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A NOTE ON EXACT PARTICULAR SOLUTIONS OF THE GENERALIZED SHALLOW-WATER EQUATIONS

By

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Abstract. This note presents a set of systems of two first-order quasi-linear partial differential equations, which can be reduced to the shallow-water equations. This set includes equations describing a two-layered fluid flow.

Introduction. The shallow-water equations

$$u_t + uu_x + h_x = 0, \qquad h_t + uh_x + hu_x = 0 \tag{SW}$$

have been intensively studied analytically and numerically (see, e.g., [2], [4, §13.10]). Several exact analytical solutions of (SW) are known:

$$u = x/t, \quad h = a/t; \qquad u = (b + 2x/t)/3, \quad h = (x/t - b)^2/9 + a/t^{2/3}; \quad [2,\ §5] \tag{PSa,b}$$

$$u = f'(t)x/f(t), \qquad 4h = a(1 - x^2/f^2(t))/f(t), \quad f'(t) = (1 - a/f(t))^{1/2}; \tag{PSc}$$

$$x = 2ht^2 s + 0.5\ln(1 + 2s/(1 - s)), \quad u = 2hts, \quad s^2 = 1 - h/(b - h^2t^2); \quad [3,\ §3]. \tag{PSd}$$

Here a and b are arbitrary constants, and $'$ denotes the derivative. Consider the generalized shallow-water equations:

$$v_t + F(v,\zeta)v_x + 0.5G_1(v,\zeta)\zeta_x = 0, \qquad \zeta_t + F(v,\zeta)\zeta_x + 0.5G_2(v,\zeta)v_x = 0. \tag{GSW}$$

THEOREM. Let $u = u(x,t)$ and $h = h(x,t)$ be a solution of (SW).

a) If

$$F(v,\zeta) = f_1(v+\zeta) + f_2(v-\zeta) \quad \text{and} \quad G_1(v,\zeta) = G_2(v,\zeta) = f_1(v+\zeta) - f_2(v-\zeta), \tag{1}$$

then relations

$$f_1(v+\zeta) + f_2(v-\zeta) = u(x,t) \quad \text{and} \quad f_1(v+\zeta) - f_2(v-\zeta) = 2h^{1/2}(x,t) \tag{2}$$

give a solution of (GSW).

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b) If

$$F(v,\zeta) = f_1(v)f_2(\zeta) + q, \quad G_1(v,\zeta) = (c_1 + f_1^2)f_2'/f_1', \tag{3}$$

and

$$G_2(v,\zeta) = (c_2 + f_2^2)f_1'/f_2',$$

then relations

$$f_1(v)f_2(\zeta) + q = u(x,t) \quad \text{and} \quad (c_1 + f_1^2)(c_2 + f_2^2) = 4h(x,t) \tag{4}$$

give a solution of (GSW). Here f_1 and f_2 are arbitrary functions, and q is a constant.

Proof. Inserting $u = F(v,\zeta)$ and $h = G(v,\zeta)$ into (SW), we get

$$v_t + (F + A(v,\zeta))v_x + B_1(v,\zeta)\zeta_x = 0, \qquad \zeta_t + (F - A(v,\zeta))\zeta_x + B_2(v,\zeta)v_x = 0. \tag{5}$$

Here $A = (G_vG_\zeta - GF_vF_\zeta)/D$, $B_1 = (G_\zeta^2 - GF_\zeta^2)/D$, $B_2 = (GF_v^2 - G_v^2)/D$, and $D(v,\zeta) = F_vG_\zeta - G_vF_\zeta$.

a) Considering $F(v,\zeta) = f_1(v+\zeta) + f_2(v-\zeta)$ and $G = [f_1(v+\zeta) - f_2(v-\zeta)]^2/4$ we obtain $A = 0$, $B_1 = B_2 = [f_1(v+\zeta) - f_2(v-\zeta)]/2$. Thus (5) becomes (GSW) with F, G_1, and G_2 as given by (1).

b) Considering $F(v,\zeta) = f_1(v)f_2(\zeta) + q$ and $G = (c_1 + f_1^2)(c_2 + f_2^2)/4$ we obtain $A = 0$, $B_1 = 0.5(c_1 + f_1^2)f_2'/f_1'$, $B_2 = 0.5(c_2 + f_2^2)f_1'/f_2'$. Thus (5) becomes (GSW) with F, G_1, and G_2 as given by (3). □

EXAMPLE 1. Taking $f_1(z) = f_2(z) = z^n$ in (1) we get (GSW) with $F = (v+\zeta)^n + (v-\zeta)^n$, $G_1 = G_2 = (v+\zeta)^n - (v-\zeta)^n$. This equation has exact analytical solutions $2v(x,t) = (u/2 + h^{1/2})^{1/n} + (u/2 - h^{1/2})^{1/n}$, $2\zeta(x,t) = (u/2 + h^{1/2})^{1/n} - (u/2 - h^{1/2})^{1/n}$ with $u(x,t)$ and $h(x,t)$ from (PSa-d).

EXAMPLE 2 (Unsteady two-layer fluid flow). Taking $f_1(x) = f_2(x) = x$ and $c_1 = c_2 = -1$ in (3) we get (GSW) in the form

$$v_t + (v\zeta + q)v_x - 0.5(1 - v^2)\zeta_x = 0, \qquad \zeta_t + (v\zeta + q)\zeta_x - 0.5(1 - \zeta^2)v_x = 0. \tag{SW2}$$

This system describes a flow of two homogeneous inviscid fluids between two horizontal rigid plates in the hydrostatic and Boussinesq approximations (see (3.3.12) in [1]). The velocity u_1, u_2 and the thickness h_1, h_2 of the upper lighter (density ρ_1) and lower denser (density ρ_2) layer (see Fig. 1) are expressed in terms of v and ζ as follows:

$$u_1/c_0 = q + v(1+\zeta)/2, \quad u_2/c_0 = q - v(1-\zeta)/2,$$
$$h_1/H = (1-\zeta)/2, \quad h_2/H = (1+\zeta)/2.$$

Here H is a distance between the plates, $c_0^2 = g'H$, $g' = g(\rho_2 - \rho_1)/\rho_2$, g is the acceleration of gravity, $q = Q/c_0H$, Q is the total volume flux, and $-1 < \zeta < 1$.

If $u(x,t)$, $h(x,t)$ is a solution of (SW), then $v = w_+ + w_-$, $\zeta = w_+ - w_-$ is a solution of (SW2). Here $w_\pm = [(1 \pm u(x-qt,t))^2/4 - h(x-qt,t)]^{1/2}$. Using (PSa–d) we get exact analytical solutions of (SW2). In particular, using (PSd) with $b = 1/4$ we have

$$x - qt = \zeta(1-\zeta^2)t^2/[1 + (1 + (1-\zeta^2)^2t^2)^{1/2}] + 0.5\ln(1 + 2\zeta/(1-\zeta)),$$
$$v = (1-\zeta^2)t/[1 + (1 + (1-\zeta^2)^2t^2)^{1/2}].$$

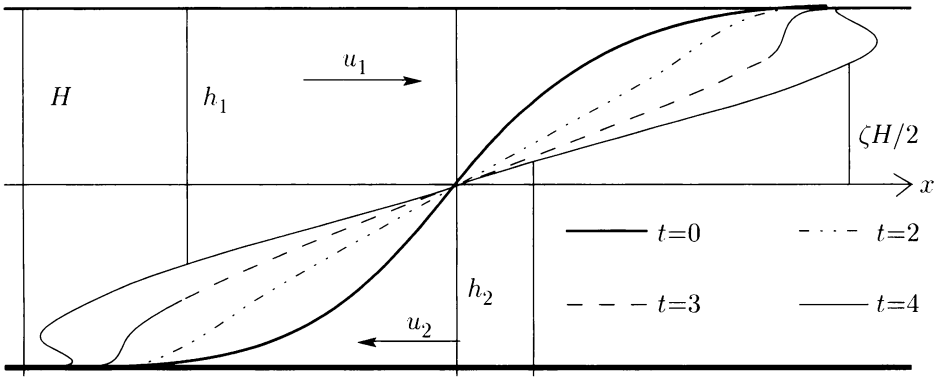

FIG. 1. Evolution of the interface described by (SW2) with $q = 0$ for initial conditions $v(x,0) = 0$, $\zeta(x,0) = \tanh x$

This is the solution of (SW2) with initial conditions $\zeta(x,0) = \tanh x$, $v(x,0) = 0$. Let $q = 0$. For $x \ll 1$ we have $\zeta \approx x(1+t^2)^{1/2}$ and $v \approx t/[1+(1+t^2)^{1/2}]$. The maximum of $v(x,t)$ is at $x = 0$ and increases from 0 to 1 when t increases from 0 to infinity. Figure 1 shows the position of the interface for $t = 0, 2, 3$, and 4.

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