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EXACT SOLUTIONS OF COMPRESSIBLE PLANE POTENTIAL FLOWS— A NEW METHOD

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1. Introduction. This work deals with exact closed-form solutions of steady plane irrotational flow of inviscid compressible fluids when the pressure is a single-valued function of density only. Three methods for finding these solutions exist in the literature. Two of these methods are well-known methods which are documented in texts (e.g., von Mises, 1958, and Courant and Friedrichs, 1948). This work describes a fourth method and it complements the recently published third method (Chandna, Husain and Latypov, 1997). In the third method, one seeks solutions for all possible flows that correspond to a chosen functional form for the streamline pattern. The present method deals with the individual flow problems, like the first two methods. The necessity for the present method arises from two serious shortcomings in the two existing methods that deal with individual flow problems. First, only a small number of problems could be dealt with by the available methods. Second, mostly incomplete solution sets were obtained by the application of these two methods.

The first method starts with a solution of the nonlinear partial differential equation for potential functions to determine the exact solutions for the flow corresponding to this solution. G. I. Taylor (1930) was probably the first who used this approach. Even though excellent analysis can be carried out by using this method, the complete solution set could not be found for any problem considered. Study of the simple radial flow is an excellent example of this weakness (cf. von Mises, 1958).

The second method, called the hodograph method due to Molenbrock (1890) and Chaplygin (1904), primarily deals with two second-order linear homogeneous partial differential equations derived in the hodograph plane. One of these two equations has the stream function in the hodograph plane as its dependent variable and the second equation has the Legendre transform function of the potential function as its dependent

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variable. We start with a solution of one of these equations and use transformation equations to obtain the dynamic variables for the corresponding flow problem in the physical plane. Radial flow is an example of a flow studied by using a simple solution of the equation in the stream function, and vortex flow is an example of a flow studied by using a simple solution of the equation in the Legendre transform function of the potential function. However, it is not possible to obtain the thermodynamic variables and the permissible gas or gases that can adopt these resulting flow patterns. A handful of other flows, using the hodograph method, have been investigated with similar partial successes (cf. von Mises, 1958, and Courant and Friedrichs, 1948).

Appreciating fully the value and the benefits of exact solutions, such as checking the accuracy of solutions obtained by numerical and approximate methods, for the advancement of the subject, and realizing the weaknesses of the two existing methods, the development of the present method was a necessity. This present method is based on a fortuitous combination of von Kármán's observations in his work (von Kármán, 1941) and M. H. Martin's idea of using new coordinate systems for expressing flow equations in the physical plane (Martin, 1971).

Three critical factors that form the underlying basis for this method are: (i) the equipotential lines and the streamlines generate an orthogonal curvilinear net in the physical plane for both compressible and incompressible steady plane irrotational flows of inviscid fluids; (ii) $\mathrm{Re}[f(z)] = \phi(x,y)$ and $\mathrm{Im}[f(z)] = \psi(x,y)$ for the incompressible case when $\phi(x,y)$ and $\psi(x,y)$ are the potential function and the stream function of the flow, and $f(z)$ is an analytic function of $z = x + iy$ with $\mathrm{Im}[f(z)] =$ constant defining the flow streamlines; (iii) the elements of the network formed by the streamlines and the equipotential lines get elongated in the flow direction or normal to the flow direction according as the density increases or decreases along the flow direction in the compressible case. As a consequence, $\mathrm{Re}[f(z)] = \gamma(\phi)$ and $\mathrm{Im}[f(z)] = \Gamma(\psi)$ when $\mathrm{Re}[f(z)] =$ constant and $\mathrm{Im}[f(z)] =$ constant coincide with the equipotential lines and the streamlines for the compressible case having the potential function $\phi(x,y)$ and the stream function $\psi(x,y)$. The unknown functions $\gamma(\phi)$ and $\Gamma(\psi)$ are problem dependent, i.e, depend upon the analytic function $f(z)$.

Exact complete closed-form solutions for four flow problems are obtained to illustrate this new method. Letting $f(z) = \xi(x,y) + i\eta(x,y) = \xi(x,y) + +i\Gamma(\psi)$, we use (ξ,ψ)-coordinates to investigate the flow problem having $\mathrm{Im}[f(z)] =$ constant as the flow streamlines. A doublet flow, a vortex flow, a spiral flow, and a source flow are studied to fully illustrate our method.

This paper is organized as follows. In Sec. 2, we recapitulate the basic governing equations for the steady plane motion of compressible fluids and recast these in Martin's form. Section 3 contains exact closed-form solutions in subsections 3.1 to 3.4.

2. Flow equations. Consider the irrotational isentropic steady plane motion of a compressible inviscid fluid in the (x,y)-plane. This motion, subject to no external forces, is governed by the continuity equation, the dynamical equations and the irrotationality condition given by (cf. Courant and Friedrichs, 1948):

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] + \frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] + \frac{\partial p}{\partial y} = 0, \tag{3}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4}$$

where the velocity components u, v, the pressure p, and the density ρ are functions of x, y.

Equations (1)–(4) constitute a determinate system of four partial differential equations for four unknown functions u, v, p, and ρ. We do not need to prescribe an additional relation, called the equation of state, and deal with the determinate system (1)–(4). However, we reserve the right to prescribe the equation of state whenever it is convenient to do so.

Using the irrotationality condition (4) in the linear momentum equations (2) and (3), integration of these equations yields

$$\frac{1}{2}(u^2 + v^2) + \int \frac{dp}{\rho} = \text{constant}. \tag{5}$$

This equation determines ρ as a function of $u^2 + v^2$ for any given equation of state $p = p(\rho)$.

2.1. *Martin's form for flow equations.* Equation (1) implies the existence of a stream function $\psi = \psi(x, y)$ for which

$$d\psi = -\rho v\, dx + \rho u\, dy, \quad \text{or } \frac{\partial \psi}{\partial x} = -\rho v, \quad \frac{\partial \psi}{\partial y} = \rho u. \tag{6}$$

Curvilinear coordinates ξ, ψ are introduced in the physical plane, in which the curves $\psi(x, y) = $ constant are the streamlines and the curves $\xi(x, y) = $ constant are chosen arbitrarily. We seek x, y, u, v, p, and ρ as functions of ξ, ψ instead of the independent Cartesian variables x, y.

Let

$$x = x(\xi, \psi), \qquad y = y(\xi, \psi) \tag{7}$$

define the curvilinear net in the (x, y)-plane, such that the transformation Jacobian J satisfies

$$0 < |J| < \infty, \qquad J = \left|\frac{\partial(x, y)}{\partial(\xi, \psi)}\right| \tag{8}$$

and the squared element of arc length along any curve is given by

$$ds^2 = E(\xi, \psi)\, d\xi^2 + 2F(\xi, \psi)\, d\xi\, d\psi + G(\xi, \psi)\, d\psi^2 \tag{9}$$

where

$$
\begin{aligned}
E &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2, \\
F &= \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \psi}, \\
G &= \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 .
\end{aligned} \tag{10}
$$

Equations (7) can be solved to determine ξ, ψ as functions of x, y so that

$$
\begin{aligned}
\frac{\partial x}{\partial \xi} &= J\frac{\partial \psi}{\partial y} \\
\frac{\partial x}{\partial \psi} &= -J\frac{\partial \xi}{\partial y}, \\
\frac{\partial y}{\partial \xi} &= -J\frac{\partial \psi}{\partial x}, \\
\frac{\partial y}{\partial \psi} &= J\frac{\partial \xi}{\partial x}.
\end{aligned} \tag{11}
$$

Using (10), we get

$$
J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \xi} = \pm\sqrt{EG - F^2} = \pm W \text{ (say)}. \tag{12}
$$

Letting δ be the local angle of inclination of the tangent to a coordinate line ψ =constant, directed in the sense of increasing ξ, we have from differential geometry the following (cf. Martin, 1971 and Watherburn, 1939):

$$
\frac{\partial x}{\partial \xi} = \sqrt{E}\cos\delta, \qquad \frac{\partial y}{\partial \xi} = \sqrt{E}\sin\delta, \tag{13}
$$

$$
\frac{\partial x}{\partial \psi} = \frac{F}{\sqrt{E}}\cos\delta - \frac{J}{\sqrt{E}}\sin\delta, \qquad \frac{\partial y}{\partial \psi} = \frac{F}{\sqrt{E}}\sin\delta + \frac{J}{\sqrt{E}}\cos\delta, \tag{14}
$$

$$
\frac{\partial \delta}{\partial \xi} = \frac{J}{E}\Gamma_{11}^{2}, \qquad \frac{\partial \delta}{\partial \psi} = \frac{J}{E}\Gamma_{12}^{2}, \tag{15}
$$

$$
K = \frac{1}{W}\left[\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^{2}\right) - \frac{\partial}{\partial \xi}\left(\frac{W}{E}\Gamma_{12}^{2}\right)\right] = 0, \tag{16}
$$

where

$$
\Gamma_{11}^{2} = \frac{1}{2W^2}\left[-F\frac{\partial E}{\partial \xi} + 2E\frac{\partial F}{\partial \xi} - E\frac{\partial E}{\partial \psi}\right], \qquad \Gamma_{12}^{2} = \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \xi} - F\frac{\partial E}{\partial \psi}\right]
$$

are Christoffel's symbols and K is the Gaussian curvature. Having recorded the above results, we transform Eqs. (1) to (4) into a new form with ξ, ψ as the new independent variables and call it Martin's form. This form was derived in a previous paper [Chandna, Husain, and Latypov, 1997] and is summed up in the following theorem:

THEOREM I (Martin's Form). If the streamlines $\psi(x, y)$ = constant of a steady, plane, inviscid, potential compressible fluid flow are taken as a set of coordinate curves in a curvilinear coordinate system (ξ, ψ) in the physical plane, the system of four partial differential equations for u, v, p, and ρ as functions of (x, y) may be replaced by the system

$$\begin{aligned}
&\frac{1}{2}\rho\frac{\partial}{\partial\xi}(q^2) + \frac{\partial p}{\partial\xi} = 0, \quad \frac{1}{2}\rho\frac{\partial}{\partial\psi}(q^2) + \frac{\partial p}{\partial\psi} = 0 && \text{(linear momentum)} \\
&\frac{\partial}{\partial\xi}\left(\frac{F}{\sqrt{E}}q\right) - \frac{\partial}{\partial\psi}(\sqrt{E}q) = 0 && \text{(irrotationality)} \\
&\rho q = \frac{\sqrt{E}}{W} && \text{(continuity)} \\
&\frac{\partial}{\partial\psi}\left(\frac{W}{E}\Gamma^2_{11}\right) - \frac{\partial}{\partial\xi}\left(\frac{W}{E}\Gamma^2_{12}\right) = 0 && \text{(Gauss)}
\end{aligned} \tag{17}$$

of five equations in six unknowns E, F, G, ρ, q, and p as functions of ξ, ψ.

Here, E, F, G are the coefficients of the first fundamental form (9), Γ^2_{11} and Γ^2_{12} are Christoffel's symbols, and $W = \sqrt{EG - F^2}$.

Given a solution

$$\begin{aligned}
E &= E(\xi, \psi), & F &= F(\xi, \psi), & G &= G(\xi, \psi), \\
q &= q(\xi, \psi), & p &= p(\xi, \psi), & \rho &= \rho(\xi, \psi)
\end{aligned}$$

of the system, the (ξ, ψ)-plane is mapped upon the physical plane by

$$z = \int \frac{e^{i\delta}}{\sqrt{E}}[E\,d\xi + (F + iW)\,d\psi], \qquad \delta = \int \frac{W}{E}(\Gamma^2_{11}\,d\xi + \Gamma^2_{12}\,d\psi)$$

and upon the hodograph plane by

$$u + iv = \frac{\sqrt{E}}{\rho W}\exp(i\delta).$$

The state equation for a gas, for a given streamline pattern, is determined from the solutions for p and ρ.

The system of five equations in six unknowns in Theorem I is an underdetermined system because of the arbitrariness inherent in the choice of the coordinate lines $\xi =$ constant. However, application of this theorem to find exact closed-form solutions for a given flow problem is carried out only after this system has been made determinate.

3. Exact solutions. The streamlines and the equipotential lines generate a network of mutually orthogonal families of curves for both inviscid incompressible and inviscid compressible steady plane irrotational fluid flows. All elements of the network are of square shape for an incompressible flow so that the network is isometric. However, the elements of the network are elongated in the flow direction or normal to the flow direction, for a compressible flow, according as the density increases or decreases along the flow (cf. von Kármán, 1941).

Similarly, there is an inherent distinction when complex analysis is used to study these incompressible or compressible flows. Let $\phi(x, y)$ represent the potential function of the flow. For the incompressible case, if $f(z) = \xi(x, y) + i\eta(x, y)$ is an analytic function of $z = x + iy$ such that $\xi(x, y) =$ constant and $\eta(x, y) =$ constant coincide with the equipotential lines and the streamlines respectively, then $\xi(x,y) = \phi(x,y)$ and $\eta(x,y) = \psi(z,y)$.

However, for the compressible case, $\xi(x,y)$ and $\eta(x,y)$ are, respectively, some functions of the functions $\phi(x,y)$ and $\psi(x,y)$. This functional form depends upon $f(z)$. We exploit this knowledge and the (ξ,ψ)-coordinates to investigate exact solutions.

We study four flow problems in the next section to illustrate this new approach. These problems have previously been investigated by the hodograph method, but complete results could not be obtained [cf. von Mises, 1958, and A. H. Shapiro, 1953].

3.1. *Doublet flow.* Craggs (1951) investigated this flow, when pressure is a single-valued function of density only. This mathematically beautiful work, also documented in von Mises (1958), is carried out in the hodograph plane. Due to the inherent weakness of the hodograph method, the complete exact solution set in the physical plane could not be obtained. Furthermore, the question of whether this flow was possible for a real gas remained unanswered. The present method overcomes these shortcomings.

Taking the analytic function $f(z) = \xi(r,\theta) + i\eta(r,\theta) = \frac{1}{z}$, where $z = x + iy = r\exp(i\theta)$, we discuss the doublet flow with circles $\xi(r,\theta) = \frac{1}{r}\cos\theta = $ constant as the equipotential lines and circles $\eta(r,\theta) = -\frac{1}{r}\sin\theta = $ constant as the streamlines. As indicated above, for this compressible flow, we have

$$\begin{aligned} \xi(r,\theta) &= \frac{1}{r}\cos\theta = \beta(\phi), \\ \eta(r,\theta) &= -\frac{1}{r}\sin\theta = \Gamma(\psi) \end{aligned} \tag{18}$$

so that

$$\begin{aligned} r &= \frac{1}{\sqrt{\xi^2+\eta^2}} = \frac{1}{\sqrt{\xi^2+\Gamma^2(\psi)}}, \\ \theta &= \arctan\left(-\frac{\eta}{\xi}\right) = \arctan\left[\frac{-\Gamma(\psi)}{\xi}\right]. \end{aligned}$$

Here, $\Gamma(\psi)$ is an unknown function of the stream function $\psi(r,\theta)$ for the flow. In the limit, when the fluid is incompressible, $\beta(\phi) = \phi$ and $\Gamma(\psi) = \psi$.

Considering (ξ,ψ) as a set of independent variables, we have

$$\begin{aligned} ds^2 &= dr^2 + r^2\,d\theta^2 \\ &= \frac{1}{(\xi^2+\eta^2)^2}[d\xi^2 + d\eta^2] = \frac{1}{\{\xi^2+\Gamma^2(\psi)\}^2}[d\xi^2 + \Gamma'^2(\psi)\,d\psi^2]. \end{aligned} \tag{19}$$

The metric coefficients for the chosen net and the transformation Jacobian are

$$E(\xi,\psi) = \frac{1}{\{\xi^2+\Gamma^2(\psi)\}^2}, \quad F(\xi,\psi) = 0, \quad G(\xi,\psi) = \frac{\Gamma'^2(\psi)}{\{\xi^2+\Gamma^2(\psi)\}^2},$$

and

$$J(\xi,\psi) = \left|\frac{\partial(x,y)}{\partial(\xi,\psi)}\right| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\frac{\partial(r,\theta)}{\partial(\xi,\psi)}\right| = \frac{\Gamma'(\psi)}{\{\xi^2+\Gamma^2(\psi)\}^2}. \tag{20}$$

Furthermore,

$$W(\xi,\psi) = \sqrt{EG-F^2} = \frac{|\Gamma'(\psi)|}{\{\xi^2+\Gamma^2(\psi)\}^2}. \tag{21}$$

We note that $J > 0$ if $\Gamma'(\psi) > 0$. In this case, the fluid flows along a streamline in the direction of increasing ξ. Without loss of generality, we assume that $\Gamma'(\psi) > 0$.

Employing (20) and (21) in Eqs. (17), we observe that the Gauss equation is identically satisfied and the flow along a family of curves $\mathrm{Im}(\frac{1}{z}) =$ constant is governed by the system of four equations

$$\begin{aligned} \rho q \frac{\partial q}{\partial \xi} + \frac{\partial p}{\partial \xi} &= 0, \\ \rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} &= 0, \\ \frac{\partial}{\partial \psi}\left[\frac{1}{\xi^2 + \Gamma^2(\psi)} q\right] &= 0, \\ \rho q &= \left[\frac{\xi^2 + \Gamma^2(\psi)}{\Gamma'(\psi)}\right] \end{aligned} \tag{22}$$

in four unknown functions $\rho(\xi,\psi)$, $q(\xi,\psi)$, $p(\xi,\psi)$, and $\Gamma(\psi)$. Defining the flow intensity

$$\rho q = \left[\frac{\xi^2 + \Gamma^2(\psi)}{\Gamma'(\psi)}\right] = \alpha(\xi,\psi) \tag{23}$$

we observe that

$$\begin{aligned} \frac{\partial \alpha}{\partial \xi} &= \frac{2\xi}{\Gamma'(\psi)}, \\ \frac{\partial \alpha}{\partial \psi} &= \left[\frac{2\Gamma(\psi)\Gamma'^2(\psi) - \Gamma^2(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)}\right] - \xi^2\left[\frac{\Gamma''(\psi)}{\Gamma'^2(\psi}\right]. \end{aligned} \tag{24}$$

Employing (23) in the first two equations of (22), multiplying these by $d\xi$, $d\psi$ respectively and adding, yields

$$\alpha\, dq + dp = 0. \tag{25}$$

Using the integrability condition $\frac{\partial^2 p}{\partial\xi\partial\psi} = \frac{\partial^2 p}{\partial\psi\partial\xi}$ gives

$$\frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} - \frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} = \left|\frac{\partial(\alpha, q)}{\partial(\xi,\psi)}\right| = 0. \tag{26}$$

Since $\frac{\partial \alpha}{\partial \xi} \neq 0$, $\frac{\partial \alpha}{\partial \psi} \neq 0$, and the flow is non-uniform, it follows from (26) and (23) that

$$q = q(\alpha), \qquad \rho = \frac{\alpha}{q(\alpha)}. \tag{27}$$

Using (23) and (24) in the third equation of (22) and simplifying, we have

$$\frac{\Gamma''(\psi)}{2\Gamma(\psi)\Gamma'(\psi)} = \frac{\alpha q'(\alpha) - q(\alpha)}{\alpha^2 q'(\alpha)}. \tag{28}$$

Since $J^\star = |\frac{\partial(\alpha,\psi)}{\partial(\xi,\psi)}| = \frac{\partial\alpha}{\partial\xi} = \frac{2\xi}{\Gamma'(\psi)} \neq 0$ in the flow domain, it follows that the variables α, ψ may be regarded as two independent variables. Using this fact, each side of Eq. (28) is equal to a separation constant k_0 (say). Therefore, Eq. (28) yields

$$\begin{aligned} \Gamma''(\psi) - 2k_0\Gamma(\psi)\Gamma'(\psi) &= 0, \\ \alpha(1 - k_0\alpha)q'(\alpha) - q(\alpha) &= 0. \end{aligned}$$

Integrating these equations, we obtain

$$\Gamma'(\psi) = k_0\Gamma^2(\psi) + k_1,$$
$$q(\alpha) = \frac{k_2\alpha}{(1 - k_0\alpha)}, \tag{29}$$

where k_0, k_1, and $k_2 \neq 0$ are arbitrary constants. These constants are such that $k_0 \to 0$, $k_1 \to 1$, and $k_2 \to 1$ as the flow approaches the incompressible case of unit density, in which case $\Gamma(\psi) = \psi$.

Returning to Cartesian coordinates and using (18), (29), (23), and (25), we get

$$\eta = \frac{-y}{x^2 + y^2}, \tag{30}$$

$$\Gamma'(\psi) = \frac{k_0 y^2 + k_1(x^2 + y^2)^2}{(x^2 + y^2)^2}, \tag{31}$$

$$\alpha = \frac{(x^2 + y^2)}{k_0 y^2 + k_1(x^2 + y^2)^2}, \tag{32}$$

$$q = \frac{k_2(x^2 + y^2)}{k_1(x^2 + y^2)^2 - k_0 x^2}, \tag{33}$$

$$\rho = \frac{1 - k_0\alpha}{k_2} = \frac{1}{k_2}\left[\frac{k_1(x^2 + y^2)^2 - k_0 x^2}{k_1(x^2 + y^2)^2 + k_0 y^2}\right], \tag{34}$$

$$\begin{aligned} p &= p_0 - \int_0^\alpha \alpha q'(\alpha)\, d\alpha \\ &= \begin{cases} p_0 - \dfrac{k_2}{k_0^2}\left[\ln(1 - k_0\alpha) + \dfrac{1}{1 - k_0\alpha} - 1\right]; k_0 \neq 0, \\ p_0 - \frac{1}{2}\alpha^2; k_0 = 0, \end{cases} \\ &= \begin{cases} p_0 - \dfrac{k_2}{k_0^2}\left[\ln\left\{\dfrac{k_1(x^2 + y^2)^2 - k_0 x^2}{k_1(x^2 + y^2)^2 + k_0 y^2}\right\} + \dfrac{k_1(x^2 + y^2)^2 + k_0 y^2}{k_1(x^2 + y^2)^2 - k_0 x^2} - 1\right]; k_0 \neq 0, \\ p_0 - \dfrac{1}{2(x^2 + y^2)^2}; k_0 = 0, \end{cases} \end{aligned} \tag{35}$$

where p_0 is the stagnation pressure and the two expressions for $p(x, y)$ are for the compressible case when $k_0 \neq 0$ and for the incompressible case when $k_0 = 0$.

Differentiating (30) with respect to y and x, using $\frac{\partial\psi}{\partial y} = \rho u$, $\frac{\partial\psi}{\partial x} = -\rho v$ and Eq. (31), we get

$$\rho u = \frac{y^2 - x^2}{k_0 y^2 + k_1(x^2 + y^2)^2}$$

and

$$\rho v = \frac{-2xy}{k_0 y^2 + k_1(x^2 + y^2)^2}.$$

Employing (34) in these equations, the velocity components are given by

$$u = u(x, y) = \frac{k_2(y^2 - x^2)}{k_1(x^2 + y^2)^2 - k_0 x^2} \tag{36}$$

and

$$v = v(x,y) = \frac{-2k_2xy}{k_1(x^2+y^2)^2 - k_0x^2}. \tag{37}$$

Summing up, we have:

THEOREM II. If $\frac{1}{r}\sin\theta$ = constant are the flow streamlines for the steady plane irrotational doublet flow of an inviscid isentropic compressible fluid, then the solution set for the determinate system (1) to (4) of four partial differential equations for the four unknown functions (u, xy), $v(x,y)$, $p(x,y)$, and $\rho(x,y)$ is given by (36), (37), (35), and (34) respectively.

Eliminating α from (34) and (35), it can be shown that the equation of state for the flow must be

$$p = p^\star - \frac{k_2}{k_0^2}\ln\rho - \frac{1}{k_0^2\rho}$$

where $p^\star = p_0 - \frac{k_2}{k_0^2}\ln k_2$. Therefore, a doublet flow is not possible for a polytropic gas as was presumed.

Since $\Gamma(\psi) = \psi$ for the limiting case of incompressible flow and $\rho q = \alpha$ in (23) reduces to $q = \alpha$ if $\rho = 1$, it follows from (29) that $k_0 = 0$, $k_1 = k_2 = 1$. Using these in the solution set (34) to (37) for the compressible flow, the known solution set for the incompressible flow is also recovered:

$$u(x,y) = \frac{(y^2-x^2)}{(x^2+y^2)^2}, \qquad v(x,y) = \frac{-2xy}{(x^2+y^2)^2},$$

and

$$p = p_0 - \frac{1}{2(x^2+y^2)^2}.$$

3.2. *Vortex flow.* This flow was previously investigated by the hodograph method and is documented in texts (cf. von Mises, 1958). According to this method, the Legendre transform function $\Phi(q,\theta)$ of the potential function satisfies the linear partial differential equation

$$q^2\frac{\partial^2\Phi}{\partial q^2} + (1-M^2)\left[q\frac{\partial\Phi}{\partial q} + \frac{\partial^2\Phi}{\partial\theta^2}\right] = 0$$

when

$$\frac{\partial\Phi}{\partial q} = x\cos\theta + y\sin\theta, \qquad \frac{\partial\Phi}{\partial\theta} = q(y\cos\theta - x\sin\theta).$$

Vortex flow corresponds to the solution $\Phi(q,\theta) = c\theta = c\tan^{-1}(\frac{v}{u})$ of this partial differential equation. Only dynamic variables can be determined in the physical plane. Excellent analysis based on the knowledge of these dynamic variables has been carried out. However, the achieved analysis is incomplete. It is not possible to determine the two thermodynamic variables and we are therefore prevented from knowing the gases for which the vortex flow is permissible.

According to our new approach, we consider the analytic function $i \ln z$ such that $\mathrm{Im}(i \ln z) =$ constant are the flow streamlines and $\mathrm{Re}(i \ln z) =$ constant are their orthogonal trajectories. By this choice, we get

$$\mathrm{Im}(i \ln z) = \ln r = \Gamma(\psi) \tag{38}$$

so that

$$x = r\cos\theta = e^{\Gamma(\psi)}\cos\theta, \qquad y = r\sin\theta = e^{\Gamma(\psi)}\sin\theta. \tag{39}$$

Using (39) in (10) and (12), and employing (θ, ψ)-coordinates, we get

$$\begin{aligned}
E(\theta,\psi) &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = e^{2\Gamma(\psi)}, \\
F(\theta,\psi) &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \psi} = 0, \\
G(\theta,\psi) &= \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 = e^{2\Gamma(\psi)}\Gamma'^2(\psi), \\
W(\theta,\psi) &= \sqrt{EG - F^2} = e^{2\Gamma(\psi)}|\Gamma'(\psi)|,
\end{aligned} \tag{40}$$

and

$$J(\theta,\psi) = \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \theta} = -e^{2\Gamma(\psi)}\Gamma'(\psi).$$

Since $J(\theta,\psi) = -e^{2\Gamma(\psi)}\Gamma'(\psi) < 0$ or > 0 accordingly as $\Gamma'(\psi)$ is positive or negative, respectively, it follows that the circular fluid motion is clockwise in the direction of decreasing θ or anti-clockwise in the direction of increasing θ according as $\Gamma'(\psi)$ is positive or negative, respectively.

Considering the clockwise motion, so that $W(\theta,\psi) = e^{2\Gamma(\psi)}\Gamma'(\psi)$, and writing the equations of Theorem I in (θ,ψ)-coordinates, we use (40) to find that the Gauss equation is identically satisfied and

$$\rho q \frac{\partial q}{\partial \theta} + \frac{\partial p}{\partial \theta} = 0, \tag{41}$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0, \tag{42}$$

$$\frac{\partial}{\partial \psi}[e^{\Gamma(\psi)} q] = 0, \tag{43}$$

$$\rho q = \frac{1}{e^{\Gamma(\psi)}\Gamma'(\psi)}. \tag{44}$$

Employing the integrability condition $\frac{\partial^2 p}{\partial \theta\, \partial \psi} = \frac{\partial^2 p}{\partial \psi\, \partial \theta}$, we find that $\left|\frac{\partial(\rho q, q)}{\partial(\theta,\psi)}\right| = 0$, i.e., q is a function of ρq. Since ρq is only a function of ψ given by (44), it follows that q is a function of ψ only.

Using $q = q(\psi)$ in Eqs. (43), (44), (41), (42) and employing (38) yields

$$q = \frac{B}{e^{\Gamma(\psi)}} = \frac{B}{r} = \frac{B}{\sqrt{(x^2+y^2)}}, \tag{45}$$

$$\rho = \frac{1}{B\Gamma'(\psi)}, \tag{46}$$

and

$$p = \int \frac{B}{e^{2\Gamma(\psi)}}\, d\psi, \tag{47}$$

where $B > 0$ is an arbitrary constant.

Differentiating equation (38) with respect to y and x, respectively, and using (6), (46), we obtain the solution for the dynamic variables, given by

$$\begin{aligned} u(x,y) &= \frac{y}{\rho\Gamma'(\psi)(x^2+y^2)} = \frac{By}{x^2+y^2}, \\ v(x,y) &= \frac{-x}{\rho\Gamma'(\psi)(x^2+y^2)} = \frac{-Bx}{x^2+y^2}. \end{aligned} \tag{48}$$

To proceed further and obtain the complete solution set in the physical plane, it is necessary that we prescribe the state equation for the gas. Consider a polytropic gas as an example. The following approach can be taken for any other gas. The state equation for a polytropic gas is

$$p = k\rho^\gamma \tag{49}$$

where $k > 0$ and the adiabatic constant γ are known constants.

Using (47) and (46) in (49), we obtain

$$\int \frac{B}{e^{2\Gamma(\psi)}}\, d\psi = k\left[\frac{1}{B\Gamma'(\psi)}\right]^\gamma.$$

Differentiating this equation with respect to ψ and simplifying gives

$$\frac{\Gamma''(\psi)}{[\Gamma'(\psi)]^\gamma} + \frac{B^{\gamma+1}\Gamma'(\psi)}{k\gamma e^{2\Gamma(\psi)}} = 0.$$

Integrating this equation and using (38) implies that

$$\Gamma'(\psi) = \left[C - \left(\frac{\gamma-1}{2k\gamma}\right)\frac{B^{\gamma+1}}{e^{2\Gamma(\psi)}}\right]^{\frac{-1}{\gamma-1}} = \left[C - \frac{(\gamma-1)B^{\gamma+1}}{2k\gamma r^2}\right]^{-\frac{1}{\gamma-1}} \tag{50}$$

where C is an arbitrary constant.

Using (50) in (46), the density is given by

$$\rho(x,y) = \frac{1}{B}\left[C - \frac{(\gamma-1)B^{\gamma+1}}{2k\gamma(x^2+y^2)}\right]^{\frac{1}{\gamma-1}}. \tag{51}$$

Employing (51) in (49) yields the pressure

$$p(x,y) = k\left\{\left(\frac{1}{B}\right)^\gamma\left[C - \frac{(\gamma-1)(B)^{\gamma+1}}{2k\gamma(x^2+y^2)}\right]^{\frac{\gamma}{\gamma-1}}\right\}. \tag{52}$$

Summing up, we have:

THEOREM III. If $x^2 + y^2 =$ constant are the flow streamlines for the clockwise steady, plane, irrotational, inviscid compressible vortex flow of a polytropic gas, then the solution set in the physical plane is given by (48), (51), and (52).

The vortex flow is permissible for every gas with dynamic variables given by (48). However, the thermodynamic variables are dependent upon the choice of gas.

Since $\Gamma(\psi) = \psi$ in (38) and $q = \frac{1}{e^{\psi}}$ in (44) in the limiting case of an incompressible flow of unit density, it follows that $B = 1$ and the solution set for the incompressible flow is realized to be

$$u = \frac{y}{x^2 + y^2}, \qquad v = \frac{-x}{x^2 + y^2},$$

and

$$p = p^{\star} - \tfrac{1}{2}q^2 = p^{\star} - \frac{1}{2(x^2 + y^2)}$$

where $p^{\star}$ is the constant stagnation pressure.

3.3. *A spiral flow.* This flow was previously investigated by the existing methods (cf. von Mises, 1958 and Courant and Friedrichs, 1948). The present study provides additional information that could not be obtained by these earlier methods. We deal with the flow problem that has spirals $\ln r + \theta =$ constant as its family of streamlines when (r, θ) are the polar coordinates of a point whose Cartesian coordinates are (x, y). For this problem, consider the analytic function

$$w = \xi(r, \theta) + i\eta(r, \theta) = \ln z = \ln r + i\theta$$

so that

$$(x, y) = e^{\xi}(\cos \eta, \sin \eta), \tag{53}$$

$$\xi + \eta = \ln r + \theta = \Gamma(\psi), \Gamma'(\psi) \neq 0 \tag{54}$$

where $\Gamma(\psi)$ is some function of the stream function ψ. Since $|\frac{\partial(\xi, \xi+\eta)}{\partial(\xi, \eta)}| = 1$, we can use (ξ, ψ)-coordinates.

Using (53) and (54), the squared element of arc length is given by

$$\begin{aligned} ds^2 = dx^2 + dy^2 &= e^{2\xi}(d\xi^2 + d\eta^2) \\ &= e^{2\xi}[2\, d\xi^2 - 2\Gamma'(\psi)\, d\xi\, d\psi + \Gamma'^2(\psi)\, d\psi^2]. \end{aligned} \tag{55}$$

Comparing (55) and (9) gives

$$\begin{gathered} E(\xi, \psi) = 2e^{2\xi}, \qquad F(\xi, \psi) = -e^{2\xi}\Gamma'(\psi), \\ G(\xi, \psi) = e^{2\xi}\Gamma'^2(\psi), \qquad W(\xi, \psi) = \sqrt{EG - F^2} = e^{2\xi}|\Gamma'(\psi)|. \end{gathered} \tag{56}$$

Using (53) and (54), we obtain

$$J(\xi, \psi) = \left|\frac{\partial(x, y)}{\partial(\xi, \psi)}\right| = \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\frac{\partial(\xi, \eta)}{\partial(\xi, \psi)}\right| = e^{2\xi}\Gamma'(\psi). \tag{57}$$

Since the fluid flows along a streamline in the direction of increasing ξ if $J > 0$, it follows from (57) that $\Gamma'(\psi) > 0$ for such a flow.

Employing (56) in (17), the Gauss equation is identically satisfied and the outward spiral flow is governed by the following system:

$$\rho q \frac{\partial q}{\partial \xi} + \frac{\partial p}{\partial \xi} = 0, \tag{58}$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0, \tag{59}$$

$$\rho q = \frac{\sqrt{2}}{e^{\xi}\Gamma'(\psi)}, \tag{60}$$

$$\Gamma'(\psi)\frac{\partial q}{\partial \xi} + 2\frac{\partial q}{\partial \psi} + \Gamma'(\psi) q = 0 \tag{61}$$

of four equations in four unknown functions $\rho(\xi,\psi)$, $q(\xi,\psi)$, $p(\xi,\psi)$, and $\Gamma(\psi)$. Using (60) in (58) and (59) to eliminate ρ and employing the integrability condition $\frac{\partial^2 p}{\partial \xi\,\partial \psi} = \frac{\partial^2 p}{\partial \psi\,\partial \xi}$, we get

$$\Gamma'(\psi)\frac{\partial q}{\partial \psi} - \Gamma''(\psi)\frac{\partial q}{\partial \xi} = 0. \tag{62}$$

Equations (61) and (62) form a system of two equations in two unknown functions $\Gamma(\psi)$ and $q(\xi,\psi)$.

The general solution of the first-order linear partial differential equation (62) is given by

$$q(\xi,\psi) = q(\alpha), \qquad \alpha = e^{\xi}\Gamma'(\psi) \tag{63}$$

where $q(\alpha)$ is an arbitrary function of its argument. Employing (63) in (61), $\Gamma(\psi)$ and $q(\alpha)$ must satisfy the ordinary differential equation

$$\left[\Gamma'(\psi) + \frac{2\Gamma''(\psi)}{\Gamma'(\psi)}\right]\alpha q'(\alpha) + \Gamma'(\psi) q(\alpha) = 0. \tag{64}$$

Since $\left|\frac{\partial(\alpha,\psi)}{\partial(\xi,\psi)}\right| = e^{\xi}\Gamma'(\psi) \neq 0$ in the flow domain, it follows that α, ψ may be regarded as the two independent variables. Applying separation of variables, (64) yields

$$\left[\frac{2\Gamma''(\psi)}{\Gamma'^2(\psi)} + 1\right] = -\frac{q(\alpha)}{\alpha q'(\alpha)} = A \tag{65}$$

where A is a nonzero separation constant.

Integrating the two differential equations given by (65) yields

$$\Gamma'(\psi) = B\exp\left[\left(\frac{A-1}{2}\right)\Gamma(\psi)\right] \tag{66}$$

and

$$q(\alpha) = C\alpha^{-1/A} \tag{67}$$

where B and C are nonzero arbitrary constants.

Using (66) and (54) in (63), we get

$$\alpha = Br\exp\left[\left(\frac{A-1}{2}\right)(\ln r + \theta)\right] \tag{68}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$.

Using (63), (67) in (60) gives

$$\rho = \left(\frac{\sqrt{2}}{C}\right)(\alpha)^{\frac{1}{A}-1}. \tag{69}$$

Differentiating (54) with respect to y, x and using (63), (69), and (54), the velocity components $u(x,y)$, $v(x,y)$ are given by

$$u(x,y) = \frac{1}{r\rho\Gamma'(\psi)}(\cos\theta+\sin\theta) = \frac{1}{\rho\alpha}(\cos\theta+\sin\theta) = \frac{C(\cos\theta+\sin\theta)}{\sqrt{2}}(\alpha)^{\frac{-1}{A}} \tag{70}$$

and

$$v(x,y) = \frac{C(\sin\theta-\cos\theta)}{\sqrt{2}}(\alpha)^{\frac{-1}{A}}. \tag{71}$$

Using (67), (69) in (58), (59), we get

$$dp = \frac{\partial p}{\partial \xi}\,d\xi + \frac{\partial p}{\partial \psi}\,d\psi = -pq\,dq = -\frac{\sqrt{2}}{(C)^A}(q)^A\,dq.$$

Integrating over $[p_s, p]$, i.e. $[0, q]$, we have

$$\begin{aligned} p &= p_s - \frac{\sqrt{2}}{(A+1)(C)^A}(q)^{A+1} \\ &= p_s - \left(\frac{\sqrt{2}C}{A+1}\right)\left(\frac{1}{\alpha}\right)^{\frac{A+1}{A}} \end{aligned} \tag{72}$$

where p_s is the stagnation pressure.

Eliminating α from (72) by using (69) in (72), the equation of state for this flow problem is given by

$$p = p_s - \frac{1}{(A+1)}\left[\frac{C^{2A}}{2}\right]^{\frac{1}{A-1}}[p]^{\frac{A+1}{A-1}}. \tag{73}$$

THEOREM IV. If $\ln r + \theta =$ constant is the family of streamlines for a steady plane irrotational isentropic spiral flow of an inviscid compressible fluid, then the solution set for the determinate system (1) to (4) of the four partial differential equations for the four unknown functions $\rho(x,y)$, $u(x,y)$, $v(x,y)$, and $p(x,y)$ is given by (69) to (72), respectively, when α is given by (68). Also, the state equation for the gas that corresponds to this solution set, is given by (73).

3.4. *Source flow.* This flow was previously investigated by the two well-documented methods found in texts [cf. von Mises, 1958]. The present method improves our knowledge of this flow.

We consider $f(z) = \xi(x,y) + i\eta(x,y) = \ln z = \ln r + i\theta$ so that $\mathrm{Im}[\ln z] =$ *constant* are the flow streamlines. For this flow, we have

$$\begin{aligned} r &= \sqrt{x^2+y^2} = e^{\xi}, \\ \theta &= \tan^{-1}\frac{y}{x} = \eta = \Gamma(\psi), \end{aligned} \tag{74}$$

where $\Gamma(\psi)$ is an unknown function of ψ.

Employing the (ξ, ψ)-coordinates, we get

$$ds^2 = dr^2 + r^2\, d\theta^2 = e^{2\xi}[d\xi^2 + \Gamma'^2(\psi)\, d\psi^2]$$

so that

$$E(\xi,\psi) = e^{2\xi}, \qquad F(\xi,\psi) = 0, \qquad G(\xi,\psi) = e^{2\xi}\Gamma'^2(\psi),$$

$$J(\xi,\psi) = \left|\frac{\partial(x,y)}{\partial(\xi,\psi)}\right| = e^{2\xi}\Gamma'(\psi), \qquad W(\xi,\psi) = \sqrt{EG - F^2} = e^{2\xi}\Gamma'(\psi).$$

We have used $|\Gamma'(\psi)| = \Gamma'(\psi)$ since $J > 0$ for the flow.

Proceeding as in the previous flow examples, we obtain

$$\rho q = \frac{1}{e^{\xi}\Gamma'(\psi)} = \alpha \text{ (say)}, \tag{75}$$

$$\alpha\frac{\partial q}{\partial \xi} + \frac{\partial p}{\partial \xi} = 0, \tag{76}$$

$$\alpha\frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0, \tag{77}$$

$$\frac{\partial q}{\partial \psi} = 0, \tag{78}$$

and

$$\left|\frac{\partial(q,\alpha)}{\partial(\xi,\psi)}\right| = 0, \tag{79}$$

where (79) is obtained from (76) and (77) by using the integrability condition $\frac{\partial^2 p}{\partial\xi\,\partial\psi} = \frac{\partial^2 p}{\partial\psi\,\partial\xi}$.

Analysis of these equations yields

$$q = q(\alpha), \quad q = q(\xi), \quad \alpha = \alpha(\xi), \quad p = p(\alpha) = p(\xi),$$

and

$$\theta = \tan^{-1}\frac{y}{x} = \Gamma(\psi) = C_1\psi + C_2, \tag{80}$$

where $C_1 \neq 0$ and C_2 are arbitrary constants.

Employing (74) and (80) in (75) and (6), we have

$$\rho q = \frac{1}{C_1 r}, \qquad \rho u = \frac{x}{C_1 r^2}, \qquad \rho v = \frac{y}{C_1 r^2}. \tag{81}$$

To obtain the complete solution set in the physical plane, it is required, at this stage, that we prescribe the state equation, i.e., we choose the gas. We take a tangent gas to illustrate. The state equation of a tangent gas is [cf. Woods, 1961]

$$p = C - \frac{D}{\rho}, \quad D > 0, \tag{82}$$

where C and D are known constants.

Employing (75), (76), (80), and (81), we have

$$\int_{p^\star}^{p} dp = p - p^\star = C - \frac{Dq}{\alpha} - p^\star$$
$$= \begin{cases} -\int_0^\alpha \alpha q'(\alpha)\, d\alpha; & \text{compressible flow} \\ -\frac{q^2}{2}; & \text{incompressible flow of unit density} \end{cases} \tag{83}$$

where $p^\star$ is the constant stagnation pressure.

Differentiating (83) with respect to α for the compressible case, we get

$$\frac{q'(\alpha)}{q(\alpha)} = \frac{D}{\alpha(D-\alpha^2)}. \tag{84}$$

Integrating (84) and using (81), (82), we have

$$q = \frac{C_3\alpha}{2\sqrt{D-\alpha^2}} = \frac{C_3}{2\sqrt{C_1^2Dr^2-1}},$$
$$\rho = \frac{2\sqrt{C_1^2Dr^2-1}}{C_1C_3r},$$
$$u = \frac{C_3x}{2\sqrt{C_1^2Dr^3-r}},$$
$$v = \frac{C_3y}{2\sqrt{C_1^2Dr^3-r}},$$

and

$$p = C - \frac{C_1C_3Dr}{2\sqrt{C_1^2Dr^2-1}}, \tag{85}$$

where $C_3 \neq 0$ is an arbitrary constant and $r = \sqrt{x^2+y^2}$. Summing up, we have:

THEOREM V. If $\tan^{-1}\frac{y}{x}$ = constant are the flow streamlines for the outward steady, plane, irrotational, inviscid compressible radial flow of a tangent gas, then the solution set in the physical plane for the tangent gas is given by (85).

Results for the horizontal and the vertical flow intensities ρu and ρv, respectively, are given by (81) for the source flow of every gas. However, the solution set given by (85) is for the tangent gas and will be different for another gas.

4. Conclusions. Whereas the documented methods yielded only incomplete solutions, complete exact closed-form solutions are obtained by using our new approach when $\mathrm{Re}[f(z)]$ = constant, where $f(z)$ is an analytic function of z, defines the streamline pattern. This approach for compressible flows is only an extension of the very well established elegant complex variable technique for finding the exact closed-form solutions of incompressible flows. Doublet flow, vortex flow, spiral flow, and radial flow are fully solved as illustrations of this approach. It is hoped that this work leads us to the determination of exact solutions of compressible flows that is comparable to our understanding of exact solutions of incompressible flows.

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