

A DERIVATION OF DEGENERATE VON KÁRMÁN EQUATIONS FOR STRONGLY ANISOTROPIC PLATES

BY

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1. Introduction. This paper derives a boundary value problem for a nonlinear, degenerate elliptic system of partial differential equations modeling the deformation of an anisotropic plate. The principal result is that stress and displacement for the three-dimensional object can be approximately reconstructed from the solutions of a two-dimensional problem that approximates the three-dimensional problem with error of the order of the thickness of the plate. The derivation is based on that for the von Kármán plate model by Ciarlet in [2]. The classical von Kármán equations are a bidimensional model for the deflection of isotropic plates dating back to von Kármán's work [15] in 1910. Ciarlet's derivation for the isotropic case is the most convincing justification to date for the validity of these equations based on the equations of three-dimensional elasticity. The derivation hinges on the assumption that the Young's modulus of the plate is large with respect to the thickness. Young's modulus is a constitutive constant measuring the stiffness of the plate. For isotropic plates, and plates with only "mild" anisotropy (with slightly different stiffnesses in different directions), the assumption that the Young's modulus is adequately large is not an imposition. If the Young's modulus is unacceptably small, then the "plate" should be thought of as a membrane. On the other hand, consider the case of a "strongly" anisotropic plate, that is, a plate whose stiffness in one direction is very small in comparison with that in the perpendicular direction. A common example is a sheet of corrugated cardboard, which bends much more easily with the corrugations than across them. In this case it may be that the stiffness in one direction is of sufficient magnitude to demand a plate model, while that in the other is more appropriate to a membrane model. This is the situation treated here.

The resulting model includes a fourth-order partial differential operator with positive semi-definite characteristic form. This badly degenerate form of ellipticity we call here elliptic-parabolic, after Fichera's second-order equations, presented in [5], and widely known. Such operators have appeared in linear models for plates that lose stiffness in Weinacht's work [17], and the author's [12]. Greenlee in [7] has examined the related case of a plate/membrane system. To the author's knowledge, this paper is the first derivation

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of a nonlinear model of this form. The companion article, *Boundary value problems for degenerate von Kármán equations*, by the author, presents regularity results that permit a weak formulation of the problem amenable to the application of monotone operator theory like that presented in [1].

In our derivation, we begin with the equations of three-dimensional elastic equilibrium. Readers unfamiliar with these equations might read an introduction to finite elasticity. Stoker's book [14] has attained the stature and longevity of a classic. Marsden and Hughes provide a superb introduction in [9], although with a more geometric flavor than the work at hand. More succinct is Gurtin's monograph [6]. Finally, one should not overlook Ciarlet and Rabier's French exposition in [4]. The first two sections of the first chapter are an admirable introduction to this paper, since the work here is inspired by the derivation that completes the chapter. The work done there is generalized and extended in the important monograph [3].

2. Notation. The region $\Omega^\epsilon \in \mathbf{R}^3$ is the volume occupied by the plate in its reference state. It can be decomposed as $\Omega^\epsilon := \omega \times (-\epsilon, \epsilon)$ where $\omega \in \mathbf{R}^2$ is the outline of the plate: a bounded, open, simply connected set that we assume has a piecewise smooth boundary, and 2ϵ is the uniform thickness of the plate (assumed to be small). We denote the boundary of ω by $\partial\omega =: \gamma$, and decompose the boundary of Ω^ϵ by:

$$\begin{aligned} \Gamma_+^\epsilon &:= \bar{\omega} \times \{\epsilon\}, & \Gamma_\pm^\epsilon &:= \Gamma_+^\epsilon \cup \Gamma_-^\epsilon, \\ \Gamma_0^\epsilon &:= \gamma \times (-\epsilon, \epsilon), & \Gamma_0^\epsilon \cup \Gamma_+^\epsilon \cup \Gamma_-^\epsilon &= \partial\Omega^\epsilon. \\ \Gamma_-^\epsilon &:= \bar{\omega} \times \{-\epsilon\}, \end{aligned}$$

On the boundary of any region (two- or three-dimensional) we denote the unit outer normal by \mathbf{n} with individual components n_i . The corresponding directional derivative is $\frac{\partial}{\partial \mathbf{n}}$. The tangential derivative in the counterclockwise direction around γ is $\frac{d}{ds} := -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2}$.

The three-dimensional vector field $\mathbf{u}(x_1, x_2, x_3) := (u_1, u_2, u_3)$ is the displacement function describing the deformed plate. The component $u_i(x_1, x_2, x_3)$ is the displacement of the point (x_1, x_2, x_3) in the x_i direction in the deformed state. The domain of \mathbf{u} is Ω^ϵ . We use tensor notation for differentiation: $u_{i,j} := \frac{\partial u_i}{\partial x_j}$.

The Cauchy stress tensor associated with the deformation \mathbf{u} is denoted by σ . We represent it in the given coordinates by the symmetric three by three matrix with entries σ_{ij} . The symbol ϵ denotes the strain; the exact definition is given below in the constitutive assumptions in Sec. 4. Note that ϵ and ε are *not* equivalent in our notation.

We indicate the range of an index by the alphabet from which it comes. Greek letters range over 1 and 2, while Latin letters range over 1, 2, and 3. The convention is particularly useful when we employ the summation convention; terms with a repeated index should be summed over the range of that index. We assume this convention below, except where specifically noted. When working with Greek indices we will find it convenient to adopt the notation $\sim\alpha$ for the index not represented by α , e.g., if $\alpha = 1$, then $\sim\alpha = 2$.

3. Statement of the model, and outline of the derivation. The model presented here is based on the classical von Kármán equations for an isotropic plate; they are

$$\frac{2E\epsilon^3}{3(1-\nu^2)}\Delta^2 u - [u, w] = f \quad \text{in } \omega, \quad (1)$$

$$\Delta^2 w + \frac{E}{2}[u, u] = 0 \quad \text{in } \omega. \quad (2)$$

The bilinear form is defined by

$$[u, w] := (-1)^{\alpha+\beta} u_{,\alpha\beta} w_{,\sim\alpha\sim\beta}, \quad (3)$$

while Δ^2 represents the biharmonic operator

$$\Delta^2 u = u_{,1111} + u_{,2222} + 2u_{,1212}.$$

The boundary conditions, assuming a clamped plate, are

$$u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{and} \quad -\frac{d}{ds} w_{,1} = h_2, \quad \frac{d}{ds} w_{,2} = h_1 \quad \text{on } \partial\omega. \quad (4)$$

Here, u is the vertical deflection of the centerplane of the plate under the load f with the forces h_1 and h_2 being applied in the x_1 and x_2 directions respectively, along the edge of the plate. The function w is the accompanying Airy stress function, from which the in-plane (meaning in the centerplane) stress tensor may be deduced. The constants E and ν are the Young's modulus and Poisson's ratio (another constitutive constant; see the constitutive assumptions in the next section), respectively, and ϵ is half the thickness of the plate.

In this paper we assume that the Young's modulus in the x_1 direction is less than that in the x_2 direction by a factor ϵ , half the plate's thickness; we refer to this as an order of magnitude. We derive a two-dimensional problem similar to the von Kármán problem above. The model constructed here is:

$$\frac{\epsilon^2}{3}(Eu_{,2222} + 4Gu_{,1212}) - [u, w] = f \quad \text{in } \omega, \quad (5)$$

$$\frac{1}{E}(w_{,1111} + w_{,2222}) + \frac{1-\nu}{G}w_{,1212} = -(u_{,1}u_{,22})_{,1} \quad \text{in } \omega, \quad (6)$$

under the boundary conditions:

$$\begin{aligned} u &= 0 && \text{on } \partial\omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 && \text{on } \partial\omega \text{ where } n_2 \neq 0, \\ -\frac{d}{ds} w_{,1} &= h_2, && \text{and } \frac{d}{ds} w_{,2} = h_1 \quad \text{on } \partial\omega. \end{aligned} \quad (7)$$

Here we use the same notation as in the von Kármán equations above, but E is the Young's modulus in the x_2 direction, and G is the in-plane shear modulus. The first equation above is an elliptic-parabolic fourth-order equation in the deflection (u here, referred to as u_3 in the derivation) with second-order coefficients given by the Airy stress function. The second equation equates a uniformly elliptic fourth-order operator of the stress function to a degenerate quadratic form acting on u . This form omits exactly

those derivatives over which the elliptic-parabolic equation in the deflection does not give us control. This lends credence to the possibility of solving the derived problem under appropriate conditions; a weak formulation is given in [13].

The three-dimensional problem is carefully described in Sec. 4. The derivation of the two-dimensional problem consists of three steps. We begin in Sec. 5 with a change of variables that maps the plate onto a body with the same outline but with thickness two. In making this change of variables we place weights on the solutions of the problem in the new domain. These weights carry implicit assumptions about the relative sizes of the various components of the solutions. Because of the anisotropy, we cannot simply apply the same weights Ciarlet used for the isotropic von Kármán case.

There are two changes of weights that seem fruitful. One is to assume that the tensile stress in the x_1 direction, σ_{11} , is larger than the other in-plane components of the stress tensor by a factor of the square root of the thickness. It is possible to derive a two-dimensional model with this assumption, but it leaves us with an untenable difficulty. Either we are faced with a doubly degenerate system of equations, for which it is doubtful that a suitable existence theorem can be found, or we must perform a change of variables elongating the plate's outline by a factor of ϵ . Since the model is founded on the assumption that ϵ grows small asymptotically, the change of variables is undesirable. Instead, we assume that the displacement in the x_1 direction, u_1 , is an order of magnitude larger than in the x_2 direction, u_2 . Physically, this is reasonable, since a strongly anisotropic plate undergoes more stretching in the x_1 direction than an isotropic plate under the same load, in order to generate sufficient tension to support the load. At first this seems fruitless, since the von Kármán and related two-dimensional plate models rely on the ability to approximate the deformation with a so-called Kirchhoff-Love displacement. This generally follows from an assumption that the in-plane components of the displacement are an order of magnitude smaller than the deflection, u_3 . However, we find that even with u_1 and u_3 being of the same magnitude, the anisotropy allows us to approximate the displacement with a degenerate sort of Kirchhoff-Love displacement (see Eq. (21)), resulting in a robust bidimensional model.

In the anisotropic problem asymptotic expansions are not useful, since the resulting problem is over simplified. By isolating the terms in which different powers of ϵ appear, the expansions obscure the dependence of the deflection of the plate, u_3 , upon x_1 . So, rather than introduce asymptotic expansions, we construct functions that asymptotically approximate the solutions to the given problem, i.e., an approximate stress σ^0 and approximate displacement \mathbf{u}^0 that solve the three-dimensional problem up to an error of order ϵ .

After the introduction of the approximate solutions, in Sec. 6 we reduce to two dimensions by considering the restriction of the approximating functions to the centerplane of the plate. The approximate displacement is shown to have degenerate Kirchhoff-Love form, and the approximate displacement u_1^0 is found to be independent of x_2 . Finally, the approximating functions are constructed from their restrictions to the centerplane. The approximating nature of the construction is stated in a theorem at the end of the section; it is stated in terms of classical rather than weak solutions since the appropriate spaces for weak solutions are not clear. The boundary conditions for the deflection differ

from the isotropic case (compare (4) with (7)) because of the degeneracy of the model, but these follow naturally from the three-dimensional conditions in this case. The construction of the degenerate boundary value problem described above is carried out in Sec. 7. The independence of u_1^0 from x_2 permits the construction of the equation for the Airy stress function that is uniformly elliptic under a mild restriction on the coefficients of the constitutive equations (see Eq. (15)).

The paper closes with some remarks on the model in Sec. 8. In particular, conditions that must be imposed in order to assure existence of solutions to the derived problem are presented. Further, difficulties in obtaining regularity results for boundary value problems of this type are mentioned, along with some modifications that may circumvent these obstacles.

4. Assumptions. We break our assumptions into three categories: the basic equilibrium assumptions of finite elasticity, the boundary conditions we impose on our deformation, and the constitutive assumptions that describe the “strong” anisotropy under consideration.

1. *Equilibrium assumptions.* We assume that the deformed state of the plate is in equilibrium with the forces described here. We assume the deformation does not alter the force densities given (dead loads). The only body force assumed to be acting on the plate is gravity (this could be generalized without difficulty). Its force density per unit volume will be denoted by $G^\epsilon := (0, 0, -g^\epsilon)$ defined on Ω^ϵ . We assume the boundary forces acting on the faces of the plate are normal to the reference plane: their density per unit area vector is $F^\epsilon := (0, 0, f^\epsilon)$ on Γ_\pm^ϵ . The boundary forces on Γ_0^ϵ are assumed to act only in the plane of the undeformed plate, and are uniformly distributed in height. Thus they can be given as a force density per unit length on γ by $H^\epsilon := (h_1^\epsilon, h_2^\epsilon, 0)$.

We use the standard equilibrium condition for nonlinear elasticity: The forces acting on the body are in equilibrium with the elastic forces described by the Piola-Kirchhoff stress tensor. The mathematical formulation consists of the following two conditions.

- a) In the interior of Ω^ϵ the elastic forces are given by the divergence of the Piola-Kirchhoff stress tensor:

$$G_i^\epsilon + (\sigma_{ij} + \sigma_{kj}u_{i,k})_{,j} = 0 \quad \text{in } \Omega^\epsilon. \quad (8)$$

- b) On the boundary, the resultant force across the boundary is given by the dot product of the Piola-Kirchhoff stress tensor with the unit outer normal of the boundary. On Γ_\pm^ϵ the equilibrium condition is:

$$(\sigma_{i3} + \sigma_{k3}u_{i,k})n_3 = F_i^\epsilon \quad \text{on } \Gamma_\pm^\epsilon. \quad (9)$$

On Γ_0^ϵ the condition can be written

$$\frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} (\sigma_{i\beta} + \sigma_{j\beta}u_{i,j}) dx_3 n_\beta = H_i^\epsilon \quad \text{on } \Gamma_0^\epsilon. \quad (10)$$

2. *Assumptions on the deformation/boundary conditions.* The only assumptions we make on the nature of the deformation are the necessary boundary conditions on Γ_0^ϵ . We

use Ciarlet's conditions for a clamped plate:

$$u_\alpha \text{ independent of } x_3 \text{ for } \alpha = 1, 2 \quad \text{and} \quad u_3 = 0 \text{ on } \Gamma_0^\epsilon. \quad (11)$$

We assume the plate will not undergo any translation as a result of the forces imposed. Movement of the plate as a whole in the x_3 direction is prohibited by the boundary conditions above, but to avoid in-plane translation or rotation, we must impose constraints on the boundary forces:

$$\int_\gamma h_\alpha^\epsilon ds = 0 \quad \text{for } \alpha = 1, 2, \quad \text{and} \quad \int_\gamma h_1^\epsilon x_2 - h_2^\epsilon x_1 ds = 0. \quad (12)$$

These conditions arise only in the introduction of the Airy stress function.

3. *Constitutive assumptions.* We use the usual Almansi strain tensor to describe the strain associated with the deformation \mathbf{u} :

$$\bar{\epsilon}_{ij}(\mathbf{u}) := \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \quad (13)$$

We assume a linear elastic constitutive law; however, we assume the material is “strongly” anisotropic, that is, the stiffness in one of the in-plane directions is “much less” than in the other principal directions. The simplest anisotropy (commonly called “transversely isotropic”, see, e.g., [8]) that permits this phenomenon has the following generalized Hooke's laws:

$$\begin{aligned} \bar{\epsilon}_{11} &= \frac{1}{\epsilon \bar{E}}(\sigma_{11} - \nu \epsilon \sigma_{22} - \nu \epsilon \sigma_{33}), & \bar{\epsilon}_{12} &= \frac{1}{2\bar{G}}\sigma_{12}, \\ \bar{\epsilon}_{22} &= \frac{1}{\bar{E}}(-\nu \sigma_{11} + \sigma_{22} - \bar{\nu} \sigma_{33}), & \bar{\epsilon}_{13} &= \frac{1}{2\bar{G}}\sigma_{13}, \\ \bar{\epsilon}_{33} &= \frac{1}{\bar{E}}(-\nu \sigma_{11} - \bar{\nu} \sigma_{22} + \sigma_{33}), & \bar{\epsilon}_{23} &= \frac{1 + \bar{\nu}}{\bar{E}}\sigma_{23}. \end{aligned} \quad (14)$$

The unbarred constants of the previous section are replaced by “constants” depending on ϵ : $\bar{E} := (1/\epsilon^3)E$ is the Young's modulus in a plane of isotropy (parallel to the (x_2, x_3) -plane); $\epsilon \bar{E}$ is the Young's modulus in the x_1 direction; $\bar{G} := (1/\epsilon^3)G$ is the shear modulus across a plane of isotropy; and $\bar{\nu}$ and ν are the Poisson ratios, in and across a plane of isotropy, respectively. In the corrugated cardboard example suggested in the Introduction, this loss of stiffness in the x_1 direction corresponds to the corrugations running parallel to the x_2 -axis. The introduction of the factor ϵ^{-3} formalizes a physical notion arrived at by comparing plates of various thicknesses under identical loads. A thinner plate must be stiffer than a thicker one to undergo a deformation of the same magnitude. So, we introduce the heuristically chosen weight here, as we do not wish the solution to our problem to change magnitude as ϵ grows small.

The laws given assume a hyperelastic material; the coefficients of the equations are symmetric. Notice that the contributions of σ_{22} and σ_{33} to the strain $\bar{\epsilon}_{11}$ are of the order $O(\epsilon^{-1})$, while the contribution of σ_{11} to the strains $\bar{\epsilon}_{22}$ and $\bar{\epsilon}_{33}$ is of the order $O(1)$. It is possible for hyperelastic constitutive relations to have the strong anisotropy stipulated here, but with Poisson ratios of different magnitudes. The relations above can be modified by replacing ν with $\nu \epsilon^{-r}$. For $0 \leq r \leq 1$, the treatment below obtains, with modifications. A similar statement can be made about the magnitude of the shear modulus G .

For technical reasons we require that

$$\nu < 4G^2 + E. \quad (15)$$

This condition assures the ellipticity of the equation governing the Airy stress function. We will abbreviate Eq. (14) by $\bar{\varepsilon}_{ij} = (A\sigma)_{ij}$, where A can be thought of as the fourth-order tensor of coefficients ($\bar{\varepsilon}_{ij} = A^{ijkl}\sigma_{kl}$). We also assume that all data is square integrable on its domain of definition, that is,

$$g^\epsilon \in L^2(\Omega^\epsilon), \quad h_\alpha^\epsilon \in L^2(\gamma), \quad F_i^\epsilon \in L^2(\Gamma_\pm^\epsilon).$$

5. Formulation of the three-dimensional problem. We translate the problem Eqs. (8), (9), (10), (11), and (14) into a weak problem on a domain independent of epsilon. We begin the weak formulation by introducing the following spaces of test functions. The test space for tensors is

$$\Sigma^\epsilon := \{L^2(\Omega^\epsilon)\}_{\text{symm}}^{3 \times 3},$$

that is, the space of all symmetric three by three matrices with square integrable entries. For deformations we use the space

$$V^\epsilon := \{\mathbf{v} \in \{W^{(1,4)}(\Omega^\epsilon)\}^3 : \mathbf{v} \text{ satisfies (11)}\},$$

in other words, the space of $W^{(1,4)}$ vector fields on Ω^ϵ satisfying the boundary conditions (11), in the sense of trace.

If \mathbf{u} and the stress σ solve Eq. (14), then

$$\int_{\Omega^\epsilon} (u_{i,j} + \frac{1}{2}u_{k,i}u_{k,j}) \tau_{ij} - (A\sigma)_{ij}\tau_{ij} dV = 0 \quad \text{for all } \tau \in \Sigma^\epsilon, \quad (16)$$

and solutions \mathbf{u} and σ of this equation need only belong to V^ϵ and Σ^ϵ , respectively. By integrating Eq. (8) against $\mathbf{v} \in V^\epsilon$, applying the divergence theorem, and using Eqs. (9) and (10) to substitute for boundary values, we arrive at

$$\begin{aligned} & \int_{\Omega^\epsilon} (\sigma_{ij} + \sigma_{kj}u_{i,k})v_{i,j} dV \\ &= - \int_{\Omega^\epsilon} g^\epsilon v_3 dV + \int_{\Gamma_\pm^\epsilon} f^\epsilon v_3 dA + \int_\gamma h_\alpha^\epsilon \left(\int_{-\epsilon}^\epsilon v_\alpha dx_3 \right) dx \quad \text{for all } \mathbf{v} \in V^\epsilon, \end{aligned} \quad (17)$$

and this formulation also makes sense for $\mathbf{u} \in V^\epsilon$, $\sigma \in \Sigma^\epsilon$.

We convert the weak formulation of the problem (16), (17), with boundary conditions (11), into a problem on $\Omega := \Omega^1 = \omega \times (-1, 1)$, with boundary components $\Gamma_0 := \Gamma_0^1$, $\Gamma_+ := \Gamma_+^1$, $\Gamma_- := \Gamma_-^1$, and $\Gamma_\pm := \Gamma_\pm^1$. We introduce the spaces $V := V^1$, $\Sigma := \Sigma^1$, and the linear bijection between $\bar{\Omega}$ and $\bar{\Omega}^\epsilon$ given by associating any $x := (x_1, x_2, \chi) \in \Omega$ to $x^\epsilon = (x_1, x_2, \epsilon\chi) \in \Omega^\epsilon$. This correspondence induces the following transformations of vector fields and symmetric tensors from the domain Ω^ϵ to Ω . For any vector field \mathbf{v} defined on Ω^ϵ we can construct a \mathbf{v}^ϵ defined on Ω so that

$$v_2(x^\epsilon) = \epsilon^2 v_2^\epsilon(x): \quad v_i(x^\epsilon) = \epsilon v_i^\epsilon(x) \quad \text{for } i = 1, 3.$$

Similarly, any symmetric tensor τ with domain Ω^ϵ can be expressed in terms of a corresponding tensor τ^ϵ on Ω using the weights

$$\begin{aligned}\tau_{\alpha\beta}(x^\epsilon) &= \frac{1}{\epsilon} \tau_{\alpha\beta}^\epsilon(x) \quad \text{for } \alpha, \beta = 1, 2; & \tau_{\alpha 3}(x^\epsilon) &= \tau_{\alpha 3}^\epsilon(x) \quad \text{for } \alpha = 1, 2; \\ \tau_{33}(x^\epsilon) &= \epsilon \tau_{33}^\epsilon(x).\end{aligned}$$

These weights are based on those used in [2], as mentioned in Sec. 3. We assume that the various components of the solutions of the transformed problem, \mathbf{u}^ϵ and σ^ϵ are of comparable magnitudes. Notice, this is an implicit assumption about the relative magnitudes of the original displacement's and stress's components. For instance, assuming that $u_2^\epsilon = \frac{1}{\epsilon^2} u_2$ is comparable in size to $u_3^\epsilon = \frac{1}{\epsilon} u_3$ is an assertion that u_2 is smaller than u_3 by a factor of ϵ . Under the change in variables, integrals in Ω^ϵ are equal to the integrals of the corresponding functions in Ω provided we use the appropriate differentials.

We now apply this change of coordinates to Eq. (16). Recall $\bar{E} = E/\epsilon^3$ and $\bar{G} = G/\epsilon^3$. Dividing through by ϵ^2 , we arrive at

$$\begin{aligned}& \int_{\Omega} \frac{1}{E} \left[\tau_{11}^\epsilon \left(\frac{1}{\epsilon} \sigma_{11}^\epsilon - \nu \sigma_{22}^\epsilon \right) + \tau_{22}^\epsilon (\sigma_{22}^\epsilon - \nu \sigma_{11}^\epsilon) \right] \\ & \quad + \frac{1}{2G} (\tau_{12}^\epsilon \sigma_{12}^\epsilon + \tau_{21}^\epsilon \sigma_{21}^\epsilon) - \epsilon^2 \mathcal{B}_2(\tau^\epsilon, \sigma^\epsilon) - \epsilon^4 \tau_{33}^\epsilon \sigma_{33}^\epsilon dV \\ & = \int_{\Omega} \tau_{ij}^\epsilon \left[\frac{1}{\epsilon_i^1} u_{i,j}^\epsilon + \frac{1}{2} (u_{1,i}^\epsilon u_{1,j}^\epsilon + u_{3,i}^\epsilon u_{3,j}^\epsilon) + \frac{\epsilon^2}{2} u_{2,i}^\epsilon u_{2,j}^\epsilon \right] dV\end{aligned}\tag{18}$$

for all $\tau \in \Sigma$, where the bilinear form is

$$\begin{aligned}\mathcal{B}_2(\tau, \sigma) &= \frac{\nu}{E} (\tau_{11} \sigma_{33} + \sigma_{11} \tau_{33}) + \frac{\bar{\nu}}{E} (\tau_{22} \sigma_{33} + \sigma_{22} \tau_{33}) \\ & \quad - \frac{1}{2G} (\tau_{13} \sigma_{13} + \tau_{31} \sigma_{31}) - \frac{1 + \bar{\nu}}{E} (\tau_{23} \sigma_{23} + \tau_{32} \sigma_{32}),\end{aligned}$$

and ϵ_i^1 is defined by

$$\epsilon_i^1 = \begin{cases} \epsilon & \text{if } i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

We now treat Eq. (17) in the same way; first defining analogues of g^ϵ , f^ϵ , and h_α^ϵ on Ω by

$$g^\epsilon(x^\epsilon) = g(x), \quad f^\epsilon(x^\epsilon) = \epsilon f(x), \quad h_\alpha^\epsilon(x^\epsilon) = \frac{1}{\epsilon} h_\alpha(x).$$

The new functions h_α will satisfy the conditions (12), since h_α^ϵ did. Converting the equilibrium equation (17) into its corresponding equation on Ω , we arrive at

$$\begin{aligned}& \int_{\Omega} \sigma_{ij}^\epsilon \left(\frac{1}{\epsilon_i^1} v_{i,j}^\epsilon + u_{1,i}^\epsilon v_{1,j}^\epsilon + u_{3,i}^\epsilon v_{3,j}^\epsilon + \epsilon^2 u_{2,i}^\epsilon v_{2,j}^\epsilon \right) dV \\ & = - \int_{\Omega} g v_3^\epsilon dV + \int_{\Gamma_{\pm}} f v_3^\epsilon dA \\ & \quad + \int_{\gamma} \frac{1}{\epsilon} h_1 \int_{-1}^1 v_1^\epsilon d\chi + h_2 \int_{-1}^1 v_2^\epsilon d\chi ds \quad \text{for all } \mathbf{v} \in V.\end{aligned}\tag{19}$$

The weak formulation and change of variables just completed places all dependence of the problem upon ϵ in the solutions, $\sigma^\epsilon \in \Sigma$ and $\mathbf{u}^\epsilon \in V$. We now introduce a two-dimensional problem from whose solutions we can reconstruct approximations to the solutions of this three-dimensional problem. The asymptotic nature of this approximate solution to the given problem is: the approximating functions solve the three-dimensional problem up to an error $O(\epsilon)$. The approximations will be denoted \mathbf{u}^0 and σ^0 .

6. Derivation of the two-dimensional problem. From the three-dimensional problem Eqs. (18), (19), with boundary conditions (11) imposed on \mathbf{u}^ϵ , we wish to construct the two-dimensional problem, which we can then convert to the system of equations presented in Sec. 3. We begin by introducing the restrictions of the three-dimensional approximations to the center-plane of the plate:

$$\bar{u}_i(x_1, x_2) := u_i^0(x_1, x_2, 0) \quad \text{and} \quad \bar{\sigma}_{ij}(x_1, x_2) := \sigma_{ij}^0(x_1, x_2, 0).$$

Now we express the three-dimensional approximations in terms of the two-dimensional functions, and find an appropriate bidimensional problem that these new functions solve. Notice that Eq. (18), by considering test functions for which $\tau_{13} = \tau_{31}$ are the only nonzero entries, implies

$$u_{1,3}^\epsilon + \epsilon(u_{3,1}^\epsilon + u_{1,3}^\epsilon u_{1,1}^\epsilon + u_{3,1}^\epsilon u_{3,3}^\epsilon) = O(\epsilon^3) \quad \text{a.e. in } \Omega,$$

after multiplying through by ϵ . Solving for $u_{1,3}^\epsilon$, and applying the expansion

$$\frac{1}{1 + \epsilon u_{1,1}^\epsilon} = \sum_{k=0}^{\infty} (-\epsilon u_{1,1}^\epsilon)^k,$$

(which holds formally for ϵ sufficiently small) we find

$$u_{1,3}^\epsilon = -\frac{\epsilon u_{3,1}^\epsilon (1 + u_{3,3}^\epsilon)}{1 + \epsilon u_{1,1}^\epsilon} + O(\epsilon^3) = -\epsilon u_{3,1}^\epsilon (1 + u_{3,3}^\epsilon) + O(\epsilon^2). \quad (20)$$

In short, it must be that $u_{1,3}^\epsilon = O(\epsilon)$. If we return to Eq. (18), inserting test functions for which only τ_{33} is nonzero gives

$$u_{3,3}^\epsilon (2 + u_{3,3}^\epsilon) = -(u_{1,3}^\epsilon)^2 + O(\epsilon^2) = O(\epsilon^2)$$

by (20). Thus we require of our approximating function, $u_{33}^0 = 0$ or $u_{3,3}^0 = -2$ a.e. in Ω . Assuming $u_{3,3}^0$ is continuous (and not merely in $L^4(\Omega)$), the boundary conditions (11) imply that it must vanish in $\bar{\Omega}$ (since it is zero on Γ_0). So, $u_{3,3}^0$ is constant in the x_3 direction, or $u_{3,3}^0(x_1, x_2, x_3) = \bar{u}_3(x_1, x_2)$. Using the fact that $u_{3,3}^\epsilon = O(\epsilon^2)$ in Eq. (20) we have $u_{1,3}^\epsilon = -\epsilon u_{3,1}^\epsilon + O(\epsilon^2)$, which permits us to assume that $u_{1,3}^0 = -\epsilon \bar{u}_{3,1}$. By inserting test functions with only $\tau_{23} = \tau_{32}$ nonzero in (18), we find that $u_{2,3}^0 = -\bar{u}_{3,2}$ by analogous reasoning. So, the three-dimensional displacement \mathbf{u}^ϵ can be asymptotically approximated with error $O(\epsilon^2)$ by a “degenerate” Kirchhoff-Love displacement satisfying

$$\begin{aligned} u_1^0(x_1, x_2, x_3) &= \bar{u}_1(x_1, x_2) - \epsilon x_3 \bar{u}_{3,1}(x_1, x_2), \\ u_2^0(x_1, x_2, x_3) &= \bar{u}_2(x_1, x_2) - x_3 \bar{u}_{3,2}(x_1, x_2), \\ u_3^0(x_1, x_2, x_3) &= \bar{u}_3(x_1, x_2). \end{aligned} \quad (21)$$

We will eventually take $u_1^0(x_1, x_2, x_3) = \bar{u}_1(x_1, x_2)$ as our approximating function, since we regard ϵ as asymptotically small. We use the form in Eq. (21) through the derivation of the in-plane stress components (where an $\epsilon^{-1}u_{1,2}^\epsilon$ term appears). Thus, we have in the end approximated u_1^ϵ only up to order ϵ .

Further, applying the boundary conditions (11) to the approximate displacement, we have $u_{\alpha,3}^0 = 0$ on Γ_0 (in the sense of trace); so $u_{3,2}^0 = \bar{u}_{3,2} = 0$, as well as $u_3^0 = \bar{u}_3 = 0$ on Γ_0 or γ as appropriate. Since ϵ is considered asymptotically small, the boundary conditions give us no control over $\bar{u}_{3,1}$ on γ . Instead, knowing that $\bar{u}_3 \equiv 0$ on γ we use that its tangential derivative, $d\bar{u}_3/ds$, vanishes there also. Thus we have

$$0 = \frac{d\bar{u}_3}{ds} = -n_2\bar{u}_{3,1} + n_1\bar{u}_{3,2} = -n_2\bar{u}_{3,1},$$

since $\bar{u}_{3,2} = 0$ on γ . So, $\bar{u}_{3,1} = 0$ on portions of the boundary where $n_2 \neq 0$. Our boundary conditions on \bar{u}_3 can be written:

$$\begin{aligned} \bar{u}_3 &= 0 \quad \text{on all of } \gamma, \\ \frac{\partial \bar{u}_3}{\partial \mathbf{n}} &= 0 \quad \text{on } \gamma \text{ where } n_2 \neq 0. \end{aligned} \tag{22}$$

These are precisely the boundary conditions necessary for a well-posed problem in the linear case for an equation of the form (5).

Now, we use the cases where all entries but a fixed $\tau_{\alpha\beta}$ in (18) vanish to express the approximate in-plane stress, $\sigma_{\alpha\beta}^0$, in terms of $\bar{\mathbf{u}}$. Direct substitution yields weak forms of

$$\begin{aligned} \frac{1}{\epsilon}\sigma_{11}^\epsilon - \nu\sigma_{22}^\epsilon &= E \left[\frac{1}{\epsilon}u_{1,1}^\epsilon + \frac{1}{2}(u_{1,1}^\epsilon)^2 + (u_{3,1}^\epsilon)^2 \right] + O(\epsilon^2), \\ -\nu\sigma_{11}^\epsilon + \sigma_{22}^\epsilon &= E \left[u_{2,2}^\epsilon + \frac{1}{2}(u_{1,2}^\epsilon)^2 + (u_{3,2}^\epsilon)^2 \right] + O(\epsilon^2), \\ \sigma_{12}^\epsilon &= G \left[\frac{1}{\epsilon}u_{1,2}^\epsilon + u_{2,1}^\epsilon + u_{1,1}^\epsilon u_{1,2}^\epsilon + u_{3,1}^\epsilon u_{3,2}^\epsilon \right] + O(\epsilon^2). \end{aligned}$$

So that we may substitute \mathbf{u}^0 for \mathbf{u}^ϵ as defined in (21) into these equations, we must assume more regularity on \bar{u}_3 , namely $\bar{u}_3 \in H^2(\omega)$. Making this assumption and substitution will lower the order of the error in the first and third of these equations to $O(\epsilon)$. Solving the third equation above (after substitution) for $u_{1,2}^0$, we find that it is $O(\epsilon)$. Since this is the order of u_1^0 's approximation to u_1^ϵ , we assume that $0 = u_{1,2}^0(x_1, x_2, x_3) = \bar{u}_{1,2}(x_1, x_2)$, i.e., \bar{u}_1 depends only on x_1 . Using this fact, and Eq. (21), we find that the stresses $\sigma_{\alpha\beta}^\epsilon$ are approximated up to order ϵ by linear functions of x_3 . We are led to define $\sigma_{\alpha\beta}^0 = \bar{\sigma}_{\alpha\beta} + x_3 3m_{\alpha\beta}$ with the coefficients

$$\begin{aligned} \bar{\sigma}_{11} &:= E\bar{u}_{1,1}, & m_{11} &:= 0, \\ \bar{\sigma}_{22} &:= E \left[\bar{u}_{2,2} + \nu\bar{u}_{1,1} + \frac{1}{2}(\bar{u}_{3,2})^2 \right], & m_{22} &:= -\frac{E}{3}\bar{u}_{3,22}, \\ \bar{\sigma}_{12} &:= G(\bar{u}_{2,1} + \bar{u}_{3,1}\bar{u}_{3,2}), & m_{12} &:= -\frac{2G}{3}\bar{u}_{3,12}. \end{aligned} \tag{23}$$

Notice that these equations make sense for $\bar{\mathbf{u}}$ satisfying $\bar{u}_\alpha \in H^1(\omega)$, $\bar{u}_3 \in W^{(1,4)}(\omega)$, and $\bar{u}_{3,2} \in H^1(\omega)$. Further, the boundary conditions (22) hold in the sense of trace for all \bar{u}_3

in the closure of $C_0^\infty(\omega)$ under the norm $\|v\|_{W^{(1,4)}(\omega)} + \|v,2\|_{H^1(\omega)}$; call this Banach space U , and define

$$U = \{\mathbf{u}: u_\alpha \in H^1(\omega) \text{ for } \alpha = 1, 2, \text{ and } u_3 \in U\}.$$

We will see that this is the appropriate space for solutions of the two-dimensional problem.

Now consider $\tau \in \Sigma$, a test function; decompose it into its even and odd parts

$$\begin{aligned} \tau_{\alpha\beta}^e(x_1, x_2, x_3) &:= \frac{1}{2}(\tau_{\alpha\beta}(x_1, x_2, x_3) + \tau_{\alpha\beta}(x_1, x_2, -x_3)), \\ \tau_{\alpha\beta}^o(x_1, x_2, x_3) &:= \frac{1}{2}(\tau_{\alpha\beta}(x_1, x_2, x_3) - \tau_{\alpha\beta}(x_1, x_2, -x_3)), \end{aligned}$$

with average values defined by

$$\begin{aligned} \bar{\tau}_{\alpha\beta}^e(x_1, x_2) &:= \frac{1}{2} \int_{-1}^1 \tau_{\alpha\beta}^e(x_1, x_2, x_3) dx_3, \\ \bar{\tau}_{\alpha\beta}^o(x_1, x_2) &:= \frac{3}{2} \int_{-1}^1 x_3 \tau_{\alpha\beta}^o(x_1, x_2, x_3) dx_3. \end{aligned}$$

Using this decomposition, and the linearity of the in-plane stress tensor in x_3 , yields

$$\begin{aligned} &\int_{\Omega} \tau_{\alpha\beta}(x_1, x_2, x_3) \sigma_{\alpha\beta}^0(x_1, x_2, x_3) dV \\ &= 2 \int_{\omega} \bar{\tau}_{\alpha\beta}^e(x_1, x_2) \bar{\sigma}_{\alpha\beta}(x_1, x_2) dA + 2 \int_{\omega} \bar{\tau}_{\alpha\beta}^o(x_1, x_2) m_{\alpha\beta}(x_1, x_2) dA. \end{aligned}$$

Thus, Eq. (18), with the approximate stress and displacement substituted, has an equivalent statement in two dimensions, and the given approximations satisfy it up to $O(\epsilon)$.

To determine equilibrium conditions imposed on the bidimensional functions, $\bar{\mathbf{u}}$, $\bar{\sigma}_{\alpha\beta}$, and $m_{\alpha\beta}$, insert a test function whose only nonzero entry is $v_1(x_1, x_2, x_3) = \bar{v}(x_1, x_2)$ into (19), to find

$$\int_{\Omega} \sigma_{11}^\epsilon \left(\frac{1}{\epsilon} \bar{v},_1 + u_{1,1}^\epsilon \bar{v},_1 \right) + \sigma_{12}^\epsilon \left(\frac{1}{\epsilon} \bar{v},_2 + u_{1,1}^\epsilon \bar{v},_2 \right) dV = 2 \int_{\gamma} \frac{h_1}{\epsilon} \bar{v} ds.$$

Here \bar{v} can be any $H^1(\omega)$ function. Multiplying through by ϵ , and substituting the approximating functions for \mathbf{u}^ϵ and σ^ϵ , we arrive at

$$\int_{\Omega} \sigma_{1\alpha}^0 \bar{v},_\alpha dV = 2 \int_{\gamma} h_1 \bar{v} ds + O(\epsilon).$$

We require this equality to hold exactly, that is, without the order ϵ error. Using the form for σ^0 given by Eq. (23), integrate out the x_3 variable to arrive at

$$\int_{\omega} \bar{\sigma}_{\beta\alpha} \bar{v},_\alpha dA = \int_{\gamma} h_\beta \bar{v} ds \quad (24)$$

for $\beta = 1$. The proof for $\beta = 2$ is analogous. Notice that this equation makes sense not only for $\bar{v} \in W^{(1,4)}(\omega)$, as $v \in V$ would imply, but for \bar{v} in the larger space $\bar{v} \in H^1(\omega)$. This is the space of test functions that we will use in formulating our two-dimensional problem. Notice that it follows from the definitions (23) that this equation makes sense even if $u_\alpha \in H^1(\omega)$ so long as $u_3 \in W^{(1,4)}(\omega)$.

Define a space of test functions on ω by

$$\bar{V} = \{\bar{v} \in W^{(2,4)}(\omega): (22) \text{ holds for } \bar{u}_3 = \bar{v}\}.$$

Inserting into (19) those test functions with the only nonzero entry $v_1(x_1, x_2, x_3) = -x_3 \bar{v}_{,1}(x_1, x_2)$ for some $\bar{v} \in \bar{V}$, we have

$$\int_{\Omega} -x_3 (\sigma_{11}^{\epsilon} \bar{v}_{,11} + \sigma_{12}^{\epsilon} \bar{v}_{,12}) \left(\frac{1}{\epsilon} + u_{1,1}^{\epsilon} \right) - \sigma_{13}^{\epsilon} \bar{v}_{,1} \left(\frac{1}{\epsilon} + u_{1,1}^{\epsilon} \right) dV = 0.$$

Multiplying through by ϵ , and replacing the $\sigma_{\alpha\beta}^{\epsilon}$ with their approximations we find that

$$\int_{\Omega} \sigma_{13}^{\epsilon} \bar{v}_{,1} dV = \int_{\Omega} -x_3 \sigma_{1\alpha}^0 \bar{v}_{,1\alpha} dV + O(\epsilon).$$

We require this equation to hold exactly for the approximation σ_{13}^0 , and by integrating out the x_3 on the right-hand side find

$$\int_{\Omega} \sigma_{\beta 3}^0 \bar{v}_{, \beta} dV = -2 \int_{\omega} m_{\beta\alpha} \bar{v}_{, \beta\alpha} dA \quad \text{for all } \bar{v} \in \bar{V} \quad (25)$$

for $\beta = 1$, but again the equation follows in a similar fashion for $\beta = 2$.

We now substitute $\mathbf{v}^{\epsilon} = (0, 0, \bar{v}(x_1, x_2))$, and \mathbf{u}^0 and σ^0 for \mathbf{u}^{ϵ} and σ^{ϵ} in Eq. (19), with \bar{v} from \bar{V} . Defining

$$\bar{f}(x_1, x_2) := \frac{1}{2} \left(- \int_{-1}^1 g(x_1, x_2, x_3) dx_3 + f(x_1, x_2, 1) + f(x_1, x_2, -1) \right),$$

and using (25) to eliminate the $\sigma_{3\alpha}^0$ terms, we have that, up to an error of order ϵ ,

$$\int_{\omega} -m_{\alpha\beta} \bar{v}_{, \alpha\beta} + \bar{\sigma}_{\alpha\beta} \bar{u}_{3, \alpha} \bar{v}_{, \beta} dA = \int_{\omega} \bar{f} \bar{v} dA, \quad (26)$$

for all $\bar{v} \in \bar{V}$; but we require this equation to hold exactly for our two-dimensional approximating functions.

We require: Eq. (23) to define $\bar{\sigma}$ and m ; Eq. (24) to hold for any $\bar{v} \in H^1(\omega)$; and Eq. (26) to hold for any $\bar{v} \in U$ for some $\mathbf{u} \in \mathcal{U}$. This is the two-dimensional problem approximating the original three-dimensional problem given by Eqs. (18), (19), with boundary conditions (11). It is clear that we can construct the approximating functions \mathbf{u}^0 and $\sigma_{\alpha\beta}^0$ from the solutions $\bar{\mathbf{u}}$, $\bar{\sigma}$, and m , using Eq. (21) and the definitions of $\bar{\sigma}$ and m (just before Eq. (23)). To complete the approximation, it remains to construct σ_{i3}^0 (for $i = 1, 2, 3$) in terms of $\bar{\mathbf{u}}$. To accomplish this, we must assume more regularity on \bar{u}_3 . We specify the further assumptions in the statement of the theorem following the constructions.

To construct the $\sigma_{\alpha 3}^0$, we substitute into Eq. (19) \mathbf{v} from V with only $v_1 = v$ nonzero. Decomposing v into v^e and v^o as we did with τ above, we note that (11) implies

$$\left. \begin{aligned} \bar{v}^e(x_1, x_2) = v^e(x_1, x_2, x_3) = v \quad \text{and} \\ \bar{v}^o(x_1, x_2) = v^o(x_1, x_2, x_3) \equiv 0 \end{aligned} \right\} \quad \text{on } \gamma \text{ or } \Gamma_0 \text{ as appropriate.} \quad (27)$$

Thus (19) yields

$$\int_{\Omega} \sigma_{13}^{\epsilon} v_{,3} dV = - \int_{\Omega} \sigma_{1\alpha}^{\epsilon} v_{, \alpha} dV + \int_{\gamma} h_1 \int_{-1}^1 v dx_3 ds + O(\epsilon).$$

If we require this equation to hold without the order ϵ error for the approximating functions σ_{1j}^0 , then we can integrate out the x_3 variable on the right-hand side to arrive

at:

$$\int_{\Omega} \sigma_{13}^0 v_{,3} dV = -2 \int_{\omega} \bar{\sigma}_{1\alpha} \bar{v}_{,\alpha}^e + m_{1\alpha} \bar{v}_{,\alpha}^o dA + 2 \int_{\gamma} h_1 \bar{v}^e ds.$$

By Eq. (24), the area integral against \bar{v}^e and the boundary integral cancel, and assuming m is sufficiently regular we can apply the divergence theorem to show

$$\begin{aligned} \int_{\Omega} \sigma_{13}^0 v_{,3} dV &= -3 \int_{\Omega} x_3 m_{1\alpha}(x_1, x_2) \frac{1}{2} (v_{,\alpha}(x_1, x_2, x_3) - v_{,\alpha}(x_1, x_2, -x_3)) dV \\ &= 3 \int_{\Omega} x_3 m_{1\alpha,\alpha}(x_1, x_2) v(x_1, x_2, x_3) dV. \end{aligned} \quad (28)$$

A similar result can be derived for σ_{23}^ϵ . Since this is true for all $v \in C_0^\infty(\Omega)$ (this is a subset of V), we have that $\sigma_{\beta 3,3}^0(x_1, x_2, x_3) = -3[x_3 m_{\beta\alpha,\alpha}(x_1, x_2)]$ on Ω . Applying the divergence theorem to the left side of (28) for test functions that include Γ_{\pm} in their support, we see that $\sigma_{\beta 3}^0 \equiv 0$ on Γ_{\pm} . Thus, we have

$$\sigma_{\beta 3}^0(x_1, x_2, x_3) = -3 \int_{-1}^{x_3} \chi m_{\beta\alpha,\alpha}(x_1, x_2) d\chi = \frac{3}{2}(1 - x_3^2) m_{\beta\alpha,\alpha}(x_1, x_2). \quad (29)$$

This leaves only σ_{33}^0 to be computed. We begin by inserting a test function $\mathbf{v} \in V$ into Eq. (19) whose only nonzero entry is $v_3 = v$ satisfying $v|_{\Gamma_0} \equiv 0$. Isolating the σ_{33}^ϵ term, we have

$$\int_{\Omega} \sigma_{33}^\epsilon v_{,3} dV = \int_{\Omega} \sigma_{3\alpha}^\epsilon v_{,\alpha} + \sigma_{\alpha j}^\epsilon u_{3,\alpha}^\epsilon v_{,j} - gv dV + \int_{\Gamma_{\pm}} f v dA + O(\epsilon).$$

Using σ^0 and \mathbf{u}^0 in place of the solutions on the right-hand side does not increase the order of the error; so we require that the approximating function σ_{33}^0 satisfies

$$\int_{\Omega} \sigma_{33}^0 v_{,3} dV = \int_{\Omega} [\sigma_{3\alpha}^0 + (\sigma_{\alpha j}^0 u_{3,\alpha}^0)_{,j} - g] v dV + \int_{\Gamma_{\pm}} f v dA,$$

which follows from the substitution via the divergence theorem, using that $\sigma_{\alpha 3}^0 \equiv 0$ on Γ_{\pm} . So,

$$\begin{aligned} \sigma_{33,3}^0 &= -[\sigma_{3\alpha}^0 + \sigma_{\alpha 3,3}^0 u_{3,\alpha}^0 + (\sigma_{\alpha\beta}^0 u_{3,\alpha}^0)_{,\beta} - g] \quad \text{in } \Omega, \\ \sigma_{33}^0 &= \pm f \quad \text{on } \Gamma_+ \text{ and } \Gamma_-, \text{ respectively.} \end{aligned}$$

Substitute into the first equation above Eq. (29) and the definition of $\bar{\sigma}$ and m to find

$$\sigma_{33,3}^0 = - \left[\frac{3}{2} m_{\alpha\beta,\alpha\beta} (1 - x_3^2) + 3x_3 m_{\alpha\beta} \bar{u}_{3,\alpha\beta} - (\bar{\sigma}_{\alpha\beta} \bar{u}_{3,\alpha})_{,\beta} \right].$$

By substituting this into the trivial identity

$$\begin{aligned} \sigma_{33}^0(x_1, x_2, x_3) &= \frac{1}{2} \left(\sigma_{33}^0(x_1, x_2, -1) + \sigma_{33}^0(x_1, x_2, 1) + \int_{-1}^{x_3} \sigma_{33,3}^0(x_1, x_2, \chi) d\chi \right. \\ &\quad \left. - \int_{x_3}^1 \sigma_{33,3}^0(x_1, x_2, \chi) d\chi \right), \end{aligned}$$

we arrive at the expression

$$\sigma_{33}^0(x_1, x_2, x_3) = \frac{1}{2} \left[f(x_1, x_2, 1) - f(x_1, x_2, -1) + \left(\int_{-1}^{x_3} - \int_{x_3}^1 \right) g(x_1, x_2, \chi) d\chi \right. \\ \left. + (x_3^3 - 3x_3)m_{\alpha\beta, \alpha\beta} + 3(1 - x_3^2)m_{\alpha\beta} \bar{u}_{3, \alpha\beta} - 2x_3(\bar{\sigma}_{\alpha\beta} \bar{u}_{3, \alpha})_{, \beta} \right].$$

This used the desired boundary values: $\sigma_{33}^0(x_1, x_2, \pm 1) = \pm f(x_1, x_2, \pm 1)$, for substitution into the identity. However, it is not immediately evident that the given expression satisfies these conditions. By evaluating at $x_3 = 1$, we find

$$\sigma_{33}^0(x_1, x_2, 1) = f(x_1, x_2, 1) - \bar{f}(x_1, x_2) - [m_{\alpha\beta, \alpha\beta} + (\bar{\sigma}_{\alpha\beta} \bar{u}_{3, \alpha})_{, \beta}].$$

Now, from Eq. (26), by an application of the divergence theorem, we find that the bracketed term above is $-\bar{f}$; so $\sigma_{33}^0(x_1, x_2, 1) = f(x_1, x_2, 1)$. The boundary condition at $x_3 = -1$ follows in the same way, and we have consistent constructions for all of the three-dimensional approximating functions.

Using the identities (23) we can write all these functions in terms of the displacement $\bar{\mathbf{u}}$. The following theorem summarizes our result.

THEOREM. Let $\bar{\mathbf{u}}$ be a vector field in \mathcal{U} with \bar{u}_1 independent of x_2 , and suppose that $\bar{\mathbf{u}}$ solves Eqs. (24) for any $v \in H^1(\omega)$ and (26) for any $v \in \mathcal{U}$, using the identities (23) to define $\bar{\sigma}_{\alpha\beta}$ and $m_{\alpha\beta}$. Then the following functions on Ω can be constructed:

$$u_1^0 = \bar{u}_1, \quad u_2^0 = \bar{u}_2 - x_3 \bar{u}_{3,2}, \quad u_3^0 = \bar{u}_3, \\ \sigma_{11}^0 = E \bar{u}_{1,1}, \quad \sigma_{12}^0 = G(\bar{u}_{2,1} + \bar{u}_{3,1} \bar{u}_{3,2} - 2x_3 \bar{u}_{3,12}), \\ \sigma_{22}^0 = E(\bar{u}_{2,2} + \nu \bar{u}_{1,1} + \frac{1}{2}(\bar{u}_{3,2})^2 - x_3 \bar{u}_{3,22}).$$

If further $\bar{\mathbf{u}}$ has the following regularity: $\bar{u}_\alpha \in W^{(1,4)}(\omega)$ for $\alpha = 1, 2$; $\bar{u}_{1,11}, \bar{u}_{2,12}, \bar{u}_{2,22}$, and $\bar{u}_{3,11} \in L^4(\omega)$; and $\nabla v, \nabla v_{,2} \in L^2(\omega)$ for both $v = \bar{u}_{3,12}$ and $\bar{u}_{3,22}$, then the missing approximate stresses are given by

$$\sigma_{13}^0 = -G(1 - x_3^2) \bar{u}_{3,122}, \quad \sigma_{23}^0 = -(1 - x_3^2) \left(\frac{E}{2} \bar{u}_{3,222} + G \bar{u}_{3,112} \right), \\ \sigma_{33}^0 = \frac{1}{2} \left\{ f(x_1, x_2, 1) - f(x_1, x_2, -1) + \left(\int_{-1}^{x_3} - \int_{x_3}^1 \right) g(x_1, x_2, \chi) d\chi \right. \\ \left. + x_3 \left[(3 - x_3^2) \left(\frac{E}{3} \bar{u}_{3,2222} + \frac{4G}{3} \bar{u}_{3,1122} \right) \right. \right. \\ \left. \left. - 2 \left(E \left\{ (\bar{u}_{1,1} \bar{u}_{3,1})_{,1} + [(\bar{u}_{2,2} + \nu \bar{u}_{1,1} + \frac{1}{2}(\bar{u}_{3,2})^2) \bar{u}_{3,2}]_{,2} \right\} \right. \right. \right. \\ \left. \left. \left. + G \{ [(\bar{u}_{2,1} + \bar{u}_{3,1} \bar{u}_{3,2}) \bar{u}_{3,1}]_{,2} + [(\bar{u}_{2,1} + \bar{u}_{3,1} \bar{u}_{3,2}) \bar{u}_{3,2}]_{,1} \} \right) \right] \right. \\ \left. - (1 - x_3^2)(E(\bar{u}_{3,22})^2 + 4G(\bar{u}_{3,12})^2) \right\}.$$

These are approximate solutions to (18), (19), and (11) in that $\sigma^0 \in \Sigma$ and $\mathbf{u}^0 \in V$ satisfy the boundary data exactly, and they satisfy the variational equations up to an error of the order of ϵ .

7. Construction of the degenerate von Kármán problem. We now simplify the two-dimensional problem, reducing the number of unknowns. This is a “one way” procedure in that we will lose the fact that \bar{u}_1 is independent of x_2 along the way. We introduce the notation $\sim 1 = 2$ and $\sim 2 = 1$. Our principal tool is the Airy stress function, which we use to eliminate $\bar{\sigma}$ and \bar{u}_α , replacing these with a scalar function. To begin, note that, assuming $\bar{\mathbf{u}}$ is sufficiently regular, applying the divergence theorem to Eq. (24) yields

$$\bar{\sigma}_{\alpha\beta,\beta} = 0 \quad \text{in } \omega \quad \text{and} \quad \bar{\sigma}_{\alpha\beta}n_\beta = h_\alpha \quad \text{on } \gamma. \quad (30)$$

It is well known that, so long as the h_α satisfy Eq. (12), there exists a so-called Airy stress function, $\bar{w}(x_1, x_2)$ for which

$$\bar{\sigma}_{\alpha\beta} = (-1)^{(\alpha+\beta)} \bar{w}_{,\sim\alpha\sim\beta} \quad (31)$$

and satisfying boundary conditions:

$$(-1)^\alpha \frac{d}{ds} \bar{w}_{,\alpha} = h_{\sim\alpha} \quad \text{on } \gamma.$$

These conditions do not specify the function uniquely, but, by distinguishing a point on γ (making it the origin of our coordinate system), these boundary conditions and the assumption $\bar{w}(\mathbf{0}) = \frac{\partial}{\partial n} \bar{w}(\mathbf{0}) = 0$ give Dirichlet conditions:

$$\left. \begin{aligned} \bar{w}(y) = \varphi(y) &:= -y_1 \int_{\gamma(y)} h_2 ds + y_2 \int_{\gamma(y)} h_1 ds \\ &\quad + \int_{\gamma(y)} (x_1 h_2 - x_2 h_1) ds \\ \frac{\partial \bar{w}}{\partial \mathbf{n}}(y) = \psi(y) &:= -n_1(y) \int_{\gamma(y)} h_2 ds + n_2(y) \int_{\gamma(y)} h_1 ds \end{aligned} \right\} \quad \text{for all } y \in \gamma, \quad (32)$$

where $\gamma(y)$ is the contour of γ connecting $\mathbf{0}$ to y in a counterclockwise fashion. Notice that Eq. (12) must hold for φ and ψ to be well-defined. For a discussion of the existence of the Airy stress function, the reader is referred to Zeidler [18, §65.7] for a classical approach, or the discussion in [4, §1.5], where the development is done in the greater generality of Sobolev spaces. Using \bar{w} and the expressions for m in terms of \bar{u}_3 in (23), we can rewrite Eq. (26) as

$$\begin{aligned} \int_\omega \frac{E}{3} \bar{u}_{3,22} \bar{v}_{,22} + \frac{4G}{3} \bar{u}_{3,12} \bar{v}_{,12} \\ + (-1)^{(\alpha+\beta)} \bar{w}_{,\sim\alpha\sim\beta} \bar{u}_{3,\alpha} \bar{v}_{,\beta} dA = \int_\omega \bar{f} \bar{v} dA \quad \text{for all } \bar{v} \in U. \end{aligned} \quad (33)$$

This equation is analogous to the first of the von Kármán equations, in that it is, in variational form, a fourth-order elliptic-parabolic operator acting on the vertical deflection of the centerplane of the plate, and a bilinear form acting on the deflection and the stress function (exactly the form $[u, w]$ defined in Eq. (3)).

To eliminate the \bar{u}_α , we begin by noticing that by substituting the definition (31) into Eq. (23), all the components of $D^2 \bar{w}$ can be written in terms of $\bar{\mathbf{u}}$. Motivated by

the second von Kármán equation (2), we seek a fourth-order, symmetric, and uniformly elliptic operator whose action on \bar{w} can be described by a quadratic form acting on \bar{u}_3 . Here we use that $\bar{w}_{,22} = \bar{\sigma}_{11}^0 = E\bar{u}_{1,1}$ is a function of x_1 only; so $\bar{w}_{,222} \equiv 0$. It is easy to see that the identity

$$G(\bar{w}_{,1111} + \bar{w}_{,2222}) + E\bar{w}_{,1212} - \nu\bar{w}_{,2211} = EG \left[\frac{1}{2}(\bar{u}_{3,2})_{,11}^2 - (\bar{u}_{3,1}\bar{u}_{3,2})_{,12} \right]$$

has the appropriate properties, provided the technical condition (15) is satisfied by the engineering constants of Hooke's Law (so that the left-hand side will be a uniformly elliptic operator). Multiplying this equation by some $v \in H_0^2(\omega)$ and integrating by parts leaves the weak formulation

$$\int_{\omega} \frac{1}{E} \bar{w}_{,\alpha\alpha} v_{,\alpha\alpha} + \frac{1 - \frac{\nu}{E}}{G} \bar{w}_{,12} v_{,12} dA = \int_{\omega} \bar{u}_{3,1} \bar{u}_{3,22} v_{,1} dA \quad \text{for all } v \in H_0^2(\omega) \quad (34)$$

with boundary conditions

$$\bar{w} = \varphi \quad \text{and} \quad \frac{\partial \bar{w}}{\partial \mathbf{n}} = \psi \quad \text{on } \gamma. \quad (35)$$

The equations (33), (34), and (35) are consistent for $\bar{u}_3 \in U$, and $\bar{w} \in H^2(\omega)$. They are nothing more than weak formulations of the equations announced in Sec. 3, Eqs. (5) and (6). To demonstrate the equivalence we need only apply the divergence theorem in the usual way to apply all derivatives to the solutions, \bar{u}_3 and \bar{w} , and remove the weighting applied in the original change of variables ($\bar{u}_3 = \epsilon^{-1}u$, $\bar{w} = \epsilon w$, and $\bar{f} = \epsilon^{-1}f$ for an appropriately defined f). Recall that the E and G used here are smaller by a factor of ϵ^3 than the Young's modulus and shear modulus denoted by the same letters in the summary. Thus we have completed our derivation.

When the in-plane displacement functions are reconstructed from \bar{u}_3 and \bar{w} , they are determined only up to a rigid motion, just as in the isotropic case; solve Eq. (23) for the $\bar{u}_{\alpha,\beta}$ in terms of \bar{u}_3 and $\bar{\sigma}_{\alpha\beta}$. Recall the assumption $\bar{u}_{1,2} \equiv 0$, even though we have no assurance that $\bar{w}_{,222} = 0$. Using (31) we have the gradients of the \bar{u}_α in terms of the solutions. The displacement in the plane of any point in the simply connected $\bar{\omega}$ (say the point $\mathbf{0}$ on γ), in conjunction with these gradients, completely defines the displacement $\bar{\mathbf{u}}$.

8. Conclusion. We are left with the question of existence of solutions to the problem we have derived. Based on the existence theory for the linear case, which has been examined in some detail in the works of Weinacht [16], and the author [10, 11, 12], and on physical considerations, there are reasonable conditions that must be met before this boundary value problem can be assured of solutions. In the linear case, all existence results for elliptic-parabolic equations of any order rely on the appearance of a uniformly elliptic lower-order operator. For the problem at hand, the appearance of a second-order uniformly elliptic operator in Eq. (5) (the elliptic-parabolic equation) can be assured by imposing appropriate restrictions on the boundary conditions for w . Writing Eq. (6) as $E[w] = (u_{,1}u_{,22})_{,1}$, we can decompose w into $w = w_0 + w_1$ in the usual way (see [4] or [18]). We will require w_0 to solve $E[w_0] = 0$ with the boundary conditions (7), while w_1 solves the inhomogeneous equation $E[w_1] = (u_{,1}u_{,22})_{,1}$ with homogeneous boundary conditions. Notice that this decouples w_0 from the nonlinear system; so it is determined

by the boundary data alone. For suitable data, we can be assured that the bilinear form $[u, w_0]$ will be a uniformly elliptic operator acting on u . It is easy to see that a necessary and sufficient condition is simply that $w_{0,11}w_{0,22} - w_{0,12}^2 \geq c > 0$ on all of ω . Physically this corresponds to a requirement that the force being applied along the boundary generate tension in the interior of the plate, so as to support the load, despite the lack of stiffness. With this restriction on the boundary data in place, it may be possible to demonstrate existence of solutions to the given problem. The proof in the isotropic case, first demonstrated by Berger in [1] (for ω of unspecified shape), uses the theory of monotone operators. In particular, the von Kármán problem is restated as a fixed point problem for a pseudomonotone operator in an appropriate Hilbert space. Such an approach may be fruitful here; however, because of the degeneracy, the proper space is not obvious. This is the topic of the companion article [13]. The author expects that, once existence is demonstrated, uniqueness will follow for sufficiently small loads and large tension by the inverse function theorem. The bifurcation theory for this boundary value problem should also be analogous to that of the classical von Kármán problem, but, as [16] indicates, the spectral theory for these degenerate operators is fundamentally different from that for uniformly elliptic operators.

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