A NOTE ON EXACT PARTICULAR SOLUTIONS
OF THE GENERALIZED SHALLOW-WATER EQUATIONS

BY

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Abstract. This note presents a set of systems of two first-order quasi-linear partial
differential equations, which can be reduced to the shallow-water equations. This set
includes equations describing a two-layered fluid flow.

Introduction. The shallow-water equations
\[
\begin{align*}
\frac{u_t}{u_x} + \frac{u}{h} + \frac{h}{x} &= 0, \\
\frac{h_t}{u_x} + \frac{u}{h} + \frac{h}{x} &= 0
\end{align*}
\] (SW)
have been intensively studied analytically and numerically (see, e.g., [2], [4, §13.10]).
Several exact analytical solutions of (SW) are known:
\[
\begin{align*}
\frac{u}{x/t} &= \frac{a}{t} \quad \frac{h}{(x/t - b)^2/9 + a/t^2}; \quad [2, \S 5] \\
\frac{u}{f'(t)x/f(t)} &= \frac{4h}{a(1 - x^2/f^2(t))/f(t)} \quad \frac{f'(t)}{(1 - a/f(t))^{1/2}}; \\
x &= 2ht^2s + 0.5\ln(1 + 2s/(1 - s)) \quad \frac{u}{2hts}; \quad \frac{s^2}{1 - h/(b - h^2t^2)}; \quad [3, \S 3].
\end{align*}
\]
(PSa,b)
(PSc)
(PSd)

Here a and b are arbitrary constants, and ' denotes the derivative. Consider the gener-
alized shallow-water equations:
\[
\begin{align*}
\frac{v_t}{F(v, \zeta)v_x} + 0.5G_1(v, \zeta)\zeta_x &= 0, \\
\zeta_t + F(v, \zeta)\zeta_x + 0.5G_2(v, \zeta)v_x &= 0.
\end{align*}
\] (GSW)

Theorem. Let \(u = u(x,t)\) and \(h = h(x,t)\) be a solution of (SW).

a) If
\[
\begin{align*}
F(v, \zeta) &= f_1(v + \zeta) + f_2(v - \zeta) \quad \text{and} \quad G_1(v, \zeta) &= G_2(v, \zeta) = f_1(v + \zeta) - f_2(v - \zeta), \\
\end{align*}
\]
then relations
\[
\begin{align*}
f_1(v + \zeta) + f_2(v - \zeta) &= u(x,t) \quad \text{and} \quad f_1(v + \zeta) - f_2(v - \zeta) = 2h^{1/2}(x,t)
\end{align*}
\]
give a solution of (GSW).
b) If

\[ F(v, \zeta) = f_1(v)f_2(\zeta) + q, \quad G_1(v, \zeta) = (c_1 + f_1^2)f_2' / f_1', \]

and

\[ G_2(v, \zeta) = (c_2 + f_2^2)f_1' / f_2', \]

then relations

\[ f_1(v)f_2(\zeta) + q = u(x, t) \quad \text{and} \quad (c_1 + f_1^2)(c_2 + f_2^2) = 4h(x, t) \]

give a solution of (GSW). Here \( f_1 \) and \( f_2 \) are arbitrary functions, and \( q \) is a constant.

**Proof.** Inserting \( u = F(v, \zeta) \) and \( h = G(v, \zeta) \) into (SW), we get

\[ v_t + (F + A(v, \zeta))v_x + B_1(v, \zeta)c_x = 0, \quad \zeta_t + (F - A(v, \zeta))\zeta_x + B_2(v, \zeta)v_x = 0. \]

Here \( A = (G_v G_\zeta - G F_v F_\zeta) / D, \ B_1 = (G_\zeta^2 - G F_\zeta^2) / D, \ B_2 = (G F_\zeta^2 - G_\zeta^2) / D, \) and \( D(v, \zeta) = F_v G_\zeta - G_v F_\zeta. \)

a) Considering \( F(v, \zeta) = f_1(v + \zeta) + f_2(v - \zeta) \) and \( G = \{f_1(v + \zeta) - f_2(v - \zeta)\}^2 / 4 \) we obtain \( A = 0, \ B_1 = B_2 = (f_1^2(v + \zeta) - f_2^2(v - \zeta)) / 4. \) Thus (5) becomes (GSW) with \( F, G_1, \) and \( G_2 \) as given by (1).

b) Considering \( F(v, \zeta) = f_1(v)f_2(\zeta) + q \) and \( G = (c_1 + f_1^2)(c_2 + f_2^2) / 4 \) we obtain \( A = 0, \ B_1 = 0.5(c_1 + f_1^2)f_2' / f_1', \ B_2 = 0.5(c_2 + f_2^2)f_1' / f_2'. \) Thus (5) becomes (GSW) with \( F, G_1, \) and \( G_2 \) as given by (3). \( \square \)

**Example 1.** Taking \( f_1(z) = f_2(z) = z^n \) in (1) we get (GSW) with \( F = (v + \zeta)^n + (v - \zeta)^n, \ G_1 = G_2 = (v + \zeta)^n - (v - \zeta)^n. \) This equation has exact analytical solutions

\[ 2v(x, t) = \left[ u/2 + h_1'^2 \right]^{1/2} + \left[ u/2 - h_1'^2 \right]^{1/2}, \quad 2\zeta(x, t) = \left[ u/2 + h_1'^2 \right]^{1/n} - \left[ u/2 - h_1'^2 \right]^{1/n} \]

with \( u(x, t) \) and \( h(x, t) \) from (PSa-d).

**Example 2 (Unsteady two-layer fluid flow).** Taking \( f_1(x) = f_2(x) = x \) and \( c_1 = c_2 = -1 \) in (3) we get (GSW) in the form

\[ v_t + (v \zeta + q)v_x - 0.5(1 - v^2)\zeta_x = 0, \quad \zeta_t + (v \zeta + q)\zeta_x - 0.5(1 - \zeta^2)v_x = 0. \]

This system describes a flow of two homogeneous inviscid fluids between two horizontal rigid plates in the hydrostatic and Boussinesq approximations (see (3.3.12) in [1]). The velocity \( u_1, u_2 \) and the thickness \( h_1, h_2 \) of the upper lighter (density \( p_1 \)) and lower denser (density \( p_2 \)) layer (see Fig. 1) are expressed in terms of \( v \) and \( \zeta \) as follows:

\[ u_1/c_0 = q + v(1 + \zeta)/2, \quad u_2/c_0 = q - v(1 - \zeta)/2, \]
\[ h_1/H = (1 - \zeta)/2, \quad h_2/H = (1 + \zeta)/2. \]

Here \( H \) is a distance between the plates, \( c_0^2 = g'H, \ g' = g(p_2 - p_1)/p_2, \ g \) is the acceleration of gravity, \( q = Q/c_0 H, \ Q \) is the total volume flux, and \(-1 < \zeta < 1\).

If \( u(x, t), h(x, t) \) is a solution of (SW), then \( v = w_+ + w_-, \ \zeta = w_+ - w_- \) is a solution of (SW2). Here \( w_{\pm} = \{(1 \pm u(x - qt, t))^2/4 - h(x - qt, t)\}^{1/2}. \) Using (PSa–d) we get exact analytical solutions of (SW2). In particular, using (PSd) with \( b = 1/4 \) we have

\[ x - qt = \zeta(1 - \zeta^2)t^2/[1 + (1 + (1 - \zeta^2)^2t^2)^{1/2}] + 0.5 \ln(1 + 2\zeta/(1 - \zeta)), \]
\[ v = (1 - \zeta^2)t/[1 + (1 + (1 - \zeta^2)^2t^2)^{1/2}]. \]
This is the solution of (SW2) with initial conditions $\zeta(x,0) = \tanh x$, $v(x,0) = 0$. Let $q = 0$. For $x \ll 1$ we have $\zeta \approx x(1 + t^2)^{1/2}$ and $v \approx t/[1 + (1 + t^2)^{1/2}]$. The maximum of $v(x,t)$ is at $x = 0$ and increases from 0 to 1 when $t$ increases from 0 to infinity. Figure 1 shows the position of the interface for $t = 0, 2, 3,$ and 4.

**References**


