FINITE ENERGY TRAVELLING WAVES
FOR NONLINEAR DAMPED WAVE EQUATIONS

BY

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1. Introduction. We show the existence of solitary waves resulting from a dynamic balance between dissipation and repulsive nonlinearity in a semilinear damped wave equation. More specifically, we examine the long-time behaviour of solutions to the Cauchy problem:

$$\begin{cases} u_{tt} + du_t - \Delta u + mu = |u|^{\alpha-1}u, \\ u = u(x,t), \quad x \in R^N, \quad N \geq 3, \quad t \in R^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in R^N, \end{cases} \tag{1.1}$$

where d, m are strictly positive constants and the exponent α satisfies

$$1 < \alpha < \alpha^* \quad \text{with} \quad \alpha^* = 1 + \min\left\{\frac{2}{N-2}, \frac{4}{N}\right\}. \tag{1.2}$$

The basic quantity is the *energy*

$$E(u, u_t) \equiv \frac{1}{2} \int_{R^n} |\nabla u|^2 + m|u|^2 + |u_t|^2 \, dx - \frac{1}{\alpha+1} \int_{R^n} |u|^{\alpha+1} \, dx, \tag{1.3}$$

which represents a Lyapunov function of the problem, i.e., it is decreasing along any nonstationary trajectory of (1.1).

Employing the potential-well arguments of PAYNE-SATTINGER [16] one observes that any solution of (1.1) emanating from sufficiently small initial data exists globally for all $t \in R^+$ and tends to zero with growing t. On the other hand, LEVINE [13, Part IV] showed that solutions corresponding to certain large data (with negative energy) blow up at a finite time.

In this paper, we shall deal with localized, finite energy states, i.e., with solutions to (1.1) that belong to neither class discussed above. A standard example of such a solution is, of course, a nonzero solution to the *stationary problem*:

$$-\Delta w + mw = |w|^{\alpha-1}w, \quad w \in H^1(R^N). \tag{1.4}$$

According to KWONG [12], there is a positive, radially symmetric solution w_g of (1.4) unique (up to a spatial shift) in the class of positive solutions. On the other hand, the

existence of an infinite sequence of solutions (changing sign) with energy going to infinity
was proved by BERESTYCKI-LIONS [4]. All solutions of (1.4) except zero are unstable
as solutions to the evolutionary problem (1.1) (see KELLER [11], SHATAH [19]).

It is well known that solutions of (1.1) satisfy the *energy equality*:

$$\frac{d}{dt} E(u, u_t)(t) = -\int_{R^N} |u_t(t)|^2 \, dx \tag{1.5}$$

for all t belonging to the existence interval $[0, T_{\max})$. Consequently, for a *global bounded*
solution the relation (1.5) may be integrated with respect to t to deduce that $\|u_t\|_{L^2}$ must
be "small" for large values of t. This justifies a conjecture that the long-time behaviour
of bounded solutions of (1.1) should be somehow related to the stationary problem (1.4).

This is indeed the case as shown by the following result.

THEOREM 1.1. For any positive integer k there exist initial data

$$u_0 \in H^2(R^N), \qquad u_1 \in H^1(R^N)$$

such that the corresponding (strong) solution

$$u \in \bigcap_{i=1}^{2} C^i(R^+; H^{2-i}(R^N))$$

of the problem (1.1) is defined for all $t \in R^+$ and

$$\begin{cases} \|u(\cdot, t) - \sum_{j=1}^{2k} (-1)^j w_g(\cdot - x_j(t))\|_{H^1} \to 0, \\ u_t(\cdot, t) \to 0 \text{ in } L^2(R^N) \text{ as } t \to \infty, \end{cases} \tag{1.6}$$

where w_g is the (unique) positive solution of (1.4) radially symmetric with respect to the
origin and

$$|x_j(t)| \to \infty, \qquad \text{dist}(x_j(t), x_i(t)) \to \infty \quad \text{for } i \neq j \text{ as } t \to \infty. \tag{1.7}$$

Thus the problem (1.1) possesses solutions that are global in time and split into a
finite number of travelling waves that propagate in space with different velocities and
locally "look" like the stationary solution w_g. As for different k we find (different)
solutions satisfying (1.6), the asymptotic behaviour of the dynamic system associated to
(1.1) exhibits truly infinite-dimensional character in contrast with an analogous problem
on a *bounded* spatial domain (cf. GHIDAGLIA-TEMAM [8]). Such solutions have all
characteristic properties attributed to solitons as they are *localized, finite energy states
that are fundamentally nonlinear objects and so cannot be reached by perturbation theory
from any linear state* (REMOISSENET [18, Chapter 1]).

The number of waves appearing in (1.6) is even mainly because of the technique of the
proof. On the other hand, it seems interesting to note that the existence of waves with
an odd number of *travelling* components is not obvious. It is shown in [7] that if $N = 1$
and u is a solution of the problem (1.1) satisfying (1.6) with $k = 1$, then u converges to
a fixed stationary solution.

The remaining part of the paper is devoted to the proof of Theorem 1.1. In Sec. 2, we
review some standard facts concerning well-posedness of (1.1) and regularity properties
of the solutions.

In Sec. 3, we show that the asymptotic behaviour of bounded solutions may be described by a formula very similar to (1.6). The crucial observation is that any sequence of the form $\{u(\cdot, t_n)\}$, $t_n \to \infty$ where u solves (1.1) represents a Palais-Smale sequence related to the stationary problem (1.4). Consequently, the technique of BENCI-CERAMI [2] and P. L. LIONS [14] can be used to obtain the desired result.

In Sec. 4, we construct solutions with the property

$$0 < c_1 \leq \|u(\cdot, t)\|_{H^1} \leq c_2 \quad \text{for all } t \in R^+ \tag{1.8}$$

(here and always the symbols c_i, $i = 1, 2, \ldots$ will denote positive constants). It is only at this stage of the proof where the specific shape of the nonlinearity plays a significant role. Specifically, we will see that boundedness of the energy of a trajectory $[u(t), u_t(t)]$, $t \in R^+$, yields boundedness in the energy space

$$X \equiv H^1(R^N) \times L^2(R^N). \tag{1.9}$$

Note that such a trick is well known from the classical paper of AMBROSETTI-RABINOWITZ [1] on the application of critical point theory to nonlinear elliptic problems.

Finally in Sec. 5, we take advantage of the invariance of the problem with respect to a certain group of reflections and construct solutions of (1.1) satisfying both (1.8) and

$$\lim_{t \to \infty} E(u, u_t)(t) = 2kE(w_g, 0). \tag{1.10}$$

Such a solution will be shown to have all the properties claimed in the conclusion of Theorem 1.1. At this final step, the nonexistence result of ESTEBAN-LIONS [6] will play a crucial role in eliminating solutions that converge to a fixed stationary state.

2. Well-posedness and regularity of solutions.
Let us begin with the linear problem:

$$\begin{cases} v_{tt} + dv_t - \Delta v + mv = g(x, t), \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in R^N. \end{cases} \tag{2.1}$$

If $g = 0$, it is well known (see, e.g., LIONS-MAGENES [15, Chapt. 3.8–3.9]) that the problem (2.1) is well posed and generates a C_0-semigroup $\{S_t\}$ on the Hilbert scale

$$X_\omega \equiv H^\omega(R^N) \times H^{\omega-1}(R^N), \quad \omega \in [0, 1]. \tag{2.2}$$

Moreover, as $d > 0$, there are constants $M, \delta > 0$ such that

$$\|S_t\|_{\mathcal{L}(X_\omega)} \leq M \exp(-\delta t) \quad \text{for all } t \in R^+, \ \omega \in [0, 1]. \tag{2.3}$$

The solutions of the nonhomogeneous problem are given by the Duhamel formula

$$[v(t), v_t(t)] = S_t[v_0, v_1] + \int_0^t S_{t-s}[0, g(s)] \, ds \tag{2.4}$$

provided $g \in L^1_{\text{loc}}(R^+, H^{\omega-1}(R^N))$.

By virtue of the hypothesis (1.2), the nonlinear term in Eq. (1.1) may be treated as a locally Lipschitz perturbation on the space X and, consequently, the standard theory of evolution equations (see, e.g., PAZY [17, Chapt. 6]) yields the following result:

PROPOSITION 2.1. (i) For any data $[u_0, u_1] \in X$, there exists a function u unique in the class

$$u \in C([0, T_{\max}), H^1(R^N)), \qquad u_t \in C([0, T_{\max}), L^2(R^N))$$

that solves the problem (1.1) in the sense of distributions on a maximal existence interval $[0, T_{\max})$.

In case $T_{\max} < \infty$, we have

$$\lim_{t \to T_{\max}} \|[u(t), u_t(t)\|_X = \infty. \tag{2.5}$$

(ii) For any $T < T_{\max}$ there is a neighbourhood $\mathcal{U}$ of $[u_0, u_1]$ in X and a constant $\mathcal{L}(T, \mathcal{U})$ such that any solution $[v, v_t]$ of (1.1) corresponding to the data

$$[v(0), v_t(0)] \in \mathcal{U}$$

is defined on $[0, T]$ and

$$\|[v(T), v_t(T)] - [u(T), u_t(T)\|_X \leq \mathcal{L}\|[v(0), v_t(0)] - [u_0, u_1]\|_X. \tag{2.6}$$

The exponent α is subcritical in the sense that the superposition operator $u \to |u|^{\alpha-1}u$ may be differentiated both with respect to x and t to obtain a quantity bounded in $H^\beta(R^N)$ for a certain $\beta > -1$. Consequently, the standard bootstrap arguments (cf., e.g., GHIDAGLIA-TEMAM [9]) may be used to improve the regularity of the weak solutions.

PROPOSITION 2.2. Assume that the initial data belong to the class

$$u_0 \in H^2(R^N), \qquad u_1 \in H^1(R^N).$$

Then the unique weak solution of problem (1.1) the existence of which is claimed in Proposition 2.1 is, in fact, a strong solution to the problem satisfying

$$u \in \bigcap_{i=0}^{2} C^i([0, T_{\max}), H^{2-i}(R^N)). \tag{2.7}$$

Note that, by virtue of (2.6), (2.7), all formulas derived in the future by means of multiplication of (1.1) by quantities containing various derivatives of u may be justified, provided they make sense, even within the class of weak solutions by density arguments. In particular, the energy equality (1.5) holds for any weak solution of problem (1.1).

3. Asymptotic behaviour of bounded solutions.
We start with a lemma on the asymptotic behaviour of Palais-Smale sequences that may be proved using the ideas of the *concentrated compactness theory* of P. L. LIONS [14, Theorem III.4]. The same result is also proved in BENCI-CERAMI [2, Lemma 3.1].

PROPOSITION 3.1. Assume there is a sequence $\{u_n\} \subset H^1(R^N)$ such that

$$\|u_n\|_{H^1} \text{ is bounded for } n \to \infty, \tag{3.1}$$

$$-\Delta u_n + m u_n - |u_n|^{\alpha-1} u_n \to 0 \text{ in } H^{-1}(R^N) \text{ as } n \to \infty. \tag{3.2}$$

Then either u_n converges to zero in $H^1(R^N)$ or there is a subsequence (not relabeled) and an integer $m \geq 1$ such that

$$\|u_n - w_1 - \sum_{j=2}^{m} w_j(\cdot - x_n^j)\|_{H^1} \to 0 \quad \text{as } n \to \infty \tag{3.3}$$

where

$$|x_n^j| \to \infty, \qquad \text{dist}(x_n^i, x_n^j) \to \infty \quad \text{for } i \neq j \text{ as } n \to \infty \tag{3.4}$$

and w_1, $w_j \neq 0$, $j = 2, \ldots, m$, are (not necessarily distinct) solutions of problem (1.4).
   Moreover, we have

$$E(u_n, 0) \to \sum_{j=1}^{m} E(w_j, 0) \quad \text{as } n \to \infty. \tag{3.5}$$

Our next goal is to show that the sequence $u_n \equiv u(t_n)$, $t_n \to \infty$ where u is a bounded solution of (1.1) satisfies the hypotheses of Proposition 3.1. To this end, assume that a global solution of (1.1) satisfies

$$\|[u(t), u_t(t)]\|_X \leq c_3. \tag{3.6}$$

Thus the energy $E(u, u_t)$ is bounded and we deduce from (1.5) that

$$\int_0^\infty \int_{R^N} u_t^2 \, dx \, dt \leq c_4. \tag{3.7}$$

Differentiating (1.1) with respect to t and setting $v = u_t$ we obtain that v solves the linear problem (2.1) with

$$v(0) = u_t(0) \in L^2(R^N), \qquad v_t(0) = u_{tt}(0) \in H^{-1}(R^N), \tag{3.8}$$

and

$$g = \alpha |u|^{\alpha-1} u_t. \tag{3.9}$$

By virtue of (1.2), (3.6), (3.7), we have

$$g \in L^\infty(R^+, H^{-1}(R^N)) \cap L^2(R^+, H^{-1}(R^N)). \tag{3.10}$$

Now, we can use the linear theory on the space $L^2 \times H^{-1}$, specifically the formulas (2.3), (2.4) together with the estimate (3.10) to conclude

$$v(t) = u_t(t) \to 0 \text{ in } L^2(R^N), \qquad v_t(t) = u_{tt}(t) \to 0 \text{ in } H^{-1}(R^N) \quad \text{as } t \to \infty. \tag{3.11}$$

Consequently, setting $u_n = u(t_n)$ we get

$$-\Delta u_n + m u_n - |u_n|^{\alpha-1} u_n = -(u_n)_{tt} - d(u_n)_t \to 0$$
$$\text{in } H^{-1}(R^N) \text{ as } n \to \infty. \tag{3.12}$$

Thus we are allowed to apply Proposition 3.1 to obtain:

PROPOSITION 3.2. Assume there is a global solution u of (1.1) satisfying (3.6).
   Then for any sequence $t_n \to \infty$ either

$$u(t_n) \to 0 \text{ in } H^1(R^N), \qquad u_t(t_n) \to 0 \text{ in } L^2(R^N) \tag{3.13}$$

or there is a subsequence (not relabeled) such that

$$\begin{cases} \|u(t_n) - w_1 - \sum_{j=2}^{m} w_j(\cdot - x_n^j)\|_{H^1} \to 0, \\ u_t(t_n) \to 0 \text{ in } L^2(R^N) \text{ as } t_n \to \infty, \end{cases} \tag{3.14}$$

$$E(u, u_t)(t_n) \to \sum_{j=1}^{m} E(w_j, 0) \tag{3.15}$$

where the quantities w_j, x_n^j, $j = 1, \ldots, m$, have the same properties as in Proposition 3.1.

4. The existence of bounded solutions.

In this section, we construct solutions satisfying (1.8). Let us start with two auxiliary results.

LEMMA 4.1. Under the hypothesis (1.2), the zero solution is locally asymptotically stable, i.e., there is an $\varepsilon > 0$ such that for all data satisfying

$$\|[u_0, u_1]\|_X < \varepsilon,$$

the (unique) solution u of (1.1) exists globally in time and

$$u(t) \to 0 \text{ in } H^1(R^N), \qquad u_t(t) \to 0 \text{ in } L^2(R^N) \quad \text{as } t \to \infty. \tag{4.1}$$

Proof. Clearly, u is a solution of the linear problem (2.1) with $g = |u|^{\alpha-1}u$. By virtue of the Sobolev embedding theorem we have

$$\|g(t)\|_{L^2}^2 = \int_{R^N} |u(t)|^{2\alpha} \, dx \le c_5 \|u(t)\|_{H^1}^{2\alpha} \text{ for all } t \in [0, T_{\max}) \tag{4.2}$$

where $\alpha > 1$.

Consequently, if the data are sufficiently small, the relations (2.3), (2.4) imply exponential decay of the X-norm of the solution to zero.   Q.E.D.

LEMMA 4.2. Assume that a solution u of (1.1) satisfies

$$E(u, u_t)(t) \ge -c_5 \quad \text{for all } t \in [0, T_{\max}). \tag{4.3}$$

Then $T_{\max} = \infty$ and there is a constant c_6 such that

$$\|[u(t), u_t(t)]\|_X \le c_6 \quad \text{for all } t \in R^+. \tag{4.4}$$

Proof. Since the energy is bounded, the relation (1.5) yields

$$\int_0^{T_{\max}} \int_{R^N} u_t^2 \, dx \, dt \le c_7. \tag{4.5}$$

Thus we can estimate the difference

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{L^2} &\le \left\| \int_{t_1}^{t_2} u_t(s) \, ds \right\|_{L^2} \\ &\le \int_{t_1}^{t_2} \|u_t(s)\|_{L^2} \, ds \\ &\le \sqrt{t_2 - t_1} \sqrt{c_7} \quad \text{for all } t_1 < t_2 \le T_{\max}; \end{aligned} \tag{4.6}$$

hence $\|u(t)\|_{L^2}$ is bounded on $[0, T_{\max}]$.

On the other hand, integrating (1.5) with respect to t we get

$$\int_{R^N} |\nabla u(t)|^2 + |u_t(t)|^2 + m|u(t)|^2 \, dx \leq c_8 \left(1 + \int_{R^N} |u(t)|^{\alpha+1} \, dx \right) \qquad (4.7)$$

where the last term may be estimated by means of the Sobolev interpolation inequality

$$\int_{R^N} |v|^{\alpha+1} \, dx \leq c_9 \left(\int_{R^N} |v|^2 \, dx \right)^{\frac{2(\alpha+1)-N(\alpha-1)}{4}} \left(\int_{R^N} |\nabla v|^2 \, dx \right)^{\frac{N(\alpha-1)}{4}}. \qquad (4.8)$$

By virtue of (1.2), we have always $\frac{N(\alpha-1)}{4} < 1$ and, consequently, the relations (4.6), (4.7) yield

$$\|[u(t), u_t(t)]\|_X \leq c_{10} \quad \text{on } [0, T_{\max}]. \qquad (4.9)$$

In accordance with Proposition 2.1 (i), the solution u is defined globally for all t and (4.5) reads

$$\int_0^\infty \int_{R^N} u_t^2 \, dx \, dt \leq c_7. \qquad (4.10)$$

To complete the proof, we have to show that the solution remains bounded in the energy norm. By virtue of (4.7), (4.8), it suffices to prove that

$$\|u(t)\|_{L^2} \leq c_{11} \quad \text{for all } t \in R^+. \qquad (4.11)$$

To this end, we multiply (1.1) by u and integrate by parts to obtain

$$\int_{t_1}^{t_2} \int_{R^N} |\nabla u|^2 + m|u|^2 - |u|^{\alpha+1} \, dx \, dt$$

$$= \int_{R^N} u(t_1)u_t(t_1) - u(t_2)u_t(t_2) \, dx \qquad (4.12)$$

$$+ \int_{t_1}^{t_2} \int_{R^N} u_t^2 \, dx \, dt - d \int_{t_1}^{t_2} \int_{R^N} uu_t \, dx \, dt.$$

Making use of (4.10) and the mean-value theorem we deduce that for an arbitrary time interval $I \subset R^+$ of length 1 there are $t_1 < t_2$ such that

$$\begin{cases} I \subset [t_1, t_2], \quad t_2 - t_1 \leq 3, \\ \|u_t(t_1)\|_{L^2}, \ \|u_t(t_2)\|_{L^2} \leq \sqrt{c_7}. \end{cases} \qquad (4.13)$$

As the energy is nonincreasing, we have

$$\int_{t_1}^{t_2} \int_{R^N} \frac{1}{2}(|\nabla u|^2 + m|u^2| + |u_t|^2) - \frac{1}{\alpha+1}|u|^{\alpha+1} \, dx \, dt \leq (t_2 - t_1)E(u_0, u_1) \leq c_{12}. \qquad (4.14)$$

Combining (4.10), (4.12), and (4.13) together with (4.14) we obtain

$$\int_{t_1}^{t_2} \int_{R^N} |\nabla u|^2 + |u|^2 + |u_t|^2 \, dx \, dt$$

$$\leq c_{13} \left((\|u_t(t_1)\|_{L^2} + \|u_t(t_2)\|_{L^2}) \max_{t \in [t_1, t_2]} \|u(t)\|_{L^2} + 1 \right) \qquad (4.15)$$

$$\leq c_{14} \left(1 + \max_{t \in [t_1, t_2]} \|u(t)\|_{L^2} \right).$$

On the other hand, by means of (4.6), (4.10) and the mean-value theorem we obtain

$$\max_{t \in [t_1, t_2]} \|u(t)\|_{L^2} \leq c_{15} \left(1 + \int_{t_1}^{t_2} \|u(s)\|_{L^2} \, ds \right). \qquad (4.16)$$

Combining (4.15), (4.16) we get the desired estimate

$$\int_{t_1}^{t_2} \int_{R^N} |\nabla u|^2 + |u|^2 + |u_t|^2 \, dx \leq c_{16} \qquad (4.17)$$

for t_1, t_2 as in (4.13). Consequently, any time interval of length 3 contains a point ξ such that

$$\|u(\xi)\|_{L^2} \leq c_{17} \qquad (4.18)$$

which, together with (4.6), completes the proof of (4.11).   Q.E.D.

Consider a couple of sets:

$$\begin{cases} \mathcal{C}_0 \equiv & \{[u_0, u_1] \in X \mid \text{the corresponding solution of (1.1)} \\ & \text{is defined for } t \in R^+ \text{ and} \\ & \qquad\qquad [u(t), u_t(t)] \to 0 \text{ in } X \text{ as } t \to \infty\}, \qquad (4.19) \\ \mathcal{C}_- \equiv & \{[u_0, u_1] \in X \mid \text{the corresponding solution of (1.1)} \\ & \qquad\qquad \text{satisfies } \lim_{t \to T_{\max}} E(u, u_t)(t) < 0\}. \end{cases}$$

As a consequence of Proposition 2.1 (ii), Lemma 4.1, and the continuity of the energy, we get

LEMMA 4.3. Both sets $\mathcal{C}_0$, $\mathcal{C}_-$ are nonvoid and open in X. Moreover,

$$\mathcal{C}_0 \cap \mathcal{C}_- = \varnothing.$$

By virtue of Lemma 4.2, we have

LEMMA 4.4. If

$$[u_0, u_1] \in X \backslash \mathcal{C}_-,$$

then the solution exists for all $t \in R^+$ and

$$\|[u(t), u_t(t)]\|_X \leq c_{18} \quad \text{for all } t \in R^+. \qquad (4.20)$$

Combining Lemma 4.3 and Lemma 4.4 along with a simple topological argument, namely that a connected set cannot be a union of two disjoint nonvoid open sets, we infer

PROPOSITION 4.5. *Let $\chi\colon [0,q] \to X$ be a continuous curve such that $\chi(0) \in \mathcal{C}_0$, $\chi(q) \in \mathcal{C}_-$.*

Then there are initial data $[u_0, u_1]$ lying on χ such that the corresponding solution of (1.1) satisfies

$$0 < c_{19} \leq \|[u(t), u_t(t)]\|_X \leq c_{20} \quad \text{for all } t \in R^+. \tag{4.21}$$

5. The proof of Theorem 1.1. In this section, we complete the proof of Theorem 1.1. Let us start with a statement concerning the solutions of the stationary problem (1.4).

PROPOSITION 5.1. *There exists a unique positive, radially symmetric solution w_g of (1.4) attaining its maximum at the origin. The solution w_g is a ground state, i.e., it minimizes the energy among all nonzero solutions of (1.4):*

$$0 < E(w_g, 0) = \min\{E(w, 0) \mid w \neq 0 \text{ solves } (1.4)\}. \tag{5.1}$$

Moreover, there is a constant $\mu > 0$ such that for any nonzero solution w of (1.4) that is not a spatial shifts of w_g or $-w_g$ we have

$$E(w, 0) > E(w_g, 0) + \mu. \tag{5.2}$$

Proof. The existence part is due to BERESTYCKI-LIONS [3]. They proved that the problem (1.4) possesses a nonzero, positive, radially symmetric and radially decreasing solution w_g that minimizes the energy in the class of nonzero solutions (in fact, their result applies to a large class of nonlinear problems including (1.4)). KWONG [12] showed that a positive, radially symmetric solution of the problem (1.4) is unique up to a spatial shift. Consequently, all we have to prove is the relation (5.2).

Arguing by contradiction we assume that there is a sequence $\{w_n\}$ of solutions of (1.4) that *are not* spatial shifts of w_g or $-w_g$ such that

$$E(w_n, 0) \to E(w_g, 0). \tag{5.3}$$

Multiplying Eq. (1.4) by w_n and integrating by parts we obtain

$$E(w_n, 0) = \left(\frac{1}{2} - \frac{1}{\alpha + 1}\right) \int_{R^N} |\nabla w_n|^2 + m|w_n|^2 \, dx. \tag{5.4}$$

By virtue of (5.3), (5.4), the sequence $\{w_n\}$ is bounded in $H^1(R^N)$ and, consequently, satisfies the hypotheses of Proposition 3.1. From (3.5), (5.3) we deduce that there is exactly one stationary solution, which we denote $\bar{w}$, appearing in (3.3). Consequently, passing to a subsequence and making a spatial shift if necessary, we may assume:

$$w_n \to \bar{w} \text{ in } H^1(R^N) \tag{5.5}$$

where $\bar{w}$ is a solution of (1.4) satisfying

$$E(\bar{w}, 0) = E(w_g, 0). \tag{5.6}$$

Using the standard bootstrap arguments and the L^p-estimates for elliptic problems we deduce that boundedness of the solutions w_n, $n = 1, 2, \ldots$ and $\bar{w}$ in $H^1(R^N)$ implies boundedness in the Sobolev space $W^{3,p}(R^N)$ with $p < \infty$ arbitrary. In particular, $\bar{w}$

is a classical solution of the problem minimizing the energy. According to COLEMAN-GLAZER-MARTIN [5], the function $\bar{w}$ is radially symmetric and radially monotone; hence, by KWONG [12], $\bar{w}$ is a spatial shift of w_g or $-w_g$. Thus, using boundedness of w_n in $W^{3,p}$, we infer

$$w_n \to w_g \ (\text{or} \ -w_g) \ \text{in} \ H^1(R^N) \cap C^2(R^N). \tag{5.7}$$

In what follows, we shall assume convergence to $+w_g$, the other case being treated in a completely analogous way.

Since w_g is strictly positive, we deduce from (5.7) the existence of $r > 0$ such that

$$w_n(x) > 0 \quad \text{for all} \ |x| = r \ \text{and all} \ n \ \text{large enough} \tag{5.8}$$

and

$$-\Delta w_n + b(x) w_n = 0 \quad \text{for} \ |x| > r \tag{5.9}$$

where the potential b is bounded and strictly positive. By virtue of the maximum principle, we deduce from (5.8), (5.9) that

$$w_n(x) > 0 \quad \text{for all} \ x, |x| > r \ \text{and all large} \ n. \tag{5.10}$$

On the other hand, since w_g is positive and the convergence in (5.7) is uniform on compact sets, we get

$$w_n(x) > 0 \quad \text{for all} \ x, |x| \le r. \tag{5.11}$$

Consequently, since w_n are positive everywhere, we can use the celebrated result of GIDAS-NI-NIRENBERG [10] to conclude that w_n are radially symmetric at least for large n. But by the uniqueness result of KWONG [12], any radially symmetric positive solution of (1.4) is a spatial shift of w_g. Thus, all w_n are spatial shifts of w_g in contrast with our original assumption.   Q.E.D.

Our strategy will be to prove Theorem 1.1 using Proposition 4.5. To this end, consider a system of unit vectors

$$e_j = \left[\cos\left(\frac{\pi}{k}j\right), \sin\left(\frac{\pi}{k}j\right) \right], \quad j = 1, \ldots, 2k$$

in $R^2 \subset R^N$. Note that we write $[x_1, x_2]$ instead of $[x_1, x_2, 0, \ldots, 0]$ for simplicity. To this system, we associate a family of hyperplanes

$$\mathcal{H}_i = \{x \in R^N \mid x \cdot h_i = 0, i = 1, \ldots, k\} \tag{5.12}$$

where

$$h_i = \left[-\sin\left(\frac{\pi}{2k}(2i-1)\right), \cos\left(\frac{\pi}{2k}(2i-1)\right) \right], \quad i = 1, \ldots, k.$$

Finally, let U_i, $i = 1, \ldots, k$ denote the unitary transformation of reflection with respect to $\mathcal{H}_i$, i.e.,

$$U_i x = x - 2(x \cdot h_i) h_i, \quad x \in R^N, \ i = 1, \ldots, k.$$

Observe that

$$U_i^2 = \text{Id} \tag{5.13}$$

and

$$U_i \colon \{e_i\}_{i=1}^{2k} \to \{e_i\}_{i=1}^{2k}$$

is a one-to-one mapping such that

$$\text{if } e_l = U_i e_j \text{ for a certain } i, \text{ then } l - j \text{ is an odd integer.} \tag{5.14}$$

Now, consider classes of symmetric functions:

$$\mathcal{S}_i = \{v = v(x) \mid v(U_i x) = -v(x)\}, \quad i = 1, \dots, k.$$

Observe that the $\mathcal{S}_i$ are invariant with respect to the solution semigroup of (1.1), i.e.,

$$u_0, u_1 \in \mathcal{S}_i \text{ implies } u(t), u_t(t) \in \mathcal{S}_i \text{ for all } t \in [0, T_{\max}) \tag{5.15}$$

for any solution u of (1.1). Indeed, the Laplacian commutes with both reflections and the inversion $v \approx -v$ and so does the nonlinearity (it is odd).

Consequently, the set $\mathcal{S} = \bigcap_{i=1}^{k} \mathcal{S}_i$ is invariant, i.e.,

$$u_0, u_1 \in \mathcal{S} \text{ yields } u(t), u_t(t) \in S \text{ for } t \in [0, T_{\max}). \tag{5.16}$$

Consider a function

$$z(\lambda, x) = \sum_{j=1}^{2k} (-1)^j w_g(x - \lambda e_j), \quad \lambda > 0, \ x \in R^N.$$

LEMMA 5.2. We have $z(\lambda, \cdot) \in S$ for any fixed $\lambda > 0$. Moreover, given $\varepsilon > 0$ there is λ_0 such that

$$E(z(\lambda, \cdot), 0) \le 2k E(w_g, 0) + \varepsilon \tag{5.17}$$

for all $\lambda \ge \lambda_0$.

Proof. We have

$$z(\lambda, U_i x) = \sum_{j=1}^{2k} (-1)^j w_g(U_i x - \lambda e_j)$$

where, since w_g is radially symmetric,

$$w_g(U_i x - \lambda e_j) = w_g(U_i(U_i x - \lambda e_j)) = w_g(x - \lambda U_i e_j). \tag{5.18}$$

By virtue of (5.14), (5.18), we get

$$\begin{aligned} z(\lambda, U_i x) &= \sum_{j=1}^{2k} (-1)^j w_g(x - \lambda U_i e_j) \\ &= \sum_{l=1}^{2k} (-1)^{j-l} (-1)^l w_g(x - \lambda e_l) \\ &= -z(\lambda, x), \quad \text{for all } i = 1, \dots, k. \end{aligned} \tag{5.19}$$

To observe (5.17), one has to realize that

$$\operatorname{dist}(\lambda e_j, \lambda e_i) \to \infty, \quad i \ne j, \text{ as } \lambda \to \infty$$

and, consequently,

$$\lim_{\lambda \to \infty} E(z(\lambda, \cdot), 0) = 2kE(w_g, 0). \quad \text{Q.E.D.}$$

Keeping in mind the notation of Proposition 4.5, we consider a curve

$$\begin{cases} \chi \colon [0, q] \to X, \\ \chi(s) = [sz(\lambda, \cdot), 0], \ s \in [0, q] \end{cases} \tag{5.20}$$

with parameters λ, q to be determined below. Obviously,

$$\chi(0) = [0, 0] \in \mathcal{C}_0. \tag{5.21}$$

Computing the energy we obtain

$$E(\chi(s)) = \frac{1}{2}s^2 \int_{R^N} |\nabla z|^2 + m|z|^2 \, dx - \frac{s^{\alpha+1}}{\alpha+1} \int_{R^N} |z|^{\alpha+1} \, dx, \quad s \ge 0, \tag{5.22}$$

and letting $\lambda \to \infty$ we deduce

$$\begin{cases} \int_{R^N} |\nabla z(\lambda, x)|^2 \, dx \to 2k \int_{R^N} |\nabla w_g(x)|^2 \, dx, \\ \int_{R^N} |z(\lambda, x)|^2 \, dx \to 2k \int_{R^N} |w_g(x)|^2 \, dx, \\ \int_{R^N} |z(\lambda, x)|^{\alpha+1} \, dx \to 2k \int_{R^N} |w_g(x)|^{\alpha+1} \, dx. \end{cases} \tag{5.23}$$

On the other hand, w_g is a solution of (1.4) and, consequently,

$$0 < Q \equiv \int_{R^N} |\nabla w_g|^2 + m|w_g|^2 \, dx \equiv \int_{R^N} |w_g|^{\alpha+1} \, dx. \tag{5.24}$$

We deduce from (5.22)–(5.24) that the function

$$s \to E(\chi(s))$$

behaves for large λ like

$$s \to 2kQ \left(\frac{s^2}{2} - \frac{s^{\alpha+1}}{\alpha+1} \right),$$

i.e., attains its maximum at $s \approx 1$ and decays to $-\infty$ as $s \to \infty$. Thus given $\varepsilon > 0$ arbitrary, we can fix the parameters λ, q in such a way that

$$\chi(0) = [0, 0], E(\chi(q)) < 0, \text{ i.e., } \chi(q) \in \mathcal{C}_-,$$
$$\max_{s \ge 0} E(\chi(s)) \le 2kE(w_g, 0) + \varepsilon. \tag{5.25}$$

Applying Proposition 4.5, we find data lying on χ such that the corresponding solution u of the problem (1.1) is defined on R^+ and satisfies

$$0 < c_{19} \le \|[u(t), u_t(t)]\|_X \le c_{20} \quad \text{for all } t \in R^+. \tag{5.26}$$

Moreover, (5.25) yields

$$\lim_{t \to \infty} E(u, u_t)(t) \le 2kE(w_g, 0) + \varepsilon \tag{5.27}$$

and, by virtue of (5.16),

$$u(t), u_t(t) \in \mathcal{S} \quad \text{for all } t \in R^+. \tag{5.28}$$

Our ultimate goal is to show that the solution just obtained has all the properties claimed in the conclusion of Theorem 1.1.

Using Proposition 3.2 we obtain a sequence $t_n \to \infty$ such that $u(t_n)$, $u_t(t_n)$ satisfy (3.14), (3.15). Our first claim is that the function w_1 in (3.14) must be zero. Clearly, we have $w_1 \in \mathcal{S}$ since

$$u(t_n) \to w_1 \quad \text{in } H^1_{\text{loc}}(R^N)$$

and $u(t) \in \mathcal{S}$ for all t. In particular, w_1 is a solution of the problem (1.4) on the half-plane

$$\Omega = \{x \mid x \cdot h_i \geq 0\}$$

satisfying the boundary condition

$$w_1|_{\partial \Omega} \equiv w_1|_{\mathcal{H}_i} = 0.$$

It follows from the nonexistence results of ESTEBAN-LIONS [6] that $w_1 \equiv 0$.

Next, using the relations (3.15), (5.27) and the fact that w_g minimizes the energy in the class of nonzero stationary solutions we deduce

$$\begin{cases} \|u(t_n) - \sum_{j=1}^m w_j(\cdot - x_n^j)\|_{H^1} \to 0, \\ u_t(t_n) \to 0 \text{ in } L^2(R^N) \end{cases} \tag{5.29}$$

where

$$|x_n^j| \to \infty, \quad \text{dist}(x_n^i, x_n^j) \to \infty \quad \text{for } i \neq j \text{ as } n \to \infty, \tag{5.30}$$

$$m \leq 2k \tag{5.31}$$

and w_j are *nonzero* solutions of (1.4).

Now, our aim is to localize the points x_n^j. To this end, we introduce the sectors

$$\mathcal{V}_j = \Big\{x = [x_1, x_2, \ldots, x_N] \in R^N \mid [x_1, x_2] = r[\cos\phi, \sin\phi],$$
$$r \geq 0, \phi \in \Big[(2j-1)\frac{\pi}{2k}, (2j+1)\frac{\pi}{2k}\Big]\Big\}, \qquad j = 1, \ldots, 2k,$$

i.e., $\mathcal{V}_i$, $\mathcal{V}_{i+k}$, $i = 1, \ldots, k$ are the sectors determined by the hyperplanes $\mathcal{H}_i$, $\mathcal{H}_{i+1}$ (with the convention $\mathcal{H}_{k+1} = \mathcal{H}_1$) containing the vectors e_i, e_{i+k} respectively. Observe that

$$R^N = \Big\{\bigcup_{i=1}^k \mathcal{H}_i\Big\} \cup \Big\{\bigcup_{j=1}^{2k} \text{int } \mathcal{V}_j\Big\}, \tag{5.32}$$

$$\text{int } \mathcal{V}_i \cap \text{int } \mathcal{V}_j = \varnothing \quad \text{for } i \neq j. \tag{5.33}$$

LEMMA 5.3. There is a constant $C > 0$ such that for any l and any n there is a j such that

$$\text{dist}(x_n^j, \mathcal{V}_l) \leq C. \tag{5.34}$$

Proof. Assume the contrary, i.e., there is a sequence $n \to \infty$ and $l = l(n)$ such that

$$\min_j \text{dist}(x_n^j, \mathcal{V}_{l(n)}) \to \infty \quad \text{as } n \to \infty. \tag{5.35}$$

Since $l(n)$ ranges only in a finite set, there is an l which appears in (5.35) for infinitely many n. Passing to a subsequence, we may suppose

$$\min_j \text{dist}(x_n^j, \mathcal{V}_l) \to \infty \quad \text{as } n \to \infty. \tag{5.36}$$

On the other hand, since $u(t) \in \mathcal{S}$, we have

$$\|u(t)\|_{H^1(\mathcal{V}_i)} = \|u(t)\|_{H^1(\mathcal{V}_j)} \quad \text{for all } i, j. \tag{5.37}$$

Consequently, the relations (5.29), (5.36) yield

$$\|u(t_n)\|_{H^1(\mathcal{V}_l)} \to 0 \quad \text{as } n \to \infty,$$

which, by virtue of (5.37), implies

$$u(t_n) \to 0 \quad \text{in } H^1(R^N),$$

in contrast with (5.29).   Q.E.D.

LEMMA 5.4. We have

$$\min_{j,l} \operatorname{dist}(x_n^j, \partial \mathcal{V}_l) \to \infty \quad \text{as } n \to \infty. \tag{5.38}$$

Proof. Assume the contrary, i.e., that there is a constant K such that

$$\operatorname{dist}(x_n^{j(n)}, \partial \mathcal{V}_{l(n)}) \leq K.$$

Similarly, as above, since there is only a finite number of indexes j, l, we can find a fixed couple $\bar{j}$, $\bar{l}$ such that

$$\operatorname{dist}(x_n^{\bar{j}}, \partial \mathcal{V}_{\bar{l}}) \leq K \quad \text{for all } n,$$

in particular, there is an $\bar{i}$ such that

$$\operatorname{dist}(x_n^{\bar{j}}, \mathcal{H}_{\bar{i}}) \leq K \quad \text{for } n \to \infty. \tag{5.39}$$

Let P be the orthogonal projection, $P \colon R^N \to \mathcal{H}_{\bar{i}}$. We define a sequence

$$y_n \equiv P x_n^{\bar{j}}.$$

By virtue of (5.30), (5.39), we get

$$x_n^{\bar{j}} - y_n \text{ is convergent as } n \to \infty, \tag{5.40}$$

$$|x_n^j - y_n| \to \infty \quad \text{for } j \neq \bar{j} \text{ as } n \to \infty, \tag{5.41}$$

passing to subsequences as the case may be.

Finally, we deduce from (5.29) that

$$\left\| u(\cdot + y_n, t_n) - w_{\bar{j}}(\cdot - (x_n^{\bar{j}} - y_n)) - \sum_{j, j \neq \bar{j}} w_j(\cdot - (x_n^j - y_n)) \right\|_{H^1} \to 0. \tag{5.42}$$

By virtue of (5.40), we have

$$w_{\bar{j}}(\cdot - (x_n^{\bar{j}} - y^n)) \to \bar{w} \text{ in } H^1(R^N) \text{ as } n \to \infty$$

where $\bar{w}$ is a *nonzero* solution of (1.4).

On the other hand, if $U_{\bar{i}}$ is the reflection with respect to $\mathcal{H}_{\bar{i}}$, we have

$$u(U_{\bar{i}}x + y_n, t_n) = u(U_{\bar{i}}(x + y_n), t_n) = -u(x + y^n, t_n) \tag{5.43}$$

as $u \in \mathcal{S}$ and $U_{\bar{i}}P = P$. By virtue of (5.41), (5.42), we get

$$u(\cdot + y_n, t_n) \to \bar{w} \text{ in } H^1_{\text{loc}}(R^N)$$

which, together with (5.43), yields

$$\bar{w} \equiv 0 \text{ on } H_{\bar{i}} \text{ in the sense of traces.}$$

By the nonexistence result of ESTEBAN-LIONS [6] we get $\bar{w} \equiv 0$ on R^N—a contradiction.   Q.E.D.

It follows from Lemma 5.3, Lemma 5.4 that for all n sufficiently large, each set int $\mathcal{V}_l$ contains at least one point x_n^j, which together with (5.31), (5.33) implies

$$m = 2k. \tag{5.44}$$

By virtue of (3.15), (5.27), we have

$$E(w_j, 0) \leq E(w_g, 0) + \varepsilon \quad \text{for all } j = 1, \ldots, 2k. \tag{5.45}$$

Thus by Proposition 5.1, the relation (5.45) yields that either

$$w_j = w_g \quad \text{or} \quad w_j = -w_g$$

or w_j is a spatial shift of one of the above functions for any j. In any case, the asymptotic behaviour for $t_n \to \infty$ may be described by the formula

$$\|u(t_n) - \sum_{j=1}^{2k} (-1)^j w_g(\cdot - x_n^j)\|_{H^1} \to 0 \quad \text{as } t_n \to \infty, \tag{5.46}$$

$$u_t(t_n) \to 0 \text{ in } L^2(R^N)$$

with x_n^j as in (5.30). Note that the factor $(-1)^j$ comes from the symmetry properties of u and the fact that each sector $\mathcal{V}_l$ contains exactly one point x_n^j for n large.

Repeating the same procedure we see that the same holds for an *arbitrary* sequence $t_n \to \infty$, in particular, the number $2k$ of "waves" in (5.46) is independent of $t_n \to \infty$ and we have always $w_j = \pm w_g$. Thus we can find the functions x_j of the time t satisfying (1.6), (1.7).

Theorem 1.1 has been proved.

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