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OSCILLATION WAVES IN RIEMANN PROBLEMS FOR PHASE TRANSITIONS

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1. Introduction. In this work we are interested in one-dimensional motion of a body capable of phase transitions. Such a problem is often described by the usual equations of elasticity with non-monotone stress-strain relations [1]. This system of conservation laws is elliptic in the interval of values of the strain where the stress function is monotone decreasing, and is hyperbolic otherwise. This interval is usually referred to as the unstable region. For such systems of mixed type, it has been shown in [2] that a constant state in the elliptic region is unstable in the sense that any local disturbance of a constant state, taken as initial data for the numerical solution, gives rise to a sequence of approximate solutions converging to a *nontrivial solution* of the Cauchy problem with the constant initial data. This nontrivial solution may be a *distributional solution* (also called weak solution), or, more generally, a *measure-valued solution* (abbreviated m-v solution), a concept first introduced by DiPerna [3] (see Definition 2.1). It was also shown that a new type of Riemann waves may appear as (measure-valued) solutions of Riemann problems with data inside the elliptic region. These waves are called *oscillation waves*, and are associated with persistent oscillations in the numerical solutions of the approximation scheme.

To approach the solution of such a 2×2 system of conservation laws, we employ a scheme in which each numerical solution approximates the solution of a 3×3 system describing the one-dimensional motion of a rate-type viscoelastic body (see [4]–[8]) with a Maxwell type viscosity, which is inversely proportional to the grid spacing of the numerical scheme and hence reduces to the original 2×2 system as the grid spacing tends to zero. This scheme, based on the method of characteristics proposed in [7], has the mathematical advantage of allowing an energy estimate, which is crucial in the proof of the existence of measure-valued solutions.

We make use of an analytical result, earlier stated in [2] (see Theorem 4.1), to test convergence of approximate solutions and to obtain the limiting function. Indeed, for Riemann problems, a sequence of approximate solutions with diminishing grid spacing can be constructed from a fixed numerical solution simply by scaling in x and t variables.

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This process of scaling allows us to go to the limit of increasing viscosity by observing the asymptotic behavior of a fixed numerical scheme as the time step increases.

By another result, also stated in [2] (see Theorem 4.2), we can get expectation values of state functions of interest, with respect to the Young measures constituting the m-v solution of the Riemann problem. This allows us not only to verify, in our numerical experiments, the condition for being an m-v solution by calculating the expectation values of the state variables and the flux functions, but also to obtain more information about the limiting m-v solutions. For instance, we are able to apply DAFERMO'S entropy rate criterion (see [9]), here extended to m-v solutions, in an obvious way for expectation values, to confirm that the trivial constant solutions in the unstable restriction are less stable than the corresponding nontrivial m-v solutions constructed from the numerical scheme. However, our intention in presenting results concerning the entropy rate criterion is more to illustrate our methods than to make a serious discussion about the validity of the application of such criteria in the context considered here. A discussion about the admissibility of solutions to the system studied here may be found, although without conclusive results, in [10]–[20]. For the analogous static problem we refer to [21, 22].

Careful examination of numerical results reveals an interesting fact that the wave pattern, associated with the nontrivial m-v solution in the unstable region, appears as intense oscillations between two pure phases and tends to become stationary in the region behind the wave fronts at large time steps. These patterns can be interpreted as a phase mixture consisting of layers of two pure phases densely distributed along the length of the body, and resemble the experimentally observed band-like structure of a single crystal sample of shape memory alloy during austenitic-martensitic transformation [23, 24]. Moreover, the average phase fractions in this region are constant in the limit, which means that phase mixture is macroscopically homogeneous, and their values agree with the coefficients in the convex combination of the corresponding strains of the two pure phases for the prescribed initial strain in the unstable interval, as one would expect.

This work is organized in the following way. In Sec. 2, we present the system modeling phase transitions, the companion system of rate-type viscoelasticity, the definition of m-v solutions and recall an extension of TARTAR'S result on the existence of Young measures [25]. In Sec. 3, we present the construction of the approximate solutions to the Cauchy problem by the method of characteristics applied to the companion system of rate-type viscoelasticity. The main result of this section is the proof that the approximate solutions generate an m-v solution to the Cauchy problem. In Sec. 4, we establish two basic results concerning the Riemann problem, which are analogous to those stated in [2]. They provide a criterion for testing the convergence in $L^2_{\rm loc}$ for sequences of approximate solutions, and a method for computing expectation values of state functions of interest, with respect to the m-v solution associated with the oscillation waves. In Sec. 5, we present the results of the numerical experiments, with emphasis in nontrivial m-v solutions for initial data in the elliptic region and the appearance of oscillation waves. A comparison between the nontrivial m-v solution and the trivial constant solution is shown to confirm the instability of the constant state in the elliptic region by employing the entropy rate criterion. Average phase fractions are also determined in this region, where a mixture of two pure phases with numerous well-defined interface boundaries can be seen from the numerical solution. The results of the last two sections follow the same line of investigation set forth in [2].

2. Statement of the problem. The system of partial differential equations for one-dimensional motion of an elastic body is given, in the reference spatial coordinate x and time t, by

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial \sigma_e(\varepsilon)}{\partial x} &= 0, \\ \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} &= 0, \end{aligned} \tag{2.1}$$

where v is the particle velocity, and ε is the strain. The body is called elastic if the stress-strain relation $\sigma = \sigma_e(\varepsilon)$ is single-valued, or the stress σ_e is a monotone increasing function of the strain ε and hence the system is hyperbolic. However, for material bodies that exhibit phase changes during loading cycles, this is no longer true. The common practice (see [1]) to describe the one-dimensional motion of bodies with phase transitions is still to treat it as an elastic problem by the system (2.1) but with a non-monotone stress-strain relation $\sigma = \sigma_e(\varepsilon)$, which is increasing in the pure phases and decreasing in the interval where phase transitions may occur. In this case, the system (2.1) is elliptic if ε belongs to the interval where σ_e is monotone decreasing, and hyperbolic otherwise.

For the purpose of this work we assume that σ_e is a piecewise smooth function satisfying

$$\begin{gathered} \sigma_e'(\varepsilon) > 0, \quad \forall \varepsilon \notin [\varepsilon_A, \varepsilon_M], \\ -M \le \sigma_e'(\varepsilon) \le E_1 < E, \\ \sigma_e(0) = 0, \qquad \int_0^\varepsilon \sigma_e(s)ds \ge 0, \end{gathered} \tag{2.2}$$

for certain positive constants, $\varepsilon_A, \varepsilon_M, E, E_1, M$. For simplicity, in our numerical experiments, we will consider the simple piecewise linear function (see Fig. 1 on p. 118):

$$\sigma_e(\varepsilon) = \begin{cases} E_1\varepsilon, & \varepsilon \le \varepsilon_A, \\ E_1\varepsilon_A - M(\varepsilon - \varepsilon_A), & \varepsilon_A < \varepsilon < \varepsilon_M, \\ E_1\varepsilon_A - M(\varepsilon_M - \varepsilon_A) + E_2(\varepsilon - \varepsilon_M), & \varepsilon \ge \varepsilon_M, \end{cases} \tag{2.3}$$

where $0 < \varepsilon_A < \varepsilon_M$ and $0 < E_2 \le E_1$. In general, for a function σ_e that is monotone increasing except in an interval $(\varepsilon_A, \varepsilon_M)$, where it is monotone decreasing, as is the case of (2.3), the last condition in (2.2) will be satisfied if $\varepsilon = 0$ does not belong to Maxwell's interval $(\varepsilon_\alpha, \varepsilon_\beta)$. This interval is defined by $\sigma_e(\varepsilon_\alpha) = \sigma_e(\varepsilon_\beta) = \sigma_\mu \in (\sigma_e(\varepsilon_M), \sigma_e(\varepsilon_A))$, so that the two curves $\sigma = \sigma_\mu$ and $\sigma = \sigma_e(\varepsilon)$ determine two bounded regions of equal areas in the (ε, σ)-plane.

We consider the initial-value problem for (2.1) with initial data,

$$(v, \varepsilon)(x, 0) = (v_0, \varepsilon_0)(x), \quad x \in \mathbf{R}. \tag{2.4}$$

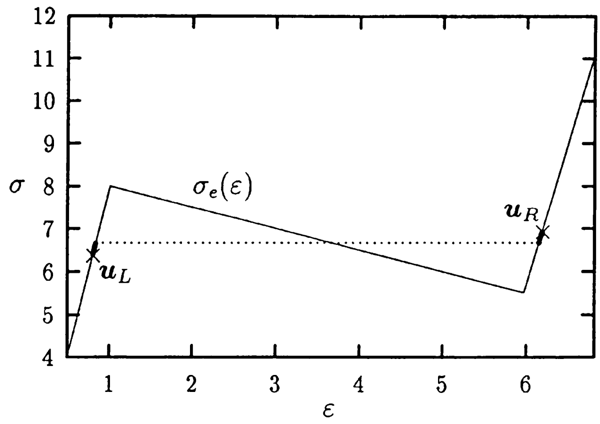

FIG. 1. A simple Riemann problem with initial data $\mathbf{u}_L = (0, 0.5)$, $\mathbf{u}_R = (0, 6.5)$

Our approach to finding the solution of the elastic problem (2.1), (2.4) will proceed by means of a numerical scheme which is primarily intended to approximate the solution of the following system which describes the one-dimensional motion of a rate-type viscoelastic body following SULICIU (see [5]–[7]):

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} &= 0, \\ \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial \sigma}{\partial t} - E\frac{\partial v}{\partial x} &= -\kappa(\sigma - \sigma_e(\varepsilon)), \end{aligned} \tag{2.5}$$

where $\kappa > 0$ is a Maxwell type viscosity constant, or sometimes $1/\kappa$ is referred to as the relaxation time parameter. The constant E is called the dynamic Young's modulus. For (2.5) we give the initial data

$$(v, \varepsilon, \sigma)(x, 0) = (v_0, \varepsilon_0, \sigma_0)(x), \quad x \in \mathbf{R}. \tag{2.6}$$

Intuitively, if one can obtain a sequence of solutions, $(v^\kappa, \varepsilon^\kappa, \sigma^\kappa)$, for the viscoelastic problem (2.5), (2.6), as parameters $\kappa \to \infty$, one would expect that the sequence $(v^\kappa, \varepsilon^\kappa)$ might converge to a solution of the elastic problem (2.1), (2.4). Following this intuitive idea, we shall consider a sequence of approximate solutions constructed by the method of characteristics with grid spacings inversely proportional to the parameter κ. In Sec. 3, we prove that, when $\kappa \to \infty$, a subsequence of such approximate solutions converges weakly and generates what is called, after DIPERNA [3], a measure-valued solution of the elastic problem (2.1), (2.4). We state below the definition of measure-valued solutions.

2.1. DEFINITION. Let $\mathbf{P}(\mathbf{R}^2)$ denote the set of all probability measures over $\mathbf{R}^2$. A measure-valued solution to (2.1), (2.4) is a mapping $\nu\colon \mathbf{R} \times [0, \infty) \to \mathbf{P}(\mathbf{R}^2)$, denoted by $\nu_{x,t}$, satisfying:

1. for each $0 \le p < 2$,

$$M_p(x, t) = \langle \nu_{x,t}, |\mathbf{u}|^p \rangle \tag{2.7}$$

is a measurable function in $L^2_{\mathrm{loc}}(\mathbf{R} \times [0,\infty))$, with $\mathbf{u} = (v, \varepsilon)$;

2. for all $\phi \in C_0^\infty(\mathbf{R} \times [0,\infty))$ we have

$$\iint_{\mathbf{R}\times[0,\infty)} \{\langle \nu_{x,t}, \mathbf{u}\rangle \phi_t + \langle \nu_{x,t}, \mathbf{F}(\mathbf{u})\rangle \phi_x\} \, dx\, dt + \int_{-\infty}^{\infty} \mathbf{u}_0(x)\phi(x,0)dx = 0, \tag{2.8}$$

where $\mathbf{F}(\mathbf{u}) = (-\sigma_e(\varepsilon), -v), -v$, $\mathbf{u}_0(x) = (v_0, \varepsilon_0)(x)$.

The concept of an m-v solution is motivated by TARTAR's result on the existence of Young measures (see [25]). We state below an extension of this theorem for sequences uniformly bounded in L^p_{loc}, whose proof can be found in [26] (see also [27, 28]).

2.2. LEMMA. Let $\mathbf{u}^\varepsilon : \Omega \subseteq \mathbf{R}^m \to \mathbf{R}^n$ be a sequence of vector functions indexed by $\varepsilon > 0$, a parameter which is to be made to vanish. Assume that, for some $p > 1$, and for each compact $K \subseteq \Omega$, $\|\mathbf{u}^\varepsilon\|_{L^p(K)} < M_K$, with $M_K > 0$ a constant not depending on ε. Then, there exists a subsequence which we still label $\mathbf{u}^\varepsilon$, and a parametrized family of probability measure $\nu_y \in \mathbf{P}(\mathbf{R}^n)$, $y \in \Omega$, such that for any continuous function $h \in C(\mathbf{R}^n)$ satisfying

$$\left\| \frac{h(\mathbf{u})}{1 + |\mathbf{u}|^q} \right\|_\infty < \infty, \tag{2.9}$$

for some $q \in [0, p)$, we have

$$\langle \nu_y, h(\mathbf{u})\rangle = \overline{h}(y), \tag{2.10}$$

where

$$h(\mathbf{u}^\varepsilon) \rightharpoonup \overline{h},$$

in the sense of distributions.

The way to construct an m-v solution in the following section is especially important for the subsequence numerical studies in Secs. 4 and 5, of the Riemann problem for (2.1) with initial data:

$$(v,\varepsilon)(x,0) = \begin{cases} (v_L, \varepsilon_L), & x < 0, \\ (v_R, \varepsilon_R), & x > 0. \end{cases} \tag{2.11}$$

3. Method of characteristics and a measure-valued solution. For the viscoelastic problem (2.5), (2.6), the change of dependent variables:

$$p = \sigma + cv, \qquad q = \sigma - cv, \qquad r = \sigma - E\varepsilon, \tag{3.1}$$

with $c = \sqrt{E}$, will put (2.5) into the more convenient characteristic form:

$$\begin{aligned} \frac{\partial p}{\partial t} - c\frac{\partial p}{\partial x} &= G(p,q,r), \\ \frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} &= G(p,q,r), \\ \frac{\partial r}{\partial t} &= G(p,q,r), \end{aligned} \tag{3.2}$$

where

$$G(p,q,r) = -\kappa \left(\frac{p+q}{2} - \sigma_e \left(\frac{1}{E} \left(\frac{p+q}{2} - r \right) \right) \right).$$

For the inverse transformation we have

$$v = \frac{p-q}{2c}, \qquad \varepsilon = \frac{1}{E}\left(\frac{p+q}{2} - r\right), \qquad \sigma = \frac{p+q}{2}. \tag{3.3}$$

The data (2.6) is transformed through (3.1) into

$$(p, q, r)(x, 0) = (p_0, q_0, r_0)(x). \tag{3.4}$$

Let Δx, Δt be grid spacings satisfying $\Delta x = c\Delta t$. We use the first-order approximation given by the method of characteristics to define the lattice functions p_i^j, q_i^j, r_i^j, with $j = 0, 1, \dots, i = \dots, -1, 0, 1, \dots$, as follows:

$$\begin{aligned} p_i^0 &= p_0(i\Delta x), \qquad & p_i^{j+1} &= p_{i+1}^j + \Delta t G_{i+1}^j, \\ q_i^0 &= q_0(i\Delta x), \qquad & q_i^{j+1} &= q_{i-1}^j + \Delta t G_{i-1}^j, \\ r_i^0 &= r_0(i\Delta x), \qquad & r_i^{j+1} &= r_i^j + \Delta t G_i^j, \end{aligned} \tag{3.5}$$

where $G_i^j = G(p_i^j, q_i^j, r_i^j)$.

Given lattice functions p_i^j, q_i^j, r_i^j defined above, we define the approximate solution to (3.2), (3.4), depending on the parameter κ,

$$\mathbf{p}(x, t; \Delta t, \kappa) = (p, q, r)(x, t; \Delta t, \kappa),$$

by setting

$$(p, q, r)(x, t, \Delta t, \kappa) = (p_i^j, q_i^j, r_i^j), \tag{3.6}$$

for $(i - \frac{1}{2})\Delta x < x < (i + \frac{1}{2})\Delta x$, and for $j\Delta t \le t \le (j+1)\Delta t$. Similarly, let us denote by

$$\mathbf{v}(x, t; \Delta t, \kappa) = (v, \varepsilon, \sigma)(x, t; \Delta t, \kappa) \tag{3.7}$$

the corresponding approximate solution to (2.5), (2.6) obtained from $\mathbf{p}(x, t; \Delta t, \kappa)$, by using the inverse relations (3.3).

At this point we remark that, using the well-known existence of global smooth solutions to (2.5), (2.6) (see [29]), and an energy estimate (see Sec. 3.1), one could easily obtain an m-v solution to (2.1), (2.4) generated by a subsequence of such smooth solutions as $\kappa \to \infty$. However, our primary interest here is to obtain a method for numerical computation of expectation values with respect o such an m-v solution. To this end, we shall prove that one can obtain an m-v solution to (2.1), (2.4) from a sequence of approximate solutions to (2.5), (2.6) constructed from (3.6), as the parameter κ tends to infinity and the grid spacings tend to zero simultaneously. This construction has the great advantage, for numerical purposes, that any member of the sequence of approximate solutions can be obtained by scaling from a fixed one, provided that the initial data are related in this manner. This scaling property will be of decisive importance for the results of the following sections concerning the numerical solution of the Riemann problem.

Now, for some fixed Δt and κ, let us construct the sequences

$$\begin{aligned}\mathbf{v}_n(x,t) &= (v_n,\varepsilon_n,\sigma_n)(x,t) \equiv \mathbf{v}\left(x,t;\frac{\Delta t}{n},n\kappa\right),\\ \mathbf{u}_n(x,t) &= (v_n,\varepsilon_n)(x,t).\end{aligned}$$

If the sequence $\mathbf{v}_n(x,t)$ is uniformly bounded in $L^2(\Omega)$, for any bounded domain $\Omega \subset \mathbf{R}\times[0,\infty)$, then so is $\mathbf{u}_n(x,t)$. We can then use Lemma 2.2 to obtain a Young measure $\nu_{x,t}$ from a subsequence of $\mathbf{u}_n(x,t)$. The question arises, whether $\nu_{x,t}$ is an m-v solution to the elastic problem (2.1), (2.4). The answer to this question is given in the following:

3.1. THEOREM. The Young measure $\nu_{x,t}$, generated from a subsequence of $\mathbf{u}_n(x,t)$, is a measure-valued solution to (2.1), (2.4) if

$$\Delta t < \frac{2}{\kappa}\frac{E-E_1}{E+M}. \tag{3.8}$$

3.2. COROLLARY. If $\mathbf{u}_n(x,t)$ converges in L^2_{loc} to $\mathbf{u}(x,t)=(v,\varepsilon)(x,t)$, then $\mathbf{u}(x,t)$ is a weak solution to (2.1), (2.4).

The proof of the above theorem will be given in the remainder of this section. The reader who is interested in seeing the numerical results first may go directly to Sec. 4.

3.1. *The existence of a measure-valued solution.* It has been from proved [4, 5] that the system (2.5) possesses a unique free energy function, $\psi(\varepsilon,\sigma)$ that satisfies

$$\begin{aligned}\frac{\partial\psi}{\partial\varepsilon}+E\frac{\partial\psi}{\partial\sigma} &= \sigma,\\ \frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_e(\varepsilon)) &\geq 0,\\ \psi(0,0) &= 0,\end{aligned} \tag{3.9}$$

and is given by

$$\psi(\varepsilon,\sigma)=\frac{\sigma^2}{2E}+\varphi(\sigma-E\varepsilon), \tag{3.10}$$

where the function φ is uniquely determined from

$$\begin{aligned}\varphi'(\tau) &= \frac{-1}{E}\sigma_e(h^{-1}(\tau)), \quad \varphi(0)=0, \quad \tau\in\mathbf{R},\\ h(\varepsilon) &= \sigma_e(\varepsilon)-E\varepsilon, \quad \varepsilon\in\mathbf{R}.\end{aligned} \tag{3.11}$$

We remark that the function φ is nonnegative for σ_e satisfying (2.2).

The density of total energy is then defined as

$$e^*(v,\varepsilon,\sigma)=\frac{v^2}{2}+\psi(\varepsilon,\sigma). \tag{3.12}$$

If $(v,\varepsilon,\sigma)(x,t)$ is a smooth solution of the system (2.5), in a domain of $\mathbf{R}\times[0,\infty)$, then the following energy identity holds:

$$\frac{\partial e^*}{\partial t}-\frac{\partial(\sigma v)}{\partial x}=-\kappa\frac{\partial\psi(\varepsilon,\sigma)}{\partial\sigma}(\sigma-\sigma_e(\varepsilon)). \tag{3.13}$$

For some $R > 0$ and $T > 0$, let

$$\Omega_T^R = \{(x,t) \mid 0 \le t \le T,\ -R - c(T-t) \le x \le R + c(T-t)\}. \tag{3.14}$$

We define the energy $e(t) = e(t;R,T)$ of the horizontal section at level t of Ω_R^T, $0 \le t \le T$, by

$$e(t) = e(t;R,T) = \int_{-R-c(T-t)}^{R+c(T-t)} e^*((v,\varepsilon,\sigma)(x,t))dx. \tag{3.15}$$

Let Ω_t^R be the part of Ω_T^R below the section at level t. Integrating (3.13) over Ω_t^R, using Green's identity, and the trivial inequality $|\sigma v| \le ce^*$, we get

$$e(t) \le e(0) - \kappa \iint_{\Omega_t^R} \frac{\partial \psi}{\partial \sigma}(\sigma - \sigma_e(\varepsilon))\, dx\, dt. \tag{3.16}$$

In particular, (3.9) implies

$$e(t) \le e(0). \tag{3.17}$$

Besides, using the inequality (see [4])

$$\frac{1}{E+M} \le \frac{1}{\sigma - \sigma_e(\varepsilon)} \frac{\partial \psi}{\partial \sigma} \le \frac{1}{E - E_1}, \tag{3.18}$$

and (3.16), for $t = T$, we get the following approach to equilibrium for smooth solutions of (2.5), (2.6):

$$\iint_{\Omega_T^R} (\sigma(x,t) - \sigma_e(\varepsilon(x,t)))^2\, dx\, dt \le \frac{E+M}{\kappa}(e(0) - e(T)). \tag{3.19}$$

We shall obtain similar estimates for the approximate solutions, in order to get *a priori* L^2_{loc} bounds for them and an approach to equilibrium when $n \to \infty$.

In addition to the approximate solutions defined in (3.6) and (3.7), we also define the auxiliary approximate solution $\overline{\mathbf{p}}(x,t;\Delta t,\kappa) = (\overline{p},\overline{q},\overline{r})(x,t;\Delta t,\kappa)$ by (3.6) for $t = j\Delta t$, and for $t_j \le t \le t_{j+1}$, $j = 0,1,\dots$, and $x \in \mathbf{R}$ by

$$\begin{aligned} \overline{p}(x,t) &= p(x + c(t-t_j), t_j) + (t - t_j)G((p,q,r)(x + c(t-t_j), t_j)), \\ \overline{q}(x,t) &= q(x - c(t-t_j), t_j) + (t - t_j)G((p,q,r)(x - c(t-t_j), t_j)), \\ \overline{r}(x,t) &= r(x,t_j) + (t-t_j)G((p,q,r)(x,t_j)), \end{aligned} \tag{3.20}$$

and the corresponding auxiliary approximate solution,

$$\overline{\mathbf{v}}(x,t;\Delta t,\kappa) = (\overline{v},\overline{\varepsilon},\overline{\sigma})(x,t;\Delta t,\kappa),$$

obtained from $\overline{\mathbf{p}}(x,t;\Delta t,\kappa)$ by the inverse relations (3.3). Let us also denote

$$\begin{aligned} e(t;\Delta t,\kappa) &= \int_{-R-c(T-t)}^{R+c(T-t)} e^*((v,\varepsilon,\sigma)(x,t;\Delta t,\kappa))dx, \\ \overline{e}(t;\Delta t,\kappa) &= \int_{-R-c(T-t)}^{R+c(T-t)} e^*((\overline{v},\overline{\varepsilon},\overline{\sigma})(x,t;\Delta t,\kappa))dx, \end{aligned} \tag{3.21}$$

with $R = I\Delta x = cI\Delta t$, $T = J\Delta t$, for certain integers I, J. We now extend (3.17) to the approximate solutions.

3.3. PROPOSITION. Assume

$$\Delta t \le \frac{2}{\kappa} \frac{E - E_1}{E + M}. \tag{3.22}$$

Then, for $0 \le t \le T$, we have

$$e(t; \Delta t, \kappa) \le e(0; \Delta t, \kappa), \qquad \overline{e}(t; \Delta t, \kappa) \le \overline{e}(0; \Delta t, \kappa). \tag{3.23}$$

Proof. We adapt the reasoning used in the proof of Theorem I.1 of [6] (see also [7]). Clearly, it suffices to prove the second inequality in (3.23). First of all, we observe that if we change the initial data in (2.6), making them equal zero outside the interval $[-R-cT, R+cT]$, and given by (2.6) inside this interval, then the auxiliary approximate solution obtained through (3.5), (3.20) for the new initial data coincides with that for the old initial data over Ω_T^R. So, if we also denote by $((\overline{v}, \overline{\varepsilon}, \overline{\sigma})(x, t; \Delta t, \kappa))$ the auxiliary approximate solution for the new initial data, it suffices to prove (3.23) for it, with $\overline{e}(x, t; \Delta t, \kappa)$ now given by

$$\overline{e}(t; \Delta t, \kappa) = \int_{-\infty}^{\infty} e^*((\overline{v}, \overline{\varepsilon}, \overline{\sigma})(x, t; \Delta t, \kappa)) dx. \tag{3.24}$$

Let $t_j \le t \le t_{j+1}$. We denote

$$\begin{aligned} \overline{p} &= \overline{p}(x, t; \Delta t, \kappa), & \overline{q} &= \overline{q}(x, t; \Delta t, \kappa), & \overline{r} &= \overline{r}(x, t; \Delta t, \kappa), \\ p_j &= p(x, t_j, \Delta t, \kappa), & q_j &= q(x, t_j; \Delta t, \kappa), & r_j &= r(x, t_j; \Delta t, \kappa), \\ G_j &= G(p_j, q_j, r_j). \end{aligned}$$

We have

$$\overline{e}(t; \Delta t, \kappa) = \int_{-\infty}^{\infty} \left(\frac{\overline{p}^2 + \overline{q}^2}{4E} + \varphi(\overline{r}) \right) dx. \tag{3.25}$$

We will need the fact that one can write

$$\varphi(r + s) = \varphi(r) + s\varphi'(r) + \frac{s^2}{2} m \tag{3.26}$$

with

$$\frac{-M}{E(E+M)} \le m \le \frac{E_1}{E(E - E_1)}, \tag{3.27}$$

which follows from well-known rules of calculus and the fact that φ' is Lipschitz continuous (cf. [7]).

From (3.25)–(3.27) we get

$$\begin{aligned} \overline{e}(t; \Delta t, \kappa) \le \overline{e}(t_j; \Delta t, \kappa) + \frac{(t - t_j)}{2} \int_{-\infty}^{\infty} \Bigg[G_j \left(\frac{p_j + q_j}{2E} + \varphi'(r_j) \right) \\ + \frac{(t - t_j)}{2} G_j^2 \left(\frac{1}{E} + m \right) \Bigg] dx. \end{aligned} \tag{3.28}$$

Set

$$E_j \equiv G_j \left(\frac{p_j + q_j}{2E} + \varphi'(r_j) \right) + \frac{(t - t_j)}{2} G_j^2 \left(\frac{1}{E} + m \right).$$

Let $(v_j, \varepsilon_j, \sigma_j)$ be obtained from (p_j, q_j, r_j) by (3.3). From (3.18), we have

$$E_j \le \kappa(\sigma_j - \sigma_e(\varepsilon_j))^2 \left(\frac{-1}{E+M} + \frac{(t-t_j)}{2}\kappa \left(\frac{1}{E} + m \right) \right); \tag{3.29}$$

so, using (3.27), if Δt satisfies (3.22), we get $E_j \le 0$, and, so, $\overline{e}(t, \Delta t, \kappa) \le \overline{e}(t_j; \Delta t, \kappa)$. Analogously, we get $\overline{e}(t_j, \Delta t, \kappa) \le \overline{e}(t_{j-1}, \Delta t, \kappa)$. So, repeating this procedure up to $j = 0$, we finally arrive at (3.23).

3.4. COROLLARY. Let $\varepsilon(x,t) = \varepsilon(x,t;\Delta t,\kappa)$, $\sigma(x,t) = \sigma(x,t;\Delta t,\kappa)$ be the last two coordinates of the approximate solution to (2.5), (2.6), obtained from (3.5) and the relations (3.3) and let Ω_T^R be given by (3.14). Then, if

$$\Delta t < \frac{2}{\kappa}\frac{E - E_1}{E+M},$$

we have

$$\iint_{\Omega_T^R} (\sigma(x,t) - \sigma_e(\varepsilon(x,t)))^2 \, dx \, dt \le \frac{C}{\kappa} e(0; \Delta t, \kappa), \tag{3.30}$$

where C is a constant depending only on E, M.

Proof. It follows from (3.29), (3.8), and (3.28), by passing the term in the left-hand side of (3.28) to the right-hand side, and the last term in the right-hand side of that inequality to the left-hand side. One then observes that the new left-hand side is greater than a multiple of the part of the integral in (3.30) corresponding to the strip from t_j to t, and the right-hand side is the difference of the energy integrals in t_j and t. Repeating this for the strip from t_{j-1} to t_j, and so on, and then summing up all the inequalities, we arrive at (3.30).

Now we are ready to prove Theorem 3.1.

Proof. It follows easily from Proposition 3.3 that the sequence $\mathbf{v}_n(x,t)$, satisfying (3.8), is uniformly bounded in $L^2(\Omega)$, for any bounded domain $\Omega \subset \mathbf{R} \times [0, \infty)$, and so is $\mathbf{u}_n(x,t)$. Consider the approximate solution and the corresponding auxiliary approximate solution,

$$(p_n, q_n, r_n)(x,t) = (p,q,r)\left(x, t; \frac{\Delta t}{n}, n\kappa\right),$$
$$(\overline{p}_n, \overline{q}_n, \overline{r}_n)(x,t) = (\overline{p}, \overline{q}, \overline{r})\left(x, t; \frac{\Delta t}{n}, n\kappa\right),$$

given by (3.6), (3.20), replacing Δt by $\Delta t/n$ and κ by $n\kappa$. It is easy to verify the following identities in the sense of distributions:

$$\begin{aligned} \frac{\partial \overline{p}_n}{\partial t} - c\frac{\partial \overline{p}_n}{\partial x} &= G_+^n(x,t) = n g_+^n(x,t), \\ \frac{\partial \overline{q}_n}{\partial t} + c\frac{\partial \overline{q}_n}{\partial x} &= G_-^n(x,t) = n g_-^n(x,t), \\ \frac{\partial \overline{r}_n}{\partial t} &= G_0^n(x,t) = n g_0^n(x,t), \end{aligned} \tag{3.31}$$

where G_+^n, G_-^n, G_0^n are the functions defined for $x \subset \mathbf{R}$, $t_j \le t \le t_{j+1}$, $j = 0, 1, \ldots$, by

$$\begin{aligned} G_+^n(x,t) &= nG((p^n, q^n, r^n)(x + c(t - t_j), t_j)), \\ G_-^n(x,t) &= nG((p^n, q^n, r^n)(x - c(t - t_j), t_j)), \\ G_0^n(x,t) &= nG((p^n, q^n, r^n)(x, t_j)). \end{aligned} \tag{3.32}$$

First, from (3.31) and Proposition 3.3 we have that $g_+^n, g_-^n, g_0^n \rightharpoonup 0$ as $n \to \infty$, in the sense of distributions. In fact, for g_0^n we have the following:

ASSERTION 1. $g_0^n \to 0$ in L^2_{loc}.

This follows immediately from Corollary 3.4. □

ASSERTION 2. $G_+^n - G_0^n \rightharpoonup 0$, $G_-^n - G_0^n \rightharpoonup 0$, as $n \to \infty$, in the sense of distributions. Indeed, given $\phi \in C_0^\infty(\mathbf{R} \times (0, \infty))$, we have

$$\begin{aligned} \left| \iint_{\mathbf{R}\times[0,\infty)} (G_+^n - G_0^n)\phi \, dx\, dt \right| &= \left| \sum_{j=0}^{N} \iint_{\mathbf{R}\times[t_j, t_{j+1}]} G_0^n(\phi(x - c(t - t_j), t) - \phi(x,t))\, dx\, dt \right| \\ &\le \sum_{j=0}^{N} \left| \iint_{\mathbf{R}\times[t_j, t_{j+1}]} G_0^n \phi_x(x - c(t - t_j), t) c(t - t_j)\, dx\, dt \right| \\ &\le n\frac{\Delta t}{n} \|\phi_x\|_\infty \iint_{\operatorname{supp}\phi} |g_0^n(x,t)| dx\, dt \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Analogously, we prove that $G_-^n - G_0^n \rightharpoonup 0$. □

We now apply the change of variables (3.3) to the system (3.31). We get

$$\begin{aligned} \frac{\partial \overline{v}_n}{\partial t} - \frac{\partial \overline{\sigma}_n}{\partial x} &= \frac{1}{2c}(G_+^n - G_-^n), \\ \frac{\partial \overline{\varepsilon}_n}{\partial t} - \frac{\partial \overline{v}_n}{\partial x} &= \frac{1}{2E}(G_+^n + G_-^n - 2G_0^n), \\ \frac{\partial \overline{\sigma}_n}{\partial t} - E\frac{\partial \overline{v}_n}{\partial x} &= \frac{1}{2}(G_-^n + G_+^n). \end{aligned} \tag{3.33}$$

Let us define

$$\begin{aligned} \tilde{v}_n(x,t) &= \frac{1}{2c}(p_n(x + c(t - t_j), t_j) - q_n(x - c(t - t_j), t_j)), \\ \tilde{\varepsilon}_n(x,t) &= \frac{1}{2E}(p_n(x + c(t - t_j), t_j) + q_n(x - c(t - t_j), t_j) - 2r_n(x, t_j)), \\ \tilde{\sigma}_n(x,t) &= \frac{1}{2}(p_n(x + c(t - t_j), t_j) + q_n(x - c(t - t_j), t_j)), \end{aligned}$$

for $x \in \mathbf{R}$, $t_j \le t \le t_{j+1}$, $j = 0, 1, \ldots$.

ASSERTION 3. $\overline{v}_n - \tilde{v}_n \rightharpoonup 0$, $\overline{\varepsilon}_n - \tilde{\varepsilon}_n \rightharpoonup 0$, $\overline{\sigma}_n - \tilde{\sigma}_n \rightharpoonup 0$, as $n \to \infty$, in the sense of distributions.

Indeed, we have from (3.20)

$$\overline{v}_n(x,t) - \tilde{v}_n(x,t) = (t - t_j)\frac{G_+^n(x,t) - G_-^n(x,t)}{2c}; \tag{3.34}$$

for $x \in \mathbf{R}$, $t_j \le t \le t_{j+1}$, $j = 0, 1, \ldots$. Now, for any $\phi \in C_0^\infty(\mathbf{R} \times (0,\infty))$,

$$\left| \iint_{\mathbf{R}\times(0,\infty)} G_+^n(x,t)\phi(x,t)\,dx\,dt \right| = \left| \sum_{j=0}^N \iint_{\mathbf{R}\times[t_j,t_{j+1}]} G_0^n(x,t)\phi(x-c(t-t_j),t)\,dx\,dt \right| \le n\|\phi\|_\infty \iint_{\operatorname{supp}\phi} |g_0^n(x,t)|\,dx\,dt,$$

and an analogous inequality holds for G_-^n. So, (3.34) and assertion 1 give that $\overline{v}_n - \tilde{v}_n \rightharpoonup 0$, in the sense of distributions. The proof of $\overline{\varepsilon}_n - \tilde{\varepsilon}_n \rightharpoonup 0$ and $\overline{\sigma}_n - \tilde{\sigma}_n \rightharpoonup 0$ follows similarly. $\square$

Now, we have

ASSERTION 4. $\tilde{v}_n - v_n \rightharpoonup 0$, $\tilde{\varepsilon}_n - \varepsilon_n \rightharpoonup 0$, $\tilde{\sigma}_n - \sigma_n \rightharpoonup 0$, as $n \to \infty$, in the sense of distributions.

Indeed, again, given $\phi \in C_0^\infty(\mathbf{R} \times (0,\infty))$, we have

$$\begin{aligned}
&\left| \iint_{\mathbf{R}\times\mathbf{R}_+} (\tilde{v}_n(x,t) - v_n(x,t))\phi(x,t)\,dx\,dt \right| \\
&\quad = \sum_{j=0}^n \iint_{\mathbf{R}\times[t_j,t_{j+1}]} \left| \frac{1}{2c}(p_n(x,t)(\phi(x-c(t-t_j),t) - \phi(x,t)) \right. \\
&\qquad\qquad \left. - q_n(x,t)(\phi(x+c(t-t_j)) - \phi(x,t))) \right| \,dx\,dt \\
&\quad \le \frac{\Delta t}{nc}\|\phi_x\|_\infty \iint_{\operatorname{supp}\phi} (|p(x,t)| + |q(x,t)|)\,dx\,dt.
\end{aligned}$$

Thus, it follows from the above estimate and Proposition 3.3 that $\tilde{v}_n - v_n \rightharpoonup 0$, in the sense of distributions. The proof of $\tilde{\varepsilon}_n - \varepsilon_n \rightharpoonup 0$ and $\tilde{\sigma}_n - \sigma_n \rightharpoonup 0$ follows identically. $\square$

From assertions 3 and 4 it follows that $\overline{\sigma}_n - \sigma \rightharpoonup 0$. On the other hand, we have from Corollary 3.4 that $\sigma_n - \sigma_e \circ \varepsilon_n \to 0$. So, we have

$$\overline{\sigma}_n \rightharpoonup \overline{\sigma}_e, \quad \text{as } n \to \infty, \qquad \overline{\sigma}_e(x,t) = \langle \nu_{x,t}, \sigma_e(\varepsilon) \rangle. \tag{3.35}$$

We also have from assertions 3 and 4 that $\overline{v}_n - v_n \rightharpoonup 0$, $\overline{\varepsilon}_n - \varepsilon_n \rightharpoonup 0$, and so $\overline{v}_n \rightharpoonup \overline{v}$, $\overline{\varepsilon}_n \rightharpoonup \overline{\varepsilon}$, where $\overline{v}(x,t) = \langle \nu_{x,t}, v \rangle$, $\overline{\varepsilon}(x,t) = \langle \nu_{x,t}, \varepsilon \rangle$. Hence, taking weak limits in the first and second equations in (3.33), we conclude the proof of the theorem. $\square$

The proof of Corollary 3.2 follows directly from (3.35) and the fact that, in this case, $\overline{\sigma}_e(x,t) = \sigma_e(\varepsilon(x,t))$.

4. Basic results for numerical schemes. Here we establish two results similar to those stated in [2], which we shall use for numerical experiments of the Riemann problem (2.1), (2.11). The first one is a convergence criterion in L^2_{loc}, for the sequence $\mathbf{u}(x,t;\Delta t/n, n\kappa)$ constructed above for the Riemann problem. By this theorem , the convergence of this sequence is associated with the convergence of certain averages along the rays of constant x/t in the (x,t)-plane, based on $\mathbf{u}(x,t;\Delta t,\kappa)$. The proof of the theorem is given in the appendix.

4.1. THEOREM. Let $\mathbf{u}_1(x,t) = \mathbf{u}(x,t;\Delta t,\kappa)$ be the approximate solution of the Riemann problem (2.1), (2.11), as defined in the last section. If

$$\ell(\xi) = \lim_{T\to\infty} \frac{2}{T^2}\int_0^T \mathbf{u}_1(\xi\tau,\tau)\tau\,d\tau \tag{4.1}$$

exists and

$$\lim_{T\to\infty} \frac{2}{T^2}\int_0^T |\mathbf{u}_1(\xi\tau,\tau) - \ell(\xi)|^2\tau\,d\tau = 0, \tag{4.2}$$

for almost every $\xi \in \mathbf{R}$, then the sequence $\mathbf{u}_n(x,t) = \mathbf{u}(x,t;\Delta t/n, n\kappa)$ converges in $L^2_{\mathrm{loc}}(\mathbf{R}\times\mathbf{R}_+)$ to a solution $\mathbf{u}(x,t)$ of the Riemann problem (2.1), (2.11), satisfying $\mathbf{u}(x,t) = \ell(x/t)$.

REMARK (Test of convergence). Associated with a certain solution of the approximation scheme (3.5), $\mathbf{u}_i^j = \mathbf{u}_1(i\Delta x, j\Delta t)$, we define the average, at time $t = j\Delta t$,

$$\bar{\mathbf{u}}(x,t) = \frac{2}{j(j+1)}\sum_{k=1}^{j} k\mathbf{u}_{i_k}^k, \quad i_k = \left[\frac{kx}{ct}\right],$$

and the mean quadratic errors

$$e^2(\mathbf{u}(x,t)) = \frac{2}{j(j+1)}\sum_{k=1}^{j} k|\mathbf{u}_{i_k}^k|^2 - |\bar{\mathbf{u}}(x,t)|^2, \quad i_k = \left[\frac{kx}{ct}\right], \tag{4.3}$$

along the rays of constant x/t, where $[a]$ denotes the integer part of the real number a. These values can easily be calculated numerically at each time step. By the above theorem, the approximate solutions $\mathbf{u}_n(x,t)$ converge in L^2_{loc} if the averages converge and the mean quadratic errors tend to zero as the time step tends to infinity.

The next theorem establishes the result from which one can numerically determine the expectation values of state space functions, with respect to m-v solutions for the Riemann problem (2.1), (2.11). The proof is given in the appendix.

4.2. THEOREM. Let $C_2(\mathbf{R}^2)$ denote the space of the continuous functions in $\mathbf{R}^2$ which grow with order < 2 when $|\mathbf{u}| \to \infty$. Let $\mathbf{u}_n(x,t) = \mathbf{u}(x,t;\Delta t/n, n\kappa)$ be the sequence defined in the preceding theorem and let $\nu_{x,t}$ be the m-v solution to (2.1), (2.11), given by Theorem 3.1. Then, given $h \in C_2(\mathbf{R}^2)$, we have

$$\langle \nu_{x,t}, h(\mathbf{u})\rangle = \lim_{T\to\infty} \frac{2}{T^2}\int_0^T h(\mathbf{u}_1((x/t)\tau,\tau))\tau d\tau, \tag{4.4}$$

whenever the limit exists for almost every $x/t \in \mathbf{R}$.

REMARK. As an application of Theorem 4.2 for numerical solutions, we define the average of a function, with domain in the state space, by

$$\bar{h}(x,t) = \frac{2}{j(j+1)}\sum_{k=1}^{j} kh(\mathbf{u}_{i_k}^k), \quad i_k = \left[\frac{kx}{ct}\right], \tag{4.5}$$

for any $h \in C_2(\mathbf{R}^2)$ along every ray of fixed x/t. Its value at each time step can easily be calculated numerically. In particular, we shall determine in our numerical experiments $\bar{\mathbf{u}}(x,t)$ and $\overline{\mathbf{F}}(x,t)$, the average values of the state variable and the flux function, which approximate the expectation values of these state functions with respect to the m-v solution, generated from $\mathbf{u}_n(x,t)$.

5. Numerical results. For our numerical experiments with the scheme proposed in Sec. 3 we take as $\sigma_e(\varepsilon)$ the function in (2.3), with $\varepsilon_A = 1$, $\varepsilon_M = 6$, $E_1 = 8$, $M = 0.5$, $E_2 = 7$, $E = 9$. We also take $\kappa\Delta t = 0.2$. The initial data for the numerical scheme are taken as $v_i^0 = v_L$, $\varepsilon_i^0 = \varepsilon_L$, $\sigma_i^0 = \sigma_e(\varepsilon_L)$, for $i < -i_0$, and $v_i^0 = v_R$, $\varepsilon_i^0 = \varepsilon_R$, $\sigma_i^0 = \sigma_e(\varepsilon_R)$, for $i > i_0$, for some positive integer i_0, say $i_0 = 1$, and any small disturbance of values for $-i_0 \le i \le i_0$. We remark that, for large time steps, the numerical solution does not depend on any particular choice of disturbance. Except when explicitly mentioned to the contrary, the graphs of functions presented in this section, at each specified time step, have as abscissa the variable x/t. We choose to plot functions in this way because, by the results in Sec. 4, it suffices to know the behavior of these functions along these rays.

5.1. *A simple Riemann problem.* We consider a Riemann problem with the left and right initial states in different phases outside the elliptic region (see Fig. 1). Specifically, we take $(v_L, \varepsilon_L) = (0, 0.5)$, $(v_R, \varepsilon_R) = (0.6.5)$. In Fig. 2, we show the graphs of the functions $v(x,t)$ and $\varepsilon(x,t)$ at the time step 50000. It is not difficult to verify that these functions solve the corresponding Riemann problem. This is confirmed by the mean quadratic error $e^2(\mathbf{u}(x,t))$ also given in Fig. 2, at time step 50000. The error clearly converges to zero almost everywhere. The averages, not shown here, also converge, and so Theorem 4.1 implies the convergence of the scheme in L^2_{loc} to a weak solution of the problem. In Fig. 1, the solution is also plotted in the stress-strain coordinates together with the non-monotone curve $\sigma_e(\varepsilon)$. From these results, it is clear that there is a unique interface boundary at $x = 0$, a single phase jump across the unstable interval, in this problem. It is interesting to observe that the solution shown here agrees with those obtained in [13, 8].

5.2. *Oscillation waves.* We now consider a Riemann problem inside the elliptic region with the same left and right initial states, $(v_L, \varepsilon_L) = (v_R, \varepsilon_R) = (0, 3)$, and a small disturbance at $x = 0$ at initial time. The small disturbance does not die out and, instead, evolves into a nontrivial approximate solution with persistent oscillations shown in Fig. 3 for $\varepsilon(x,t)$ at time step 1000000. Clearly, the sequence of approximate solutions

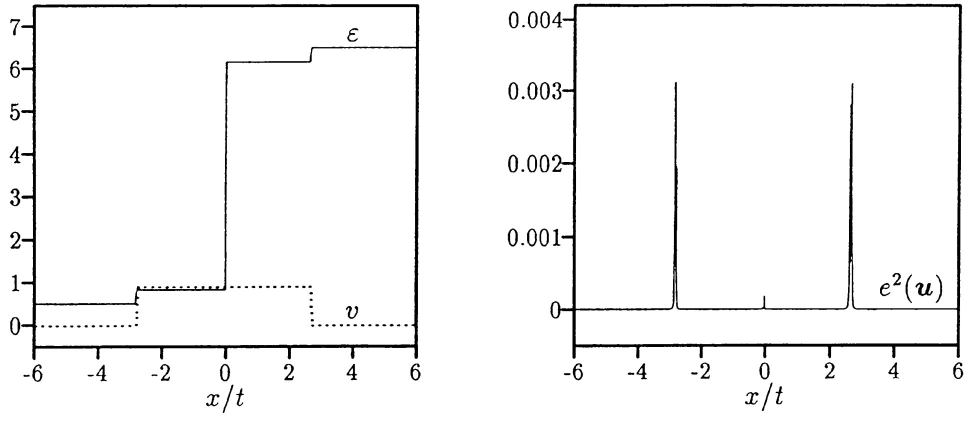

FIG. 2. A weak solution (v, ε) and quadratic errors for the Riemann problem of Fig. 1

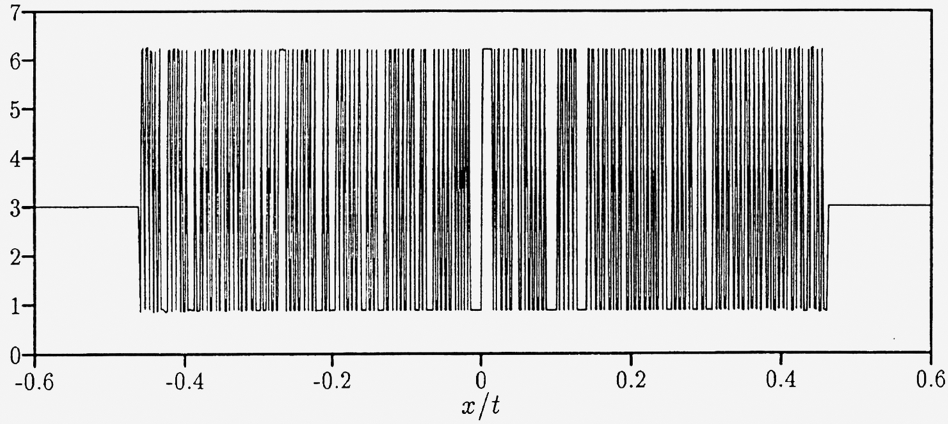

FIG. 3. Oscillation wave, $\varepsilon(x,t)$ at time step 1000000, with initial state $(0,3)$

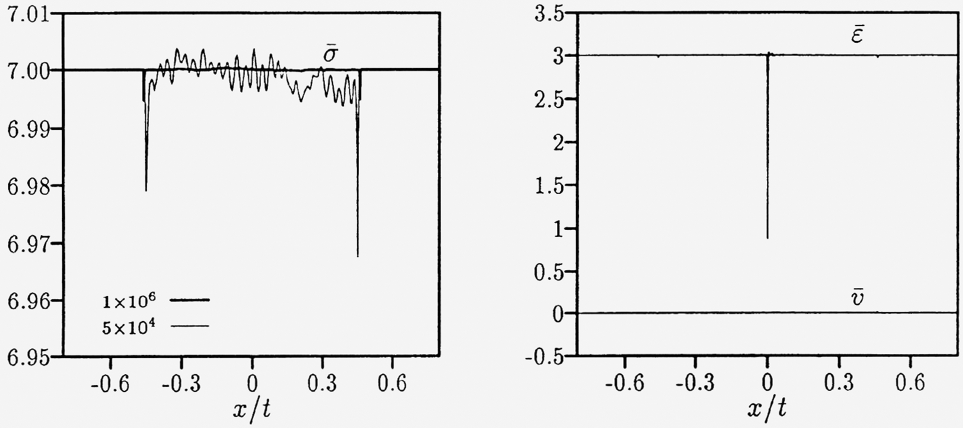

FIG. 4. Averages of σ at time steps = 50000, 1000000, and (v,ε) at 1000000, for the oscillation wave of Fig. 3

does not converge due to the appearance of oscillations. These oscillations generate a measure-valued solution, to the corresponding elastic problem according to Theorem 3.1, what we call an *oscillation wave*. In this case, by applying Theorem 4.2, we may determine expectation values, or averages, by (4.5) for h equal to the state variables, σ, v, and ε. Figure 4 shows the averages $\overline{\sigma}(x,t)$, at time steps 50000 and 1000000, as well as $\overline{v}(x,t)$ and $\overline{\varepsilon}(x,t)$ at time step 1000000. From the graphs of $\overline{\sigma}(x,t)$, at time steps 50000 and 1000000, we are convinced that the averages converge although slowly, and that $(\overline{v}(x,t),\overline{\varepsilon}(\dot{x},t))$ coincides in the limit with the trivial constant solution $(v,\varepsilon)=(0,3)$.

5.3. *The entropy rate criterion.* Here we also apply Theorem 4.2, now in connection with DAFERMOS'S entropy rate criterion for the selection of a physically relevant solution [9, 16, 17]. In the case of weak solutions, which are functions of x/t, the criterion implies that $\mathbf{u}(x,t)=V(x/t)$ is a more favorable solution (in the sense of "stability") compared to the other solution $\tilde{\mathbf{u}}(x,t) = \widetilde{V}(x/t)$ if the inequality

$$\int_{-\infty}^{\infty} \{\eta(V(\xi)) - \eta(\widetilde{V}(\xi))\} d\xi \le 0$$

holds, where the "entropy" function η can be taken as the total energy density, the sum of the kinetic and the strain energy, in the present elastic problem, given by

$$\eta(v,\varepsilon) = \frac{v^2}{2} + \int_0^{\varepsilon} \sigma_e(s)ds.$$

In other words, in this case, this criterion is a minimum energy criterion in some sense.

We can extend this criterion to the case of m-v solutions, by saying that an m-v solution $\nu_{x/t}$ is a more favorable solution compared to another m-v solution $\tilde{\nu}_{x/t}$, if the following inequality, involving the corresponding expectation values of the energy function,

$$\int_{-\infty}^{\infty} \{\langle \nu_\xi, \eta(\mathbf{u})\rangle - \langle \tilde{\nu}_\xi, \eta(\mathbf{u})\rangle\} \, d\xi \le 0, \tag{5.1}$$

holds. In order to apply this criterion to compare m-v solutions, it is necessary that the expectation value of the energy function is defined. For the m-v solution given by Theorem 3.1, in principle, expectation values are defined only for functions in $C_2(\mathbf{R}^2)$, which does not include the energy function. However, by applying Theorem 4.2, we can determine the expectation value of the characteristic function of the complement of a bounded set $\{\mathbf{u} \in \mathbf{R}^2 \,|\, |\mathbf{u}| < R\}$, for some R conveniently chosen, relative to the given m-v solution. This was done for the nontrivial m-v solution associated with the oscillation wave shown above, for the constant initial data $(0,3)$. It was seen that for $R = 4$ the averages of the corresponding characteristic function, given by (4.5), converge to zero very quickly. So, the expectation value of this characteristic function, with respect to the m-v solution of this problem, is identically zero, as a consequence of Theorem 4.2. This test indicates that the support of the Young measures, which constitute the m-v solution of the problem, is contained in a bounded set $\{\mathbf{u} \in \mathbf{R}^2 \,|\, |\mathbf{u}| < R\}$, for almost every x/t. Therefore, the expectation values relative to this m-v solution are well-defined for any continuous functions.

In Fig. 5, we show the average $\overline{\eta}(x,t)$, at time step 1000000, from which it is clear that the m-v solution associated with the oscillation wave is a more favorable solution, according to the entropy rate criterion, compared to the trivial m-v solution $\tilde{\nu}_{x/t} = \delta_{(0,3)}$, where the right-hand side is the Dirac measure concentrated at $(0,3)$, i.e., the constant solution (whose total energy is indicated as $\eta(0,3)$ in Fig. 5). Therefore, it also confirms that constant states in the elliptic region are unstable.

5.4. *Phase fractions.* Finally, we apply Theorem 4.2 to obtain expectation values of phase fractions associated with the oscillation waves. In the simplified model considered here, the functions that provide the fractions corresponding to each phase are given by

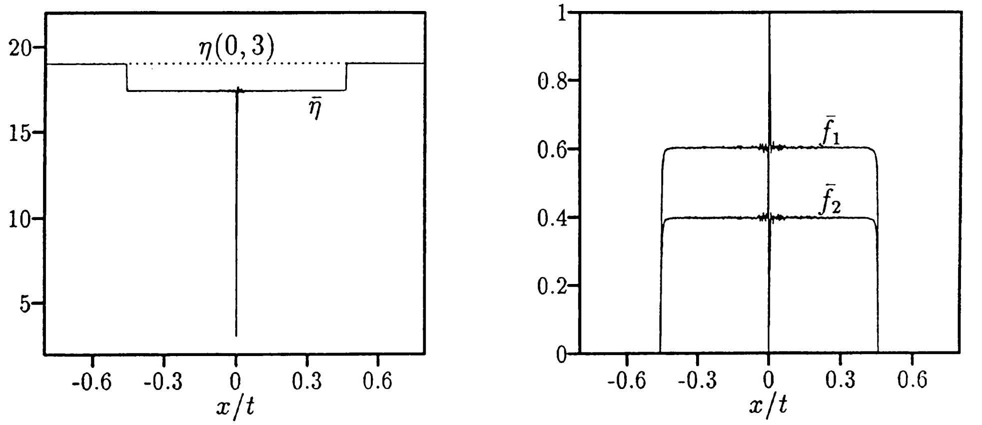

FIG. 5. Entropy rate criterion and phase fractions, for the oscillation wave of Fig. 3

$$f_1(\varepsilon) = \begin{cases} 1, & \text{if } \varepsilon \leq \varepsilon_A, \\ 0, & \text{otherwise,} \end{cases} \qquad f_2(\varepsilon) = \begin{cases} 1, & \text{if } \varepsilon \geq \varepsilon_M, \\ 0, & \text{otherwise.} \end{cases}$$

We have done several experiments to determine the expectation values of these functions, with respect to the nontrivial m-v solutions associated with the states $(0, \varepsilon)$, for $\varepsilon \in [\varepsilon_A, \varepsilon_M]$, the interval of the unstable states. Each strain ε in this interval can be written as a convex combination of the corresponding strains, $\varepsilon_1 \leq \varepsilon_A$ and $\varepsilon_2 \geq \varepsilon_M$, in the two pure phases at the same stress level, namely, $\sigma_e(\varepsilon) = \sigma_e(\varepsilon_1) = \sigma_e(\varepsilon_2)$,

$$\begin{aligned} \varepsilon &= x_1\varepsilon_1 + x_2\varepsilon_2, \\ x_1 + x_2 &= 1. \end{aligned}$$

The experiments show that expectation values of the phase fractions, with respect to the nontrivial m-v solutions associated with the states $(0, \varepsilon)$, agree with the coefficients x_1 and x_2 in the above convex combination. As an example, we have shown in Fig. 5 the averages $\overline{f}_1(x,t), \overline{f}_2(x,t)$, at time step 100000, for $\varepsilon = 3$. In this case, we have $x_1 = 0.602$ and $x_2 = 0.398$, which agree with the values found in the figure. This conclusion is what one would expect intuitively. Another important aspect of these experiments is that the average phase fractions are constant throughout the sector behind the wave front in the limit, which means that the phase mixture is macroscopically homogeneous.

On the other hand, careful examination of the oscillation pattern in the approximation solution reveals an interesting fact that the oscillation pattern tends to become stationary in the limit, as shown in Fig. 6 (see p. 132) for a typical example: the two graphs of $\varepsilon(x,t)$, at time steps 25000 and 50000, restricted to the interval of 100 grid points between 300 and 400, corresponding to the approximate solution for the initial state $(0, 3)$. They are

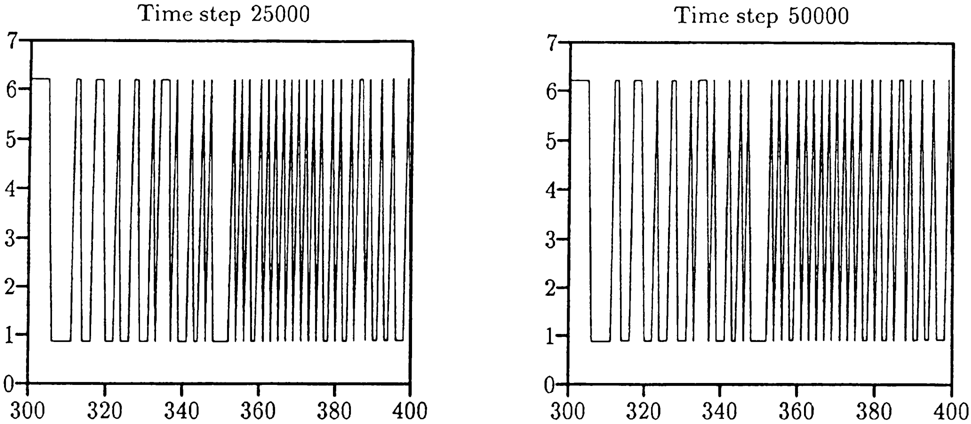

FIG. 6. Stationary phase pattern, $\varepsilon(x,t)$

identical at every grid point and oscillate intensely between two pure phases. They can be interpreted as the microscopic structure of the phase mixture with stationary phase boundaries densely distributed along the length of the body. This structure resembles the experimentally observed band-like structure of a single crystal sample of shape memory alloy during austenitic-martensitic transformation [23, 24].

Appendix.

A.1. *Proof of Theorem* 4.1. First, we easily see that $\mathbf{u}_n(x,t) = \mathbf{u}_1(nx, nt)$. Now, given any compact $K \subset \mathbf{R} \times \mathbf{R}_+$, for k, l sufficiently large, with $k \le l$, we have

$$\begin{aligned}
&\iint_K |\mathbf{u}_k(x,t) - \mathbf{u}_l(x,t)|^p \, dx\, dt \\
&\quad = \iint_K |\mathbf{u}_1(kx,kt) - \mathbf{u}_1(lx,lt)|^p \, dx\, dt \\
&\quad = \frac{1}{k^2} \iint_{kK} \left| \mathbf{u}_1(x,t) - \mathbf{u}_1\left(\frac{l}{k}x, \frac{l}{k}t\right) \right|^p \, dx\, dt \\
&\quad \le \frac{(\operatorname{diam} K)^2}{(k \operatorname{diam} K)^2} \iint_{\substack{0 \le t < k \operatorname{diam} K \\ |x| \le k \operatorname{diam} K}} \left| \mathbf{u}_1(x,t) - \mathbf{u}_1\left(\frac{l}{k}x, \frac{l}{k}t\right) \right|^p \, dx\, dt.
\end{aligned} \tag{A.2}$$

To prove the theorem, we shall show, in the following, that the right-hand side of (A.2) tends to zero as k tend to infinity.

Without any essential difference, we may assign the initial data in the following way: $\mathbf{u}_1(x,0)$ equals $\mathbf{u}_L$ for $x < -j_0\varepsilon_1$, and equals $\mathbf{u}_R$ for $x > j_0\varepsilon_1$, for some $j_0 \in N$; and for

$|x| < j_0\varepsilon_1$ arbitrary values are assigned. Now let $\alpha = \tan^{-1}\lambda$. Then we have

$$\begin{aligned}
&\frac{1}{CT^2}\iint_{\substack{0\le t<T\\|x|\le CT}} |\mathbf{u}_1(x,t)-\mathbf{u}_1(ax,at)|^p\,dx\,dt\\
&\quad= \frac{1}{CT^2}\iint_{\substack{0\le t\le T\\ \lambda t\le |x|\le j_0\varepsilon_1+\lambda t}} |\mathbf{u}_1(x,t)-\mathbf{u}_1(ax,at)|^p\,dx\,dt\\
&\qquad+\frac{1}{CT^2}\iint_{\substack{0\le t\le T\\|x|\le\lambda t}} |\mathbf{u}_1(x,t)-\mathbf{u}_1(ax,at)|^p\,dx\,dt\\
&\quad\le \frac{\text{const.}}{T}+\int_\alpha^{\pi-\alpha} d\theta \frac{1}{CT^2}\int_0^{T/\sin\theta} |\mathbf{u}_1(r,\theta)-\mathbf{u}_1(ar,\theta)|^p r\,dr\\
&\quad\le \frac{\text{const.}}{T}+\int_\alpha^{\pi-\alpha} \frac{1}{\sin^2\theta}d\theta\left\{\frac{\sin^2\theta}{CT^2}\int_0^{T/\sin\theta} |\mathbf{u}_1(r,\theta)-\ell(\theta)|^p r\,dr\right\}\\
&\qquad+\int_\alpha^{\pi-\alpha} \frac{1}{\sin^2\theta}d\theta\left\{\frac{\sin^2\theta}{a^2CT^2}\int_0^{aT/\sin\theta} |\mathbf{u}_1(r,\theta)-\ell(\theta)|^p r\,dr\right\},
\end{aligned}$$

where const. represents a constant not depending on a or T. By (4.2), it is easy to see that any of the three terms on the right-hand side of the last inequality can be made arbitrarily small, uniformly for $a \ge 1$. This completes the proof.

A.2. *Proof of Theorem* 4.2. Let $\mathbf{u}_k(x,t) = \mathbf{u}_1(kx,kt)$. By Theorem 3.1 there exists a subsequence of $\mathbf{u}_k$ generating a Young measure $\nu_{x,t}$ which is a globally defined m-v solution to the Riemann problem (2.1), (2.11), in the sense of Definition 1.1. Let us consider a point $(x,t) \in \mathbf{R}\times(0,\infty)$, and set

$$r_0 = \sqrt{x^2+t^2}, \qquad \theta_0 = \tan^{-1}\frac{t}{x}.$$

We also denote by $\Delta(r,\theta)$ the infinitesimal sector $|r-r_0| < \Delta r$, $|\theta-\theta_0| < \Delta\theta$, $\Delta r > 0$, $\Delta\theta > 0$. For the area $m(\Delta(r,\theta))$ of $\Delta(r,\theta)$ trivially we have

$$m(\Delta(r,\theta)) = 4r_0\Delta r\Delta\theta.$$

Given a continuous function $h \in C_2(\mathbf{R}^2)$, we may assume that

$$\langle \nu_{x,t}, h(\mathbf{u})\rangle = \lim_{\Delta\tau+\Delta\theta\to 0}\frac{1}{m(\Delta(r,\theta))}\iint_{\Delta(r,\theta)} \langle \nu_{y,\tau}, h(\mathbf{u})\rangle\,dy\,d\tau, \tag{A.3}$$

since this is true for $(x,t) \in \mathbf{R} \times (0,\infty)$ almost everywhere. Now, if $h \in C_2(\mathbf{R}^2)$, we have

$$\begin{aligned}
&\frac{1}{m(\Delta(r,\theta))}\iint_{\Delta(r,\theta)} \langle \nu_{y,\tau}, h(\mathbf{u})\rangle\, dy\, d\tau = \frac{1}{m(\Delta(r,\theta))}\lim_{k\to\infty}\iint_{\Delta(r,\theta)} h(\mathbf{u}_k(y,\tau))\, dy\, d\tau \\
&= \frac{1}{m(\Delta(r,\theta))}\lim_{k\to\infty}\iint_{\Delta(r,\theta)} h(\mathbf{u}_1(kr,\theta)) r\, dr\, d\theta \\
&= \frac{1}{m(k\Delta(r,\theta))}\lim_{k\to\infty}\iint_{k\Delta(r,\theta)} h(\mathbf{u}_1(r,\theta)) r\, dr\, d\theta \\
&= \frac{1}{2\Delta\theta}\int_{\theta_0-\Delta\theta}^{\theta_0+\Delta\theta} d\theta \lim_{k\to\infty}\frac{1}{2k^2 r_0\Delta r}\int_{k(r_0-\Delta r)}^{k(r_0+\Delta r)} h(\mathbf{u}_1(r,\theta)) r\, dr \\
&= \frac{1}{2\Delta\theta}\int_{\theta_0-\Delta\theta}^{\theta_0+\Delta\theta} d\theta \frac{1}{2r_0\Delta r}\left\{\frac{1}{2}(r_0+\Delta r)^2 \lim_{k\to\infty}\frac{2}{k^2(r_0+\Delta r)^2}\int_0^{k(r_0+\Delta r)} h(\mathbf{u}_1(r,\theta)) r\, dr \right.\\
&\qquad\qquad \left. -\frac{1}{2}(r_0-\Delta r)^2 \lim_{k\to\infty}\frac{2}{k^2(r_0-\Delta r)^2}\int_0^{k(r_0-\Delta r)} h(\mathbf{u}_1(r,\theta)) r\, dr\right\} \\
&= \frac{1}{2\Delta\theta}\int_{\theta_0-\Delta\theta}^{\theta_0+\Delta\theta} \ell[h](\theta)\, d\theta.
\end{aligned}$$

Since $\ell[h](\theta)$, defined by

$$\ell[h](\theta) = \lim_{R\to\infty}\frac{2}{R^2}\int_0^R h(\mathbf{u}_1(r,\theta) r\, dr,$$

whenever it exists, must be a measurable function of θ, the last equation above and (A.3) imply (4.4), and the theorem is proved.

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