta w + \sigma\theta - \eta\Delta\theta = f_3 - \alpha\Delta f_1, \tag{2.10}$$

and the boundary conditions (1.4)–(1.7).

Let $y = \{w, \theta\}$. We associate this problem with the following bilinear form on $H^2_{\Gamma_0} \cap H^1_{\Gamma_1} \times H^1$:

$$\begin{aligned}
b(y,\tilde{y}) = &\int_\Omega w\tilde{w}\,d\Omega + a(w,\tilde{w}) + \alpha\int_\Omega (\theta\Delta\tilde{w} - \tilde{\theta}\Delta w)\,d\Omega \\
&+ (\beta+\sigma)\int_\Omega \theta\tilde{\theta}\,d\Omega + \eta\int_\Omega \nabla\theta\nabla\tilde{\theta}\,d\Omega + \eta\lambda\int_\Gamma \theta\tilde{\theta}\,d\Gamma.
\end{aligned} \tag{2.11}$$

Then by the well-known Lax-Milgram theorem, there is a unique solution $y \in H^2_{\Gamma_0} \cap H^1_{\Gamma_1} \times H^1$ such that

$$b(y,\tilde{y}) = \int_\Omega [(f_1+f_2)\tilde{w} + (f_3 - \alpha\Delta f_1)\tilde{\theta}]\,d\Omega, \quad \forall \tilde{y} \in H^2_{\Gamma_0} \cap H^1_{\Gamma_1} \times H^1. \tag{2.12}$$

This implies that $\theta \in H^1$ is a weak solution of the following elliptic boundary-value problem:

$$(\beta+\sigma)\theta - \eta\Delta\theta = f_3 - \alpha\Delta f_1 + \alpha\Delta w \in L^2(\Omega), \tag{2.13}$$

$$\left(\frac{\partial\theta}{\partial\nu} + \lambda\theta\right)\bigg|_{\Gamma} = 0. \tag{2.14}$$

By the regularity theorem (see [LM]), $\theta \in H^2$.

On the other hand, (2.12) also implies that $w \in H^2_{\Gamma_0} \cap H^1_{\Gamma_1}$ is a weak solution of the following elliptic boundary-value problem:

$$w + \Delta^2 w = f_1 + f_2 - \alpha\Delta\theta \in L^2, \quad \text{in } \Omega, \tag{2.15}$$

$$w = \frac{\partial w}{\partial\nu} = 0, \quad \text{on } \Gamma_0, \tag{2.16}$$

$$w = 0, \quad \Delta w + (1-\mu)B_1 w = -\alpha\theta \in H^{\frac{3}{2}}(\Gamma_1), \quad \text{on } \Gamma_1, \tag{2.17}$$

$$\begin{cases} \Delta w + (1-\mu)B_1 w = -\alpha\theta \in H^{\frac{3}{2}}(\Gamma_2), \\ \dfrac{\partial\Delta w}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 w}{\partial\nu} = -\alpha\dfrac{\partial\theta}{\partial\nu} \in H^{\frac{1}{2}}(\Gamma_2), \end{cases} \quad \text{on } \Gamma_2. \tag{2.18}$$

Thus by the regularity theorem again, we have

$$w \in H^4 \cap H^2_{\Gamma_0} \cap H^1_{\Gamma_1}. \tag{2.19}$$

Let

$$v = w - f_1 \in H^2_{\Gamma_0} \cap H^1_{\Gamma_1}. \tag{2.20}$$

Combining (2.20) with (2.13)–(2.19) yields that $z = \{w, v, \theta\}$ belongs to $\mathcal{D}(\mathcal{A})$ and satisfies (2.8). □

THEOREM 2.2. Suppose $\Gamma_2 = \varnothing$. Then the semigroup $T(t)$ in Theorem 2.1 is exponentially stable, i.e., there exist $M, \delta > 0$ such that

$$\|T(t)\|_{\mathcal{L}(Z,Z)} \le Me^{-\delta t}, \quad t > 0. \tag{2.21}$$

Proof. By a result of Huang [H], the exponential stability of $T(t)$ is equivalent to

$$\sup\{\operatorname{Re}\lambda : \lambda \in \text{ spectrum of } \mathcal{A}\} < 0 \tag{2.22}$$

and

$$\sup\{\|(\lambda I - A)^{-1}\|_{L(Z,Z)} : \operatorname{Re}\lambda \ge 0\} = K < \infty. \tag{2.23}$$

Hence if the conclusion in Theorem 2.2 is false, then one of (2.22) and (2.23) must fail to hold. Assume that (2.23) fails. There must exist a sequence of $\lambda_n \in \mathbf{C}$ and a sequence of $z_n = \{w_n, v_n, \theta_n\} \in \mathcal{D}(\mathcal{A})$ with $\operatorname{Re}\lambda_n \ge 0, \|z_n\|_Z = 1$ such that

$$(\lambda_n I - \mathcal{A})z_n \to 0 \quad \text{in } Z \tag{2.24}$$

or

$$a(\lambda_n w_n - v_n) \to 0, \tag{2.25}$$

$$\lambda_n v_n + \Delta^2 w_n + \alpha \Delta \theta_n \to 0 \quad \text{in } L^2(\Omega), \tag{2.26}$$

$$\lambda_n \beta \theta_n - \alpha \Delta v_n + \sigma \theta_n - \eta \Delta \theta_n \to 0 \quad \text{in } L^2(\Omega). \tag{2.27}$$

By (2.7), we have

$$\operatorname{Re}\langle (\lambda_n I - \mathcal{A}) z_n, z_n \rangle_Z = \operatorname{Re} \lambda_n + \sigma \|\theta_n\|^2 + \eta \|\nabla \theta_n\|^2 + \eta \lambda \int_\Gamma \theta_n^2 \, d\Gamma. \tag{2.28}$$

Since each term on the right side of (2.28) is nonnegative, it follows from (2.24) that

$$\operatorname{Re} \lambda_n \to 0, \qquad \theta_n \to 0 \quad \text{in } H^1(\Omega), \tag{2.29}$$

which further leads to

$$a(w_n) + \|v\|^2 \to 1. \tag{2.30}$$

It is easy to see (e.g., [Re]) that

$$a(w) \geq c \|w\|_{H^2}^2, \quad \forall w \in H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega) \tag{2.31}$$

for some constant $c > 0$. Therefore (2.25) implies

$$\lambda_n w_n - v_n \to 0 \quad \text{in } L^2(\Omega). \tag{2.32}$$

Taking the complex conjugate of the inner product of (2.32) with v_n in $L^2(\Omega)$, then adding it to the inner product of (2.26) with w_n in $L^2(\Omega)$, we obtain

$$2 \operatorname{Re} \lambda_n \cdot \langle v, w \rangle + a(w_n) - \|v_n\|^2 + \alpha \langle \theta_n, \Delta w_n \rangle \to 0. \tag{2.33}$$

It follows from (2.29) that the first term in (2.33) goes to zero. The last inner product in (2.33) also converges to zero because $\|\Delta w_n\| \leq 1$ and $\|\theta_n\| \to 0$. Combining (2.33) with (2.30) yields

$$a(w_n) \to \frac{1}{2}, \qquad \|v_n\|^2 \to \frac{1}{2}. \tag{2.34}$$

In the rest of the proof, we shall show (2.34) is a contradiction. We first claim that $|\lambda_n| \geq \varepsilon > 0$ for all n large enough. Otherwise, there exists a subsequence of λ_n, still denoted by λ_n, which converges to zero. We obtain from (2.25) that

$$a(v_n) \to 0. \tag{2.35}$$

Thus v_n must converge to zero in $L^2(\Omega)$, which contradicts (2.34).

Now we divide (2.27) by λ_n and apply (2.25) and (2.29) to get

$$\alpha \Delta w_n + \frac{\eta}{\lambda_n} \Delta \theta_n \to 0 \quad \text{in } L^2(\Omega). \tag{2.36}$$

It is easy to see from (2.36) that the term $\|\frac{\Delta\theta_n}{\lambda_n}\|$ is bounded uniformly in n. Thus combining this with (2.26) yields the uniform boundedness of $\|\frac{\Delta^2 w_n}{\lambda_n}\|$ in n. Taking the inner product of (2.36) with Δw_n in $L^2(\Omega)$ yields

$$\begin{aligned}
&\alpha\|\Delta w_n\|^2 + \eta\left\langle \frac{\Delta\theta_n}{\lambda_n}, \Delta w_n\right\rangle \\
&\quad = \alpha\|\Delta w_n\|^2 + \eta\int_\Gamma \frac{\Delta w_n}{\lambda_n}\frac{\partial\theta_n}{\partial\nu}d\Gamma - \eta\int_\Gamma \frac{\theta_n}{\lambda_n}\frac{\partial(\Delta w_n)}{\partial\nu}d\Gamma + \eta\left\langle \theta_n, \frac{\Delta^2 w_n}{\lambda_n}\right\rangle \\
&\quad \to 0.
\end{aligned} \tag{2.37}$$

Now it follows from (2.29) that the last inner product of (2.37) must converge to zero. We estimate the two boundary integrals in (2.37) as follows. By the trace theorem, we have

$$\begin{aligned}
\left|\int_\Gamma \frac{\Delta w_n}{\lambda_n}\frac{\partial\theta_n}{\partial\nu}d\Gamma\right| &= \left|\lambda\int_\Gamma \frac{\Delta w_n}{\lambda_n}\theta_n\, d\Gamma\right| \\
&\le \lambda\left\|\frac{\Delta w_n}{\lambda_n}\right\|_{L^2(\Gamma)}\cdot\|\theta_n\|_{L^2(\Gamma)} \\
&\le C\cdot\left\|\frac{\Delta w_n}{\lambda_n}\right\|_{H^1(\Omega)}\cdot\|\theta_n\|_{H^1(\Omega)}.
\end{aligned} \tag{2.38}$$

Hereafter, C is a positive constant which may vary in different places. Notice that w_n satisfies the boundary conditions (1.4)–(1.6). Thus the standard estimates for the elliptic boundary-value problem and the trace theorem (see [LM]) lead to

$$\begin{aligned}
\|w_n\|_{H^4(\Omega)} &\le C(\|\Delta^2 w_n\| + \|\alpha\theta_n\|_{H^{\frac{3}{2}}(\Gamma)}) \\
&\le C(\|\Delta^2 w_n\| + \|\theta_n\|_{H^2(\Omega)}) \\
&\le C(\|\Delta^2 w_n\| + \|\Delta\theta_n\| + \|\theta_n\|_{H^1(\Omega)}),
\end{aligned} \tag{2.39}$$

$$\left\|\frac{w_n}{\lambda_n}\right\|_{H^4(\Omega)} \le C\left(\left\|\frac{\Delta^2 w_n}{\lambda_n}\right\| + \left\|\frac{\Delta\theta_n}{\lambda_n}\right\| + \left\|\frac{\theta_n}{\lambda_n}\right\|_{H^1(\Omega)}\right). \tag{2.40}$$

Thus it follows from (2.38), (2.40), and (2.29) that

$$\left|\int_\Gamma \frac{\Delta w_n}{\lambda_n}\frac{\partial\theta_n}{\partial\nu}d\Gamma\right| \to 0. \tag{2.41}$$

Similarly, we have

$$\left|\int_\Gamma \frac{\theta_n}{\lambda_n}\frac{\partial(\Delta w_n)}{\partial\nu}d\Gamma\right| \to 0. \tag{2.42}$$

It turns out from (2.37) that

$$\Delta w_n \to 0 \quad \text{in } L^2(\Omega). \tag{2.43}$$

Since $\Gamma_2 = \varnothing$, $w_n = 0$ on the whole boundary Γ. By the estimates of the elliptic boundary-value problem, we have

$$a(w_n) \le C\cdot\|w_n\|^2_{H^2(\Omega)} \le C\cdot\|\Delta w_n\|^2 \to 0. \tag{2.44}$$

This contradicts (2.34).

Now we prove that (2.22) also holds. Because $T(t)$ is a C_0 semigroup of contractions, $\rho(\mathcal{A})$ contains the set $\{\lambda \mid \operatorname{Re}\lambda > 0\}$. For any $\sigma \in [0, \frac{1}{2K}]$ and $\omega \in \mathbf{R}$, we have

$$(-\sigma + i\omega - \mathcal{A})^{-1} = \left(\frac{1}{4K} + i\omega - \mathcal{A}\right)^{-1} \left[1 - \left(\sigma + \frac{1}{4K}\right)\left(\frac{1}{4K} + i\omega - \mathcal{A}\right)^{-1}\right]^{-1}.$$

Thus,

$$\sup\{\operatorname{Re}\lambda; \lambda \in \sigma(\mathcal{A})\} < -\frac{1}{4K}.$$

Thus the proof of Theorem 2.2 is completed. □

REMARK 2.1. It is easy to see that if $w = \Delta w = \theta = 0$ on the whole boundary Γ, as considered by Rivera and Racke [RR], the conclusion of Theorem 2.2 remains true without any modifications of the proof.

If the boundary conditions are $w = \frac{\partial w}{\partial \nu} = \theta = 0$ on the whole boundary Γ, as considered by Kim [K], then we modify our proof as follows.

The proof until (2.36) still works. From (2.36), we have

$$\left\|\alpha w_n + \frac{\theta_n}{\lambda_n}\right\|_{H^2(\Omega)} \le C \left\|\Delta\left(\alpha w_n + \frac{\theta_n}{\lambda_n}\right)\right\| \to 0. \tag{2.45}$$

Combining it with (2.29) yields

$$w_n \to 0 \quad \text{in } H^1(\Omega). \tag{2.46}$$

We divide (2.26) by λ_n and then take the inner product with $-\Delta w_n$ in $L^2(\Omega)$ to obtain

$$\langle v_n, -\Delta w_n\rangle - \frac{1}{\lambda_n}\langle \Delta^2 w_n, \Delta w_n\rangle + \frac{\alpha^2}{\eta}\|\Delta w_n\|^2 \to 0. \tag{2.47}$$

Here we have also used (2.36).

Taking (2.32) into consideration and noticing that

$$-\langle \Delta^2 w_n, \Delta w_n\rangle = -\int_\Gamma \frac{\partial \Delta w_n}{\partial \nu} \cdot \overline{\Delta w_n}\, d\Gamma + \|\nabla(\Delta w_n)\|^2 \tag{2.48}$$

is a real number, we obtain

$$\operatorname{Re}\lambda_n \cdot \|\nabla w_n\|^2 + \frac{\operatorname{Re}\lambda_n}{|\lambda_n|^2}\left(\|\nabla(\Delta w_n)\|^2 - \int_\Gamma \frac{\partial \Delta w_n}{\partial \nu}\overline{\Delta w_n}\, d\Gamma\right) + \frac{\alpha^2}{\eta}\|\Delta w_n\|^2 \to 0. \tag{2.49}$$

Owing to (2.29) and (2.46), the first term on the left-hand side of (2.49) converges to zero. Since $\|\frac{\Delta^2 w_n}{\lambda_n}\|$ is bounded, by the estimates of elliptic boundary-value problems and the trace theorem,

$$\frac{1}{|\lambda_n|^2}\left(\|\nabla(\Delta w_n)\|^2 - \int_\Gamma \frac{\partial \Delta w_n}{\partial \nu}\overline{\Delta w_n}\, d\Gamma\right)$$

is also bounded. Therefore, the second term in (2.49) also converges to zero. It turns out that

$$a(w_n) \le C \cdot \|w_n\|^2_{H^2(\Omega)} \le C \cdot \|\Delta w_n\|^2 \to 0, \tag{2.50}$$

a contradiction. Thus the result by Kim [K] can also be proved by our method.

REMARK 2.2. It is still an open question whether the exponential stability remains true if $\Gamma_2 \neq \varnothing$.

3. Linear viscoelastic plate. In this section, we turn to the exponential stability of the semigroup associated with the linear viscoelastic plate (1.12)–(1.15). We choose the function spaces

$$Z = H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(0,\infty; |g'(\cdot)|; H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega)) \tag{3.1}$$

equipped with the norm

$$\|\{w,v,h\}\|_Z = \left(g_\infty a(w) + \|v\|^2 + \int_0^\infty |g'(s)| a(h)\, ds \right)^{1/2}. \tag{3.2}$$

If we introduce the variables

$$v = w', \qquad h = w(t) - w(t-s), \tag{3.3}$$

then the space Z is a "finite energy space" of the type first introduced for viscoelastic problems by Dafermos ([D1], [D2]), where the norm square of the state variables is exactly twice that of the system energy defined in (1.18).

System (1.12)–(1.15) corresponds to the abstract evolution equation

$$z' = \mathcal{A}z, \qquad z(0) = z_0. \tag{3.4}$$

Here, $z = \{w,v,h\}, z_0 = \{w^0, w^1, w_h\}$ and

$$\mathcal{A}z = \begin{pmatrix} v \\ -g_\infty \Delta^2 w + \int_0^\infty g'(s)\Delta^2 h\, ds \\ v - \frac{\partial}{\partial s} h \end{pmatrix}, \tag{3.5}$$

with

$$\mathcal{D}(\mathcal{A}) = \{z \in Z \mid \mathcal{A}z \in Z, h|_{s=0} = 0, w \text{ satisfies } (1.13)\text{–}(1.15)\}. \tag{3.6}$$

THEOREM 3.1. The operator $\mathcal{A}$ defined by (3.5)–(3.6) is an infinitesimal generator of a C_0-semigroup $T(t)$ of contractions on $H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(0,\infty; |g'(\cdot)|; H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega))$.

Proof. Actually, the fact that $T(t)$ is a C_0-semigroup has already been proved in [FI] in an abstract setting. Moreover, as in [LZ2], for any $z = \{w,v,h\} \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}
\operatorname{Re}\langle \mathcal{A}z, z\rangle_Z &= \operatorname{Re}\left\{ g_\infty a(v,w) - \left\langle g_\infty \Delta^2 w + \int_0^\infty g'(s)\Delta^2 h(s), v \right\rangle \right. \\
&\qquad \left. - \int_0^\infty g'(s) a\left(v - \frac{\partial h}{\partial s}, h \right) ds \right\} \\
&= \operatorname{Re}\left\{ -\int_\Gamma \left[\mathcal{B}_1 \left(w(t) + \int_0^\infty g'(s) w(t-s)\, ds \right) \right] \frac{\partial v}{\partial \nu}\, d\Gamma \right. \\
&\qquad \left. + \int_\Gamma \left[\mathcal{B}_2 \left(w(t) + \int_0^\infty g'(s) w(t-s)\, ds \right) \right] v\, d\Gamma + \int_0^\infty g'(s) a\left(\frac{\partial h}{\partial s}, h \right) ds \right\} \\
&= -\frac{1}{2} \int_0^\infty g''(s) a(h)\, ds \le 0. \quad \square
\end{aligned} \tag{3.7}$$

We also refer to [L] for a slightly different framework. Lagnese [L] obtained exponential stability for a slightly restrictive kernel g by imposing an appropriate feedback boundary control on Γ_2 and assuming $\Gamma_1 = \varnothing$. He also conjectured that various asymptotic decay rates for the solution of (1.12) may be obtained without recourse to boundary feedback, i.e., based only on the properties of the kernel $g(s)$. Our following theorem gives a positive answer to his conjecture.

THEOREM 3.2. Suppose $\Gamma_2 \neq \Gamma$. Then the C_0-semigroup $T(t)$ in Theorem 3.1 is exponentially stable, i.e., there exist constant $M; \delta > 0$ such that

$$\|T(t)\|_{\mathcal{L}(Z,Z)} \leq Me^{-\delta t}, \quad t > 0. \tag{3.8}$$

Proof. Suppose (2.23) is not true. Then, there exists a sequence $\lambda_n \in \mathbf{C}$ and a sequence of unit vectors $z_n = \{w_n, v_n, h_n\} \in \mathcal{D}(\mathcal{A})$ with $\operatorname{Re}\lambda_n \geq 0$ and $\|z_n\|_Z = 1$ such that

$$(\lambda_n I - \mathcal{A})z_n \to 0 \quad \text{in } Z \text{ as } n \to \infty. \tag{3.9}$$

This is equivalent to

$$a(\lambda_n w_n - v_n) \to 0, \tag{3.10}$$

$$\lambda_n v_n + g_\infty \Delta^2 w_n + \int_0^\infty g'(s)\Delta^2 h_n \, ds \to 0 \quad \text{in } L^2(\Omega), \tag{3.11}$$

$$\lambda_n h_n - v_n + \frac{\partial}{\partial s} h_n \to 0 \quad \text{in } L^2(0,\infty; |g'(s)|; H^2_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega)). \tag{3.12}$$

On the other hand, by (g4) and (3.9), we have

$$\begin{aligned} -\frac{k}{2}\int_0^\infty g'(s)a(h_n)\,ds &\leq \operatorname{Re}\lambda_n + \frac{1}{2}\int_0^\infty g''(s)a(h_n)\,ds \\ &= \operatorname{Re}\langle(\lambda_n I - \mathcal{A})z_n, z_n\rangle_Z \\ &\leq \|(\lambda_n I - \mathcal{A})z_n\|_Z \to 0. \end{aligned} \tag{3.13}$$

Therefore,

$$\int_0^\infty g'(s)a(h_n)\,ds \to 0, \tag{3.14}$$

$$\operatorname{Re}\lambda_n \to 0, \qquad \int_0^\infty g''(s)a(h_n)\,ds \to 0, \tag{3.15}$$

which implies that

$$g_\infty a(w_n) + \|v_n\|^2 \to 1. \tag{3.16}$$

Since $\Gamma_2 \neq \Gamma$, it follows from (3.10) that

$$\lambda_n w_n - v_n \to 0 \quad \text{in } H^2(\Omega). \tag{3.17}$$

Taking the inner product of (3.17) with v_n and (3.11) with w_n in $L^2(\Omega)$, and applying Green's formula and the boundary conditions, we obtain

$$\lambda_n \langle w_n, v_n \rangle - \|v_n\|^2 \to 0, \tag{3.18}$$

$$\lambda_n \langle v_n, w_n \rangle + g_\infty a(w_n) + \int_0^\infty g'(s) a(h_n, w_n)\, ds \to 0. \tag{3.19}$$

The last term in (3.19) converges to zero since

$$\begin{aligned} &\left| \int_0^\infty g'(s) a(h_n, w_n)\, ds \right| \\ &\quad \le [a(w_n)]^{1/2} \int_0^\infty -g'(s)[a(h_n)]^{1/2}\, ds \\ &\quad \le (1 - g_\infty) \int_0^\infty -g'(s) a(h_n)\, ds \to 0. \end{aligned} \tag{3.20}$$

By adding the complex conjugate of (3.18) to (3.19), we obtain

$$\operatorname{Re} \lambda_n \cdot \langle v_n, w_n \rangle + g_\infty a(w_n) - \|v_n\|^2 \to 0. \tag{3.21}$$

Now (3.16) and (3.21) imply that

$$g_\infty a(w_n) \to \frac{1}{2}, \qquad \|v_n\|^2 \to \frac{1}{2}. \tag{3.22}$$

In what follows, we shall show that (3.22) is a contradiction. We first claim that $|\lambda_n| \ge \varepsilon > 0$ for n large enough. Otherwise, there exists a subsequence, denoted by λ_n again for simplicity, such that $\lambda_n \to 0$. By (3.17), v_n must converge to zero in $L^2(\Omega)$. Contradiction.

Next, we divide (3.12) by λ_n, then take the inner product with $\frac{s v_n}{\lambda_n}$ in $L^2(0, \infty; |g'(s)|; H^2_{\Gamma_0} \cap H^1_{\Gamma_1}(\Omega))$ to obtain

$$\int_0^\infty s g'(s) a\left(h_n, \frac{v_n}{\lambda_n}\right) ds - a\left(\frac{v_n}{\lambda_n}\right) \int_0^\infty s g'(s)\, ds + \frac{1}{\lambda_n} \int_0^\infty s g'(s) a\left(\frac{\partial h_n}{\partial s}, \frac{v_n}{\lambda_n}\right) ds \to 0. \tag{3.23}$$

The first and third term in (3.23) can be estimated as follows:

$$\begin{aligned} \left| \int_0^\infty s g'(s) a\left(h_n, \frac{v_n}{\lambda_n}\right) ds \right| &\le \left[a\left(\frac{v_n}{\lambda_n}\right) \right]^{\frac{1}{2}} \int_0^\infty -s g'(s) [a(h_n)]^{\frac{1}{2}}\, ds \\ &\le \left[a\left(\frac{v_n}{\lambda_n}\right) \right]^{\frac{1}{2}} \left(\int_0^\infty -s^2 g'(s)\, ds \right)^{\frac{1}{2}} \left(\int_0^\infty -g'(s) a(h_n)\, ds \right)^{\frac{1}{2}} \\ &\le C \cdot \left(\int_0^\infty -g'(s) a(h_n)\, ds \right)^{1/2} \to 0 \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\left|\frac{1}{\lambda_n}\int_0^\infty -sg'(s)a\left(\frac{\partial h_n}{\partial s},\frac{v_n}{\lambda_n}\right)ds\right| &\le \frac{1}{\varepsilon}\left|\int_0^\infty (sg''(s)+g'(s))a\left(h_n,\frac{v_n}{\lambda_n}\right)ds\right| \\ &\le \frac{1}{\varepsilon}\left[a\left(\frac{v_n}{\lambda_n}\right)\right]^{\frac{1}{2}}\left[\left(\int_0^\infty s^2g''(s)\,ds\right)^{1/2}\left(\int_0^\infty g''(s)a(h_n)\,ds\right)^{1/2}\right. \\ &\qquad \left.+\left(\int_0^\infty -g'(s)\,ds\right)^{1/2}\left(\int_0^\infty -g'(s)a(h_n)\,ds\right)^{1/2}\right] \\ &\le C\cdot\left[\left(\int_0^\infty g''(s)a(h_n)\,ds\right)^{1/2}+\left(\int_0^\infty -g'(s)a(h_n)\,ds\right)^{1/2}\right]\to 0.\end{aligned} \tag{3.25}$$

Here we have used uniform boundedness of $a(\frac{v_n}{\lambda_n})$ in n by (3.17) and the condition on $g(s)$ as well as the convergence results in (3.14)–(3.15). Thus the second term in (3.23) must also converge to zero, which is equivalent to

$$a\left(\frac{v_n}{\lambda_n}\right)\to 0. \tag{3.26}$$

Finally, by (3.17) again, we have

$$a(w_n)\to 0. \tag{3.27}$$

This is a contradiction. Condition (2.22) can be verified by the same argument as in the proof of Theorem 2.2. □

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