QUARTERLY OF APPLIED MATHEMATICS
VOLUME LVIII, NUMBER 4
DECEMBER 2000, PAGES 613-626

QUENCHING PROFILE FOR A QUASILINEAR PARABOLIC EQUATION

By

JONG-SHENQ GUO (*Department of Mathematics, National Taiwan Normal University, Taipei 117, Taiwan, Republic of China*)

AND

BEI HU (*Department of Mathematics, University of Notre Dame, Notre Dame, Indiana*)

1. Introduction. We consider the following first initial boundary value problem:

$$u_t = (u^\alpha)_{xx} - u^{-\beta}, \quad x \in (-l, l),\ t > 0, \tag{1.1}$$

$$u(\pm l, t) = 1, \quad t > 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in [-l, l], \tag{1.3}$$

where $0 < \alpha < 1, \beta > 0, l > 0$, and $u_0(x) > 0$, $\forall x \in [-l, l]$. Without loss of generality, we may assume that $u_0(x)$ is smooth and bounded above by 1 such that $u_0(\pm l) = 1$. Since $u_0(x)$ is positive, the local (in time) existence and uniqueness of a classical solution of the problem (1.1)–(1.3) are trivial (see [8]).

Many results in quenching, such as single point quenching and profiles, are similar to those blow-up results ([3], [5] and the references therein). The system (1.1)–(1.3) in the case $\alpha < 1$ models fast diffusion and absorption (see [8]). The case $\alpha = 1$ was studied in [1], [2], [6], [7]. To study the asymptotic behavior of quenching, a simple Lyapunov function can be constructed in the case $\alpha = 1$. When $\alpha < 1$, however, such an explicit Lyapunov function is not available. In this paper, we use the idea of [4] to construct a Lyapunov function, which is rather technical.

We say that the solution u quenches if the minimum of $u(\cdot, t)$ reaches zero at some finite time. We shall always assume that u quenches at a quenching time $T < \infty$. In this case the right-hand side of Eq. (1.1) becomes singular and we no longer have a classical solution after this quenching time. A point (c, T) is said to be a quenching point if there is a sequence $\{(x_n, t_n)\}$ such that $x_n \to c$, $t_n \to T$, and $u(x_n, t_n) \to 0$ as $n \to \infty$. It is shown in [8] that there can only be finitely many quenching points that stay a positive distance away from the boundary $|x| = l$ for any positive initial data.

The purpose of this paper is to study how the solution tends to zero. For simplicity we shall only consider the symmetric case, i.e., the case that u_0 is symmetric with respect to $x = 0$ and is monotone increasing in $|x|$. It has been shown that $(0, T)$ is the only possible

Received March 23, 1998.
2000 *Mathematics Subject Classification.* Primary 35B40, 35K20, 35K55.

quenching point (cf. [8]). For simplicity, we assume that $l = 1$ and that quenching occurs. Our analysis applies to any $l > 0$.

To describe the quenching behavior near the point $(0, T)$, we introduce the following similarity variables:

$$y = \frac{x}{(T-t)^{\delta/2}}, \tag{1.4}$$

$$s = -\ln(T-t), \tag{1.5}$$

$$z(y,s) = \{u(x,t)(T-t)^{-\gamma}\}^{\alpha}, \tag{1.6}$$

where the similarity exponents are necessarily given by

$$\gamma = \frac{1}{\beta+1}, \qquad \delta = (\alpha+\beta)\gamma.$$

Then u satisfies (1.1)–(1.3) if and only if z satisfies

$$(z^{1/\alpha})_s = z_{yy} - \frac{\delta}{2} y (z^{1/\alpha})_y + \gamma z^{1/\alpha} - z^{-\beta/\alpha}, \quad (y,s) \in W, \tag{1.7}$$

$$z(y,s) = e^{\alpha\gamma s}, \quad |ye^{-\delta s/2}| = 1, \quad s \geq s_0, \tag{1.8}$$

$$z(y,s_0) = u_0^{\alpha}(yT^{\delta/2})T^{-\alpha\gamma} \equiv z_0(y), \quad |yT^{\delta/2}| \leq 1, \tag{1.9}$$

where $s_0 = -\ln(T)$ and W is given by

$$W = \{(y,s) \mid |ye^{-\delta s/2}| < 1, s > s_0\}.$$

Then the study of the quenching behavior of u near the quenching point (in the region $|x|/(T-t)^{\delta/2} \leq C$ as $t \to T^-$) is equivalent to the study of the stabilization problem as $s \to \infty$ to steady solutions of (1.7) in the whole real line $\mathbb{R}$.

Let $\varphi(y)$ be a positive solution of the equation

$$\varphi'' - \frac{\delta}{2} y (\varphi^{1/\alpha})' + \gamma\varphi^{1/\alpha} - \varphi^{-\beta/\alpha} = 0 \tag{1.10}$$

in $\mathbb{R}$. Hereafter the prime denotes the differentiation with respect to y. We shall analyze positive global solutions of (1.10) in Sec. 2. We prove that any nonconstant positive global solution of (1.10) must be monotone for all $|y|$ large; and it tends to infinity as $|y| \to \infty$. Moreover, there are only two possible growth rates. It grows either polynomially or exponentially as $|y| \to \infty$. The polynomial case is proportional to $|y|^{2\alpha/(\alpha+\beta)}$. Note that the above results have been proved in [7] for the case $\alpha = 1$. In fact, for the case $\alpha = 1$, it is easy to show that any local positive solution of (1.10) can be extended globally in $\mathbb{R}$ for $\beta \geq 1$. But, for the case $\alpha \in (0,1)$, the forward continuation in $[0,\infty)$ of the local solution of (1.10) is not available. Although the backward continuation in $[0,\infty)$ of the local solution of (1.10) holds for the case $\beta \geq \alpha$, it is not known for the case $\beta < \alpha$. From the ordinary differential equation (ODE) point of view, it is interesting to know whether (1.10) possesses a positive global solution.

We remark that for any positive global solution $\varphi(y)$ of (1.10) the function

$$\tilde{u}(x,t) = (T-t)^{\gamma/\alpha}\varphi^{1/\alpha}\left(\frac{x}{(T-t)^{\delta/2}}\right) \tag{1.11}$$

is a self-similar solution to the Cauchy problem for (1.1) for $x \in \mathbb{R}$ such that it quenches exactly at time $t = T$. Furthermore, if $\varphi(y)$ grows polynomially at infinity, then we have $\tilde{u}(x,T) = A|x|^{2/(\alpha+\beta)}$ for some positive constant A.

Motivated by a recent work of Galaktionov [4] on blow-up problems, we shall describe the quenching behavior of the solution near the quenching point by constructing a Lyapunov function (see, e.g., [9]). To construct a suitable Lyapunov function, we need to study the backward continuation property of Eq. (1.10). For the case $\beta < \alpha$, we redefine (1.10) so that the new equation has the backward continuation property (see (4.6) below).

We define the ω-limit set of the problem (1.7)–(1.9) by

$$\omega(z_0) = \{\varphi \in C(\mathbb{R}) \mid \text{there exists a sequence } s_j \to \infty \text{ such that } z(\cdot, s_j) \to \varphi(\cdot) \text{ as } j \to \infty \text{ uniformly on compact sets in } \mathbb{R}\}.$$

We show that the ω-limit set is contained in the set of nonconstant symmetric solutions of (1.10) with polynomial growth at infinity. To prove this result we need to verify two points. One is to show that the constant solution of (1.10) is not in the ω-limit set. The other is to obtain the polynomial growth property of $z(y,s)$ as $|y| \to \infty$. Notice that for $\alpha = 1$ we have $\omega(z_0) = \{(\beta+1)^\gamma\}$ (see [6], [1], and [7]).

This paper is organized as follows. In Sec. 2, we analyze positive solutions of (1.10). After giving some a priori estimates in Sec. 3, we construct and study the Lyapunov function in Sec. 4. Sections 5 and 6 are devoted to the study of the ω-limit set.

We make some remarks here. In contrast to the case $\alpha = 1$, there is no global existence result available for Eq. (1.10) for the case $0 < \alpha < 1$, from the standard existence theory of ODE. However, we proved in Sec. 5 that the ω-limit set is nonempty for the case $\beta < \alpha$. Hence, as a by-product of this fact, (1.10) does possess a nonconstant positive global solution. But, it still remains open whether there is a nonconstant positive global solution of (1.10) for the case $\beta \geq \alpha$, when $\alpha \in (0,1)$.

2. Self-similar solutions. In this section we shall analyze positive solutions of (1.10) using the method used in [7] (see also the references cited therein). We prove that any nonconstant solution of (1.10) must be monotone for all $|y|$ large; and it tends to infinity as $|y| \to \infty$. Moreover, there are only two possible growth rates. It grows either polynomially or exponentially as $|y| \to \infty$. Because the proofs are quite similar to the proofs given in [7], we shall only sketch the proofs.

Let φ be any nonconstant positive global solution $\varphi(y)$ of (1.10), i.e., φ satisfies

$$\varphi'' - \frac{\delta}{2} y(\varphi^{1/\alpha})' + \gamma\varphi^{1/\alpha} - \varphi^{-\beta/\alpha} = 0, \quad y \in \mathbb{R}, \tag{2.1}$$

where $\gamma = 1/(\beta+1)$ and $\delta = (\alpha+\beta)\gamma$. Since the case for negative y is similar to the case for positive y, we shall only consider the positive case.

Let $\kappa = (\beta+1)^{\alpha\gamma}$, $g(\varphi) = \gamma\varphi^{1/\alpha} - \varphi^{-\beta/\alpha}$, and

$$G(\varphi) = \int_\kappa^\varphi g(s)\,ds, \quad \varphi > 0.$$

Then the quantity $(\varphi'(y))^2/2 + G(\varphi(y))$ is monotone for $y > 0$ and hence the limit

$$\lim_{y\to\infty} [(\varphi'(y))^2/2 + G(\varphi(y))]$$

exists and is positive.

Set $\psi = \varphi'$. Given any positive constant a, let

$$A_a = \{(\varphi, \psi) \mid \varphi \geq \kappa, \psi \geq a\varphi\}.$$

Then by a phase plane analysis we obtain that the region A_a is a positively invariant region, i.e., there is $y_0 = y_0(a) > 0$ such that if $(\varphi(y_1), \psi(y_1)) \in A_a$ for some $y_1 \geq y_0$, then $(\varphi(y), \psi(y)) \in A_a$ for all $y \geq y_1$. We remark that here the fact $\alpha \in (0, 1]$ is used. Using this invariance property we can show that φ cannot assume the value κ at infinitely many points as $y \to \infty$ (cf. [7], Lemma 2.3). From the nonoscillation of φ, we next derive that φ must be strictly increasing to infinity as $y \to \infty$. Moreover, the limit

$$p = \lim_{y\to\infty} \frac{\varphi'(y)}{\varphi(y)}$$

exists and $p \in \{0, \infty\}$. The case $p = 0$ corresponds to the polynomial growth, and the case $p = \infty$ corresponds to the exponential growth.

We shall give the precise growth rate for the polynomial growth case.

LEMMA 2.1. Suppose that $p = 0$. Then $\lim_{y\to\infty}[y\varphi'(y)/\varphi(y)] = 2\alpha/(\alpha + \beta)$.

Proof. Let $z(y) = \varphi'(y)/\varphi(y)$ and let

$$\rho(y) = \exp\left[-\frac{\delta}{2\alpha}\int_0^y \xi\varphi(\xi)^{1/\alpha-1}d\xi\right].$$

Then we have

$$z(y) = \rho(y)^{-1}\int_y^\infty \rho(t)[\gamma\varphi(t)^{1/\alpha-1} - r(t)]dt$$

where $r(t) \to 0$ as $t \to \infty$. Applying l'Hôpital's rule, we compute that

$$\lim_{y\to\infty}[yz(y)] = \frac{2\alpha}{\alpha+\beta}.$$

Hence the proof is completed. □

From this lemma we can derive the growth rate for the case $p = 0$ and we conclude that $\varphi(y)$ behaves like $y^{2\alpha/(\alpha+\beta)}$ at $y = \infty$.

3. Some a priori estimates. In order to describe the quenching behavior of u near the quenching point $(0, T)$, we need some a priori estimates. First of all we note that z is symmetric with respect to $y = 0$ and is monotone increasing in $|y|$. Let $q = 2\alpha/(\alpha + \beta)$.

First, since 0 is the unique minimum point for each t, we have

$$u_t(0, t) \geq -u(0, t)^{-\beta}.$$

Then an integration gives that

$$z(0, s) \leq \kappa \equiv (\beta + 1)^{\alpha\gamma} \quad \text{for any } s \geq s_0. \tag{3.1}$$

Thus $\varphi(0) \leq \kappa$ for all $\varphi \in \omega(z_0)$.

Next, we claim that the constant state of (1.10) does not lie in the ω-limit set. The following lemma has been proved in [8].

LEMMA 3.1. Suppose that $u_0(x) = u_0(|x|)$ and is monotone nondecreasing in $|x|$ and that T is the quenching time. Then there is a positive constant C such that

$$(u^{\alpha+\beta})_x \geq Cx \tag{3.2}$$

in $[0,1] \times [T/2, T) \equiv Q$.

By an integration of (3.2) and returning to similarity variables, we obtain that

$$z(y,s) \geq D|y|^q \text{ in } W \tag{3.3}$$

for some positive constant D. It follows from (3.3) that the constant state of (1.10) is not in the ω-limit set.

Let $H(x,t) = (u^\alpha)_x - \alpha\eta x u^{-\beta}$, where $\eta = C/(\alpha+\beta)$. Then by (3.2) we have $H(x,t) \geq 0$ in Q and $H(0,t) = 0$ for $t > 0$. It follows that $H_x(0,t) \geq 0$ and hence $u_t(0,t) \geq -(1-\alpha\eta)u(0,t)^{-\beta}$. From this we conclude that $\varphi(0) < \kappa$ for all $\varphi \in \omega(z_0)$.

Suppose that u_0 satisfies the condition (H):

$$(u_0^\alpha)'' \leq u_0^{-\beta} \quad \text{in } [-1,1]. \tag{H}$$

We derive some estimates. Since u_0 satisfies the condition (H), $u_t < 0$ in Q_T, by the strong maximum principle. Since $u_t < 0$ in Q_T, it follows that

$$(z^{1/\alpha})_s + \frac{\delta}{2} y (z^{1/\alpha})_y - \gamma z^{1/\alpha} < 0, \tag{3.4}$$

$$z_{yy} - z^{-\beta/\alpha} < 0. \tag{3.5}$$

Multiplying the last inequality with z_y and integrating over $[0,y]$, we obtain

$$\tfrac{1}{2}|z_y(y,s)|^2 \leq \begin{cases} \frac{1}{1-\beta/\alpha}[z(y,s)^{1-\beta/\alpha} - z(0,s)^{1-\beta/\alpha}] & \text{for } \beta < \alpha, \\ \log \frac{z(y,s)}{z(0,s)} & \text{for } \beta = \alpha, \\ \frac{1}{\beta/\alpha-1}[z(0,s)^{1-\beta/\alpha} - z(y,s)^{1-\beta/\alpha}] & \text{for } \beta > \alpha. \end{cases} \tag{3.6}$$

For the case $\beta < \alpha$, using the first estimate in (3.6) and the fact that $z(0,s) < \kappa$, we can solve the ordinary differential inequality and easily obtain the polynomial bounds for z and z_y. One of the difficulties for the case $\beta \geq \alpha$ is to obtain a lower bound for $z(0,s)$. We shall overcome this difficulty by exploring the Lyapunov function in the following section.

4. Lyapunov function. We first get some estimates on the boundary $y = R(s) = \exp(\delta s/2)$. Under our assumption, any $x \in [1/2, 1]$ is not a quenching point. Thus Eq. (1.1) is uniformly parabolic in $[1/2,1] \times [0,T]$. By standard parabolic estimates (see [8]), $0 \leq u_x(1,t) \leq C^*$. It follows that

$$0 \leq z_y(R(s),s) \leq C^* \alpha \exp\left[\left(\alpha\gamma - \frac{\delta}{2}\right) s\right]. \tag{4.1}$$

Differentiating the equation

$$z(R(s),s) = \exp(\alpha\gamma s),$$

we obtain

$$\alpha\left(\gamma-\frac{\delta}{2}C^*\right)\exp(\alpha\gamma s)\le z_s(R(s),s)\le\alpha\gamma\exp(\alpha\gamma s). \tag{4.2}$$

Define, as in [9],

$$E_{R(s)}[z](s)=\int_0^{R(s)}\Phi(y,z(y,s),z_y(y,s))\,dy, \tag{4.3}$$

where $\Phi=\Phi(y,v,w)$. Then as in [9],

$$\frac{d}{ds}E_{R(s)}[z](s)=J_0+J_1+J_2, \tag{4.4}$$

where

$$\begin{aligned}
J_0&=-\int_0^{R(s)}\frac{1}{\alpha}\Phi_{ww}(y,z(y,s),z_y(y,s))z^{1/\alpha-1}|z_s|^2\,dy,\\
J_1&=\Phi_w(R(s),z(R(s),s),z_y(R(s),s))z_s(R(s),s)\\
&\quad-\Phi_w(0,z(0,s),0)z_s(0,s)\\
&\quad+\Phi(R(s),z(R(s),s),z_y(R(s),s))R'(s),\\
J_2&=\int_0^{R(s)}\left\{\Phi_v-\Phi_{wy}-\Phi_{wv}z_y-\Phi_{ww}\left[\frac{\delta}{2\alpha}yz^{1/\alpha-1}z_y-g(z)\right]\right\}z_s(y,s)\,dy\\
&\equiv\int_0^{R(s)}K(y,z(y,s),z_y(y,s))z_s(y,s)\,dy,
\end{aligned}$$

with $g(v)=\gamma v^{1/\alpha}-v^{-\beta/\alpha}$. Since $z(y,s)\ge D|y|^q$, we can change the value of $g(z)$ for $z<D|y|^q$ without changing the above calculations. Let $\zeta(\eta)$ be a smooth, nonincreasing cutoff function such that

$$\zeta(\eta)=0\quad\text{for }\eta\ge2,\qquad\zeta(\eta)=1\quad\text{for }\eta\le1,\qquad0\le\zeta(\eta)\le1.$$

For the case $\beta\ge\alpha$, we define

$$\hat g(\xi,v)=g(v); \tag{4.5}$$

and for the case $\beta<\alpha$, we define

$$\hat g(\xi,v)=g(v)\left[1-\zeta\left(\frac{2v}{D\xi^q\zeta(\xi)+D[1-\zeta(\xi)]}\right)\right]-v^{-1}\zeta\left(\frac{2v}{D\xi^q\zeta(\xi)+D[1-\zeta(\xi)]}\right). \tag{4.6}$$

We assume without loss of generality that $D<\min(1,2^{-q}\kappa)$. It is clear that with this modified definition, there is no change in the calculations in J_2.

Let

$$\Phi(y,v,w)=\int_0^w(w-\sigma)P(y,v,\sigma)d\sigma-\int_\kappa^v\hat g(y,\mu)P(y,\mu,0)\,d\mu.$$

Then

$$\begin{aligned}
\Phi_w(y,v,0)&=0,\\
\Phi_{ww}(y,v,w)&=P(y,v,w),
\end{aligned}$$

and

$$K(y,v,w) = \int_0^w \left\{ -\sigma \frac{\partial P}{\partial v}(y,v,\sigma) - \frac{\partial P}{\partial y}(y,v,\sigma) \right.$$
$$\left. + \frac{\partial}{\partial \sigma}\left[P(y,v,\sigma)\left(\hat{g}(y,v) - \frac{\delta}{2\alpha} y v^{1/\alpha - 1}\sigma \right)\right]\right\} d\sigma.$$

Let $\psi(\xi; y, v, w)$ be defined as the solution of the following:

$$\psi_{\xi\xi} - \frac{\delta}{2\alpha}\xi\psi^{1/\alpha-1}\psi_\xi + \hat{g}(\xi,\psi) = 0, \quad \xi \in (0,y), \tag{4.7}$$

$$\psi(y;y,v,w) = v, \qquad \psi_\xi(y;y,v,w) = w, \tag{4.8}$$

where the subscript ξ denotes the derivatives in the first variable ξ. We assume, for any $v > 0$, w and $y > 0$, that the solution ψ is well defined for $\xi \in [0,y]$, and that $\psi(\xi;y,v,w) > 0$ for $\xi \in [0,y]$ (these properties of the solution of the ODE (4.7) will be established later on). Differentiating (4.8) in y, we obtain $\psi_\xi(y;y,v,w) + \psi_y(y;y,v,w) = 0$, and hence

$$\psi_y(y;y,v,w) = -\psi_\xi(y;y,v,w) = -w.$$

Similarly, since $\psi_{\xi\xi}(y;y,v,w) + \psi_{\xi y}(y;y,v,w) = 0$,

$$\begin{aligned}\psi_{y\xi}(y;y,v,w) &= -\psi_{\xi\xi}(y;y,v,w)\\ &= -\frac{\delta}{2\alpha} y\psi^{1/\alpha-1}(y;y,v,w)\psi_\xi(y;y,v,w) + \hat{g}(y,\psi(y;y,v,w))\\ &= -\frac{\delta}{2\alpha} y v^{1/\alpha-1} w + \hat{g}(y,v).\end{aligned}$$

Next, differentiating (4.8) in v and w, respectively, we find that

$$\psi_v(y;y,v,w) = 1, \quad \psi_{v\xi}(y;y,v,w) = 0, \quad \psi_w(y;y,v,w) = 0, \quad \psi_{w\xi}(y;y,v,w) = 1.$$

Therefore, the functions $\psi_y(\xi;y,v,w)$ and $-w\psi_v(\xi;y,v,w) - \left[\frac{\delta}{2\alpha} y v^{1/\alpha-1} w - \hat{g}(y,v)\right] \cdot \psi_w(\xi;y,v,w)$ and their first derivatives in ξ agree at $\xi = y$.

Differentiating (4.7) and using the uniqueness of the solution of the corresponding ODE, we derive

$$\psi_y(\xi;y,v,w) = -w\psi_v(\xi;y,v,w) - \left[\frac{\delta}{2\alpha} y v^{1/\alpha-1} w - \hat{g}(y,v)\right]\psi_w(\xi;y,v,w). \tag{4.9}$$

Let

$$P(y,v,w) = \exp\left\{ -\frac{\delta}{2\alpha}\int_0^y \xi\psi(\xi;y,v,w)^{1/\alpha-1} d\xi \right\}. \tag{4.10}$$

Then a direct computation shows that (using also (4.9))

$$K(y,v,w) \equiv 0$$

and hence

$$J_2 = 0.$$

Now we study the solution ψ of the ODE (4.7). From the local existence and uniqueness theory of the initial value problem of ODE, the solution ψ is well defined in a

neighborhood of $\xi = y$. The solution can be extended as long as ψ and ψ_ξ remain bounded, and ψ is positive.

In the following discussion we assume that $(y^*, y]$ $(y^* \geq 0)$ is the maximal existence interval for the solution ψ. We want to show that $y^* = 0$.

For the case $\beta \geq \alpha$, we have $\hat{g}(y, v) \equiv g(v)$. Therefore, in this case, by multiplying Eq. (4.7) with ψ_ξ and integrating in ξ, we obtain

$$\begin{aligned} \tfrac{1}{2}\psi_\xi^2(\xi; y, v, w) + \frac{\delta}{2\alpha}\int_\xi^y \tau\psi(\tau; y, v, w)^{1/\alpha-1}\psi_\xi^2(\tau; y, v, w)\,d\tau \\ + \int_\kappa^{\psi(\xi;y,v,w)} g(\mu)\,d\mu = \tfrac{1}{2}w^2 + \int_\kappa^v g(\mu)\,d\mu. \end{aligned} \tag{4.11}$$

Hence, for $\xi \in (y^*, y)$,

$$\tfrac{1}{2}\psi_\xi^2(\xi; y, v, w) + \int_\kappa^{\psi(\xi;y,v,w)} g(\mu)\,d\mu \leq \tfrac{1}{2}w^2 + \int_\kappa^v g(\mu)\,d\mu, \tag{4.12}$$

from which it follows that ψ and ψ_ξ will remain bounded (with the bounds depending on v and w) on $(y^*, y]$. Furthermore, since $\int_0^\kappa g(\mu)\,d\mu = -\infty$ in the case $\beta \geq \alpha$, we also have $\psi(\xi; y, v, w) > 0$. Thus ψ can be extended beyond y^* if $y^* > 0$, and we must have $y^* = 0$.

Next, we consider the case $\beta < \alpha$. Multiplying Eq. (4.7) with ψ_ξ and integrating in ξ, we obtain

$$\begin{aligned} &\tfrac{1}{2}\psi_\xi^2(\xi; y, v, w) + \frac{\delta}{2\alpha}\int_\xi^y \tau\psi(\tau; y, v, w)^{1/\alpha-1}\psi_\xi^2(\tau; y, v, w)d\tau + \int_\kappa^{\psi(\xi;y,v,w)} \hat{g}(\xi, \mu)\,d\mu \\ &\quad = \tfrac{1}{2}w^2 + \int_\kappa^v \hat{g}(y, \mu)\,d\mu - \int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \hat{g}_\xi(\tau, \mu)\,d\mu\,d\tau. \end{aligned} \tag{4.13}$$

So we need to estimate the last term in the above equation. It is clear that

$$\hat{g}_\xi(\xi, v) = 0 \quad \text{in } \{v > D\xi^q\zeta(\xi) + D(1 - \zeta(\xi))\} \cup \{v < \tfrac{1}{2}[D\xi^q\zeta(\xi) + D(1 - \zeta(\xi))]\}$$

and

$$\hat{g}_\xi(\xi, v) = 2v[g(v) + v^{-1}]\zeta'\left(\frac{2v}{D\xi^q\zeta(\xi) + D[1 - \zeta(\xi)]}\right)\frac{\{D\xi^q\zeta(\xi) + D[1 - \zeta(\xi)]\}'}{\{D\xi^q\zeta(\xi) + D[1 - \zeta(\xi)]\}^2}.$$

Thus $|\hat{g}_\xi(\xi, v)|$ is uniformly bounded for $\xi > 1$ and therefore $-\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \hat{g}_\xi(\tau, \mu)\,d\mu\,d\tau$ is bounded if $1 < \xi < y$ (with the bound depending on y, and independent of w, v, ξ). For $\xi \leq 1$,

$$\hat{g}_\xi(\xi, v) = 2v[g(v) + v^{-1}]\zeta'\left(\frac{2v}{D\xi^q}\right)\frac{q}{D\xi^{q+1}}.$$

Recall that $D < 1 < \kappa$. Hence

$$[g(v) + v^{-1}] \geq 0 \quad \text{for } 0 < v < D\xi^q,\ \xi \leq 1.$$

Since $\zeta' \leq 0$, we derive

$$\hat{g}_\xi(\xi, v) \leq 0 \quad \text{for } 0 < v < D\xi^q,\ \xi \leq 1. \tag{4.14}$$

Thus for $0 \le \xi \le 1$,

$$-\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \hat{g}_\xi(\tau,\mu)\,d\mu\,d\tau = -\left[\int_\xi^{\min(1,y)} + \int_{\min(1,y)}^y\right] \int_\kappa^{\psi(\tau;y,v,w)} \hat{g}_\xi(\tau,\mu)d\mu\,d\tau;$$

the second integral in the above equality is bounded, and therefore

$$-\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \hat{g}_\xi(\tau,\mu)\,d\mu\,d\tau \le C + \int_\xi^{\min(1,y)} \int_{\psi(\tau;y,v,w)}^\kappa \hat{g}_\xi(\tau,\mu)\,d\mu\,d\tau \le C, \tag{4.15}$$

where we used (4.14) and the fact that $\hat{g}_\xi(\xi,\mu) = 0$ for $\mu > 1 \ge \xi$. We obtain

$$\tfrac{1}{2}\psi_\xi^2(\xi;y,v,w) + \int_\kappa^{\psi(\xi;y,v,w)} \hat{g}(\xi,\mu)\,d\mu \le \tfrac{1}{2}w^2 + \int_\kappa^v \hat{g}(y,\mu)\,d\mu + C, \tag{4.16}$$

where the constant C is independent of w, v, and ξ (it may depend on y). As before, this estimate implies that $|\psi_\xi|$ and ψ are bounded from above. For every $\xi > 0$, $\int_0^\kappa \hat{g}(\xi,\mu)\,d\mu = -\infty$, we also have $\psi(\xi;y,v,w) > 0$ if $\xi > 0$. Thus ψ can be extended beyond y^* if $y^* > 0$, and we must have $y^* = 0$.

We next derive some estimates for large v. In this case $\hat{g}(\xi,v) \equiv g(v)$ and the argument below applies to both the cases $\beta < \alpha$ and $\beta \ge \alpha$.

Clearly,

$$\begin{aligned}\psi(\xi;y,v,w) &\ge v - \max_{\xi\le\hat{\xi}\le y} |\psi_\xi(\hat{\xi};y,v,w)|\,(y-\xi)\\ &\ge v - \left[w^2 + 2\int_\kappa^v g(\mu)\,d\mu\right]^{1/2}(y-\xi)\\ &\ge \frac{v}{2},\end{aligned} \tag{4.17}$$

provided

$$0 < y - \xi \le \frac{v}{2}\left[w^2 + 2\int_\kappa^v g(\mu)\,d\mu\right]^{-1/2}. \tag{4.18}$$

We take ξ^* as follows:

$$\xi^* = \max\left\{\frac{y}{2},\ y - \frac{v}{2}\left[w^2 + 2\int_\kappa^v g(\mu)\,d\mu\right]^{-1/2}\right\}.$$

Then

$$\begin{aligned}P(y,v,w) &\le \exp\left\{-\frac{\delta}{2\alpha}\left(\frac{v}{2}\right)^{1/\alpha-1}\int_{\xi^*}^y \xi\,d\xi\right\}\\ &\le \exp\left\{-\min\left[\frac{\delta}{8\alpha}\left(\frac{v}{2}\right)^{1/\alpha-1}y^2,\ \frac{\frac{\delta}{4\alpha}\left(\frac{v}{2}\right)^{1/\alpha}y}{\left[w^2 + 2\int_\kappa^v g(\mu)\,d\mu\right]^{1/2}}\right]\right\}.\end{aligned} \tag{4.19}$$

It follows that

$$\begin{aligned}\Phi(y,v,w) &\le \int_0^w (w-\sigma)P(y,v,\sigma)\,d\sigma\\ &\le w^2\exp\left\{-\min\left[\frac{\delta}{8\alpha}\left(\frac{v}{2}\right)^{1/\alpha-1}y^2,\ \frac{\frac{\delta}{4\alpha}\left(\frac{v}{2}\right)^{1/\alpha}y}{\left[w^2 + 2\int_\kappa^v g(\mu)\,d\mu\right]^{1/2}}\right]\right\},\end{aligned} \tag{4.20}$$

$$
\begin{aligned}
|\Phi_w(y,v,w)| &= \left|\int_0^w P(y,v,\sigma)d\sigma\right| \\
&\le |w| \exp\left\{-\min\left[\frac{\delta}{8\alpha}\left(\frac{v}{2}\right)^{1/\alpha-1} y^2,\ \frac{\frac{\delta}{4\alpha}\left(\frac{v}{2}\right)^{1/\alpha} y}{\left[w^2+2\int_\kappa^v g(\mu)\,d\mu\right]^{1/2}}\right]\right\}.
\end{aligned}
\tag{4.21}
$$

We want to estimate J_1 from above. Thus we shall substitute $y = R(s), z(R(s),s) = v$ and $w = z_y(R(s),s)$ in the above estimates. It is clear that $z(R(s),s) \to \infty$ as $s \to \infty$ and

$$
\int_\kappa^v g(\mu)\,d\mu \sim \frac{\gamma\alpha}{1+\alpha} v^{(1+\alpha)/\alpha} \quad \text{for } v \gg 1.
$$

Thus, using the estimates at the beginning of this section, we obtain, for large s,

$$
\Phi(y, z(R(s),s), z_y(R(s),s)) \le |z_y(R(s),s)|^2 \exp\left\{-c^* z(R(s),s)^{(1-\alpha)/(2\alpha)} R(s)\right\}
$$

for some $c^* > 0$. Therefore

$$
\begin{aligned}
&\Phi(y, z(R(s),s), z_y(R(s),s))R'(s) \\
&\quad \le \frac{\delta}{2}(C^*)^2\alpha^2 \exp\left[\left(2\alpha\gamma - \frac{\delta}{2}\right)s\right] \exp\left\{-c^* \exp\left[\frac{\gamma(1-\alpha)+\delta}{2}s\right]\right\}.
\end{aligned}
\tag{4.22}
$$

Similarly,

$$
|\Phi_w(y, z(R(s),s), z_y(R(s),s))| \le |z_y(R(s),s)| \exp\left\{-c^* z(R(s),s)^{(1-\alpha)/(2\alpha)} R(s)\right\}
$$

and

$$
\begin{aligned}
&|\Phi_w(y, z(R(s),s), z_y(R(s),s))|\,|z_s(R(s),s)| \\
&\quad \le \overline{C} \exp\left[\left(2\alpha\gamma - \frac{\delta}{2}\right)s\right] \exp\left\{-c^* \exp\left[\frac{\gamma(1-\alpha)+\delta}{2}s\right]\right\}.
\end{aligned}
\tag{4.23}
$$

Thus J_1 is bounded from above by a function that decays exponentially fast.

In order to obtain the lower bound for J_1, we substitute $w = 0$ in (4.19) and obtain

$$
P(y,v,0) \le \exp\{-c^* v^{(1-\alpha)/(2\alpha)} y\} \quad \text{for } v \ge \kappa, \tag{4.24}
$$

from which we immediately derive

$$
\begin{aligned}
&\Phi(y, z(R(s),s), z_y(R(s),s))R'(s) \\
&\quad \ge -R'(s) \int_\kappa^{z(R(s),s)} g(\mu) P(R(s),\mu,0)\,d\mu \\
&\quad \ge -\gamma R'(s) z(R(s),s)^{1/\alpha+1} \exp\{-c^* \kappa^{(1-\alpha)/(2\alpha)} R(s)\} \\
&\quad \ge -\overline{C} \exp\left[\left((1+\alpha)\gamma + \frac{\delta}{2}\right)s\right] \exp\{-c^* \kappa^{(1-\alpha)/(2\alpha)} \exp(\alpha\gamma s)\}.
\end{aligned}
\tag{4.25}
$$

Thus J_1 is bounded from below by a function that decays exponentially fast.

We proved:

LEMMA 4.1.

$$
\frac{d}{ds} E_{R(s)}[z](s) = -\frac{1}{\alpha} \int_0^{R(s)} P(y, z(y,s), z_y(y,s)) z^{1/\alpha-1}(y,s) |z_s(y,s)|^2\,dy + J_1(s), \tag{4.26}
$$

where $|J_1(s)|$ decays exponentially fast to 0 as $s \to \infty$ and therefore $\int_{s_0}^{\infty} |J_1(s)| ds < \infty$.

From this lemma, it follows that

$$\frac{d}{ds}\left[E_{R(s)}[z](s) - \int_{s_0}^{s} J_1(\tau)\, d\tau\right] \leq 0,$$

and hence for every $s > s_0$,

$$E_{R(s)}[z](s) \leq E_{R(s)}[z](s_0) + \int_{s_0}^{s} J_1(\tau) d\tau \leq E_{R(s)}[z](s_0) + \int_{s_0}^{\infty} |J_1(\tau)| d\tau \equiv \overline{C}. \quad (4.27)$$

5. ω-limit for the case $\beta < \alpha$. In the sequel, we shall assume that the condition (H) holds.

THEOREM 5.1. If $\beta < \alpha$, then the ω-limit set is not empty. Furthermore, any ω-limit satisfies the ODE (1.10) for $y > 0$.

Proof. Using the first estimate in (3.6) and the fact that $z(0, s) < \kappa$, we easily obtain the polynomial bounds for z and z_y. It follows that any ω-limit is well defined and the ω-limit set is not empty (see [6], [7]).

Recall from (4.24) that, for $v \geq \kappa$,

$$P(y, v, 0) \leq \exp\{-c^* v^{(1-\alpha)/(2\alpha)} y\}. \quad (5.1)$$

It follows that (using $z(y, s) \geq Dy^q$ and $D < \min(1, 2^{-q}\kappa)$),

$$\begin{aligned} \int_0^{R(s)} \int_\kappa^{z(y,s)} \hat{g}(y, \mu) P(y, \mu, 0)\, d\mu\, dy &= \int_0^{R(s)} \int_\kappa^{z(y,s)} g(\mu) P(y, \mu, 0)\, d\mu\, dy \\ &\leq \int_0^1 \int_0^{C^*} |g(\mu)| P(y, \mu, 0)\, d\mu\, dy + \int_1^\infty \int_D^\infty |g(\mu)| P(y, \mu, 0)\, d\mu\, dy \\ &\leq C, \end{aligned} \quad (5.2)$$

where C^* is the upper bound for $z(y, s)$ over $y \in [0, 1]$. Using this estimate we immediately obtain

$$E_{R(s)}[z](s) \geq -C,$$

and therefore by using Lemma 4.1, we obtain

$$\frac{1}{\alpha} \int_{s_0}^{\infty} \int_0^{R(s)} P(y, z(y, s), z_y(y, s)) z^{1/\alpha - 1}(y, s) |z_s(y, s)|^2\, dy\, ds \leq C. \quad (5.3)$$

Using standard arguments (cf. [5], [6]) we now conclude that any ω-limit satisfies the ODE (1.10) for all $y > 0$. □

In the case that the ω-limit is positive at $y = 0$, we can further derive estimates near $y = 0$ and conclude that the ω-limit also satisfies the ODE (1.10) at $y = 0$ and $\varphi'(0) = 0$.

The case in which the ω-limit is 0 at $y = 0$ will be shown in the next theorem.

THEOREM 5.2. If φ is an element in the ω-limit and $\varphi(0) = 0$, then

$$\varphi(y) \equiv my^q,$$

where

$$m = \left\{ \frac{(\alpha+\beta)^2}{2\alpha(\alpha-\beta)} \right\}^{\alpha/(\alpha+\beta)}.$$

Proof. By the first estimate in (3.6),

$$0 \le \varphi'(y) \le \left\{ \frac{2\alpha}{\alpha-\beta} \varphi(y)^{1-\beta/\alpha} \right\}^{1/2} \quad \text{for } y > 0.$$

From this inequality, using also $\varphi(0) = 0$, it follows immediately that

$$\varphi(y) \le \left\{ \frac{(\alpha+\beta)^2}{2\alpha(\alpha-\beta)} \right\}^{\alpha/(\alpha+\beta)} y^q = my^q \quad \text{for } y > 0. \tag{5.4}$$

A direct computation shows that $\varphi_0(y) = my^q$ is a solution of the ODE (1.10). Thus

$$\begin{aligned} \varphi_{yy} - \frac{\delta}{2} y(\varphi^{1/\alpha})_y &= -\gamma\varphi^{1/\alpha} + \varphi^{-\beta/\alpha} \\ &\ge -\gamma\varphi_0^{1/\alpha} + \varphi_0^{-\beta/\alpha} \\ &= (\varphi_0)_{yy} - \frac{\delta}{2} y(\varphi_0^{1/\alpha})_y \quad \text{for } y > 0, \end{aligned}$$

and therefore

$$\begin{aligned} [\varphi(y) - \varphi_0(y)]_y &\ge \int_0^y \frac{\delta}{2} \tau(\varphi^{1/\alpha}(\tau) - \varphi_0^{1/\alpha}(\tau))_\tau \, d\tau \\ &= \frac{\delta}{2}(\varphi^{1/\alpha}(y) - \varphi_0^{1/\alpha}(y)) - \int_0^y \frac{\delta}{2}(\varphi^{1/\alpha}(\tau) - \varphi_0^{1/\alpha}(\tau)) \, d\tau \\ &\ge \frac{\delta}{2} y(\varphi^{1/\alpha}(y) - \varphi_0^{1/\alpha}(y)). \end{aligned}$$

Notice that $1/\alpha > 1$, and the function $s^{1/\alpha}$ is Lipschitz continuous. Thus by the comparison theorem for first-order ordinary differential inequalities, we get

$$\varphi(y) \ge \varphi_0(y) \quad \text{for } y > 0.$$

Combining this with (5.4), we conclude the theorem. □

6. ω-limit for the case $\beta \ge \alpha$. One of the difficulties in the case $\beta \ge \alpha$ is the lack of a lower bound for $z(0, s)$, and therefore no estimates on the derivatives of z are available.

We now consider the special case $\alpha \le \beta < 3\alpha$. In this case, if $z(y, s) \le \kappa$, then

$$\int_\kappa^{z(y,s)} g(\mu) P(y, \mu, 0) \, d\mu \le \int_{z(y,s)}^\kappa \mu^{-\beta/\alpha} d\mu \le \int_{Dy^q}^\kappa \mu^{-\beta/\alpha} d\mu.$$

Notice that, for any y^* finite, the integral

$$\int_0^{y^*} \int_{Dy^q}^\kappa \mu^{-\beta/\alpha} \, d\mu \, dy$$

is a finite number if $\beta < 3\alpha$. For each s, we take $y^* = y^*(s)$ such that $z(y^*, s) = \kappa$. (3.3) implies that y^* is bounded uniformly in s. Thus

$$\begin{aligned}\int_0^{R(s)} \int_\kappa^{z(y,s)} \hat{g}(y,\mu)P(y,\mu,0)\,d\mu\,dy &= \int_0^{R(s)} \int_\kappa^{z(y,s)} g(\mu)P(y,\mu,0)\,d\mu\,dy \\ &\le \int_0^{y^*} \int_{Dy^q}^{\kappa} |g(\mu)|P(y,\mu,0)\,d\mu\,dy + \int_{y^*}^{\infty} \int_\kappa^{\infty} |g(\mu)|P(y,\mu,0)\,d\mu\,dy \\ &\le C.\end{aligned} \tag{6.1}$$

THEOREM 6.1. Let $\alpha \le \beta < 3\alpha$. If the ω-limit set is not empty, then any ω-limit must satisfy the ODE (1.10) in $\mathbb{R}$.

Proof. From (6.1) it follows that $E_{R(s)}[z](s) \ge -C$ and therefore by using Lemma 4.1 (integrating (4.26) over $[s_0, \infty)$), we obtain

$$\frac{1}{\alpha}\int_{s_0}^{\infty} \int_0^{R(s)} P(y, z(y,s), z_y(y,s)) z^{1/\alpha - 1}(y,s)|z_s(y,s)|^2\,dy\,ds \le C. \tag{6.2}$$

Since the ω-limit set is not empty, there exists $s_j \to \infty$ such that $z(y, s_j)$ converges to an ω-limit function $\varphi(y)$ in any compact subset. Since

$$(z^{1/\alpha})_s < \gamma z^{1/\alpha} - \frac{\delta}{2} y (z^{1/\alpha})_y \le \gamma z^{1/\alpha} \quad \text{for } y > 0,$$

the solution $z(y,s)$ is uniformly bounded in $\{0 \le y \le K,\ s_j \le s \le s_j + K\}$, $\forall j$ (with the bound depending on K), for any $K > 1$.

By the standard parabolic regularity estimates, we have

$$|z_y(y,s)| + |z_{yy}(y,s)| + |z_s(y,s)| \le C_K \quad \text{for } \frac{1}{K} \le y \le K,\ s_j \le s \le s_j + K$$

(here we exclude the interval $y \in [0, 1/K]$ because on this interval, the lower bound is not established and the equation may be degenerate, and the right-hand side is unbounded). Using this estimate, a standard argument (cf. [5]) shows that the limit function $\varphi(y)$ must satisfy the ODE (1.10) in the interval $[1/K, K]$ for any $K > 0$.

Since $\beta \ge \alpha$, the argument leading to (4.12) implies that any solution of the ODE (1.10) must be positive for all $y \ge 0$, especially,

$$c^* = \varphi(0) > 0. \tag{6.3}$$

By the uniform convergence, we also have

$$z(0, s_j) \ge \tfrac{1}{2}c^* > 0 \quad \text{for } j \gg 1. \tag{6.4}$$

Using the equation

$$(z^{1/\alpha})_s(0,s) \ge \gamma z^{1/\alpha}(0,s) - z^{-\beta/\alpha}(0,s),$$

we derive that

$$z(0,s) \ge \tfrac{1}{4}c^* > 0 \quad \text{for } s_j \le s \le s_j + \mu, \tag{6.5}$$

for some small $\mu > 0$. Using this estimate and (3.6) we immediately conclude that z_y is uniformly bounded in $[0, K] \times [s_j, s_j + \mu]$. Using parabolic estimates again, we conclude that the function $\varphi(y)$ also satisfies the ODE (1.10) at $y = 0$ and $\varphi'(0) = 0$. □

Acknowledgment. The authors would like to thank the referee for some useful comments.

References

[1] M. Fila and J. Hulshof, *A note on the quenching rate*, Proc. Amer. Math. Soc. **112**, 473–477 (1991)

[2] S. Filippas and J.-S. Guo, *Quenching profiles for one-dimensional semilinear heat equations*, Quart. Appl. Math. **51**, 713–729 (1993)

[3] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equation*, Indiana Univ. Math. J. **34**, 425–447 (1985)

[4] V. A. Galaktionov, *On asymptotic self-similar behavior for quasilinear heat equation: Single point blow-up*, SIAM J. Math. Anal. **26**, 675–693 (1995)

[5] Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38**, 297–319 (1985)

[6] J.-S. Guo, *On the quenching behavior of the solution of a semilinear parabolic equation*, J. Math. Anal. Appl. **151**, 58–79 (1990)

[7] J.-S. Guo, *On the quenching rate estimate*, Quart. Appl. Math. **49**, 747–752 (1991)

[8] J.-S. Guo, *Quenching behavior for a fast diffusion equation with absorption*, Dynamic Systems and Applications **4**, 47–56 (1995)

[9] T. I. Zelenjak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differential Equations **4**, 17–22 (1968)