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REINFORCEMENT OF ELASTIC STRUCTURES IN THE PRESENCE OF IMPERFECT BONDING

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Abstract. The two-dimensional problem of plane strain is considered in the presence of imperfectly bonded elastic reinforcements. A geometric criterion on the shape and size of the elastic reinforcement is found that determines when the effects of imperfect bonding overcome the benefits of the reinforcement. The criterion is given in terms of an eigenvalue problem posed on the surface of the reinforcement.

1. Introduction. Recently a methodology has been developed to assess the effectiveness of reinforcement fibers or particles on the overall transport properties of composite materials. The methodology addresses reinforcement problems in the presence of nonstandard interfacial transmission at the surface of the reinforcement, see [6, 7, 8]. This approach has been successful in predicting the effect of particle size and shape on the enhancement of structural and thermal transport properties in the contexts of thermal contact resistance [6], coupled heat and mass transport on the interface [9], highly conducting interfaces [7], and problems of torsional rigidity with imperfectly bonded fiber reinforcements [8]. In this article we use this approach to investigate reinforcement problems in the context of two-dimensional elastic systems.

We consider the problem of plane strain for a reinforced, compressible, linearly elastic material in the presence of imperfect bonding. The elastic structure to be reinforced is a bounded subset of the plane denoted by Ω. It is assumed that the elastic reinforcement is distributed throughout the structure. The union of the reinforced regions is denoted by A_r. The part of Ω exterior to the reinforcement is referred to as the matrix and is denoted by A_m. Both the matrix and reinforcement are assumed to be made from elastically isotropic materials. The Lamé constants for the reinforcement and matrix phases are given by μ_r, λ_r and μ_m, λ_m, respectively and the associated isotropic compliance tensors are written as C_r^{-1} and C_m^{-1}. Here the reinforcement is stiffer than the matrix and we assume $C_r^{-1} < C_m^{-1}$ in the sense of quadratic forms over 2×2 strain matrices. This condition is equivalent to the requirement that $\kappa_r > \kappa_m$ and $\mu_r > \mu_m$, where κ_r and κ_m are the plane strain bulk moduli of the reinforcement and matrix respectively, i.e.,

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$\kappa_r = \mu_r + \lambda_r$ and $\kappa_m = \mu_m + \lambda_m$. The characteristic function for the reinforcement phase is written χ_{A_r} and the piece-wise constant compliance tensor is described by $C^{-1}(\chi_{A_r}) = \chi_{A_r} C_r^{-1} + (1 - \chi_{A_r}) C_m^{-1}$. The interface separating the reinforcement phase from the matrix is written as Γ, and $\Omega = A_m \cup A_r \cup \Gamma$.

Imperfect bonding is characterized by a loss of continuity in the displacement across the interface separating the reinforcement from the matrix. We consider the flexible interface model of the type used by Léné and Leguillon [5] in their treatment of the softening of effective moduli arising from damage. The stiffness of the interface is characterized by the parameter "β", relating traction forces to the relative displacement across the interface. This parameter has dimensions of shear stiffness per unit length and ranges between zero and infinity. The limiting case "$\beta = \infty$" corresponds to perfect bonding. The case of no adhesion is captured in the limit "$\beta = 0$". This model represents the effect of a very thin zone between the matrix and reinforcement that is more compliant than the matrix and reinforcement phases. Flexible interface models similar to the type treated here can be found in the work of Jones and Whitter [4]. A comprehensive treatment of interface models as they relate to imperfect bonding is provided in the recent book of Aboudi [1].

For a prescribed boundary traction g, such that $\int_{\partial\Omega} g\, dl = 0$, the compliance energy dissipated inside Ω is given by $E(A_r, g)$, where

$$E(A_r, g) = \min\{C(A_r, \tau) \mid \tau_{ji} \in L^2(\Omega), i = 1, 2, j = 1, 2, \tau_{ij} = \tau_{ji}, \\ \operatorname{div} \tau = 0, -\tau_{ij} n_j = g_i, \text{ on } \partial\Omega\} \quad (1.1)$$

and

$$C(A_r, \tau) = \frac{1}{2}\left\{\int_\Omega C^{-1}(\chi_{A_r})\tau : \tau dx + \beta^{-1}\int_\Gamma |\tau n|^2 dl\right\}. \quad (1.2)$$

Here dl is the element of arc length, $C^{-1}(\chi_{A_r})\tau : \tau = C^{-1}(\chi_{A_r})_{ijkl}\tau_{kl}\tau_{ij}$ represents the bulk compliance energy density and the vector n is the unit normal pointing into the matrix phase. The interface compliance energy density is $\beta^{-1}|\tau n|^2 = \beta^{-1}\tau_{ij} n_j \tau_{ik} n_k$, where repeated indices indicate summation. In order to include the broadest spectrum of load cases we suppose that g belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)^2$. To expedite the presentation we denote the set of all tractions $g \in H^{-\frac{1}{2}}(\partial\Omega)^2$ such that $\int_{\partial\Omega} g\, dl = 0$ by L.

The first term of the functional $C(A_r, \tau)$ is associated with the bulk compliance energy, while the second term gives the compliance energy of the interface. The minimizer τ_{A_r} is precisely the stress tensor inside the composite and is related to the elastic displacement u_{A_r} by the constitutive law: $\tau_{A_r} = C(\chi_{A_r}) e(u_{A_r})$ where $e(u_{A_r})$ is the elastic strain tensor. Here u_{A_r} is allowed to be discontinuous across the interface and belongs to the function space $H^1(\Omega \backslash \Gamma)^2$. We set $\theta = \operatorname{div} u_{A_r}$. The displacement satisfies Navier's displacement equations of equilibrium given by

$$(\lambda_r + \mu_r)\theta_{,i} + \mu_r \Delta u_{A_{ri}} = 0, \quad i = 1, 2, \quad \text{in } A_r, \quad (1.3)$$

and

$$(\lambda_m + \mu_m)\theta_{,i} + \mu_m \Delta u_{A_{ri}} = 0, \quad i = 1, 2, \quad \text{in } A_m. \quad (1.4)$$

Across the interface one has

$$[\tau_{A_{rij}} n_i] = 0 \quad \text{on } \Gamma. \tag{1.5}$$

Here the jump in a quantity q across the interface is denoted by $[q] = q_{|r} - q_{|m}$, where the subscripts indicate traces on the reinforcement and matrix sides of the interface.

In order to give the jump conditions for the displacement we decompose the traction into components normal and tangential to the interface. We let t denote the unit tangent vector obtained by counter-clockwise rotation of n by $\pi/2$ radians. The tangential component of the traction is given by $\tau_{A_{rij}} n_i t_j$ and the normal component is $\tau_{A_{rij}} n_i n_j$. The displacement discontinuity is related to the traction force on the interface through "Hooke's" law:

$$\begin{aligned} \tau_{A_{rij}} n_i n_j &= -\beta [u_{A_{ri}} n_i], \\ \tau_{A_{rij}} n_i t_j &= -\beta [u_{A_{ri}} t_i]. \end{aligned} \tag{1.6}$$

Last we note that the compliance energy is proportional to the work done against the load and is given in terms of the elastic displacement as

$$E(A_r, g) = \frac{1}{2} \int_{\partial\Omega} u_{A_{rj}} g_j \, dl. \tag{1.7}$$

The reinforcement phase can have several disconnected components. The contribution of each component to the compliance energy is investigated. A geometric criterion related to the shape and size of a component is found that determines when the effects of imperfect bonding overcome the benefits of the reinforcement. The criterion is general and applies to any simply connected component Σ of the reinforcement phase provided that the boundary of Σ does not intersect the boundary of the design domain Ω. In order to give the criterion we introduce the fourth-order tensor

$$\mathcal{T} = \beta^{-1} (C_m^{-1} - C_r^{-1})^{-1}. \tag{1.8}$$

Here each element of $\mathcal{T}$ has dimensions of length. This tensor provides a measure of the magnitude of the interfacial compliance with respect to the difference in compliance between the reinforcement and matrix phases. We will show that the components of $\mathcal{T}$ provide a natural length scale for the selection of reinforcement size. The relevant geometric parameter intrinsic to Σ is given by

$$\sigma = \min_{\substack{\varphi \in \mathcal{C} \\ \varphi \neq 0}} \left\{ \frac{\int_{\partial\Sigma} \partial_s \varphi_{,i} \, \partial_s \varphi_{,i} \, dl}{\int_\Sigma \varphi_{,ij} \, \varphi_{,ij} \, dx} \right\}, \tag{1.9}$$

where

$$\mathcal{C} = \left\{ \varphi \in H^{5/2}(\Sigma) \mid \Delta^2 \varphi = 0, \int_{\partial\Sigma} \varphi \, dl = 0, \int_{\partial\Sigma} \varphi_{,i} \, dl = 0, i = 1, 2 \right\}.$$

Here ∂_s denotes tangential differentiation on the curve $\partial\Sigma$ and Δ^2 is the biharmonic operator. The notation $\varphi_{,i}$ represents differentiation, i.e., $\varphi_{,i} = \partial_{x_i} \varphi$ and $\varphi_{,ij} = \partial^2_{x_i x_j} \varphi$. The parameter σ has dimensions of inverse length. Conditions of stationarity for (1.9)

deliver the eigenvalue problem

$$\Delta^2\phi = 0, \quad \text{on } \Sigma, \tag{1.10}$$

$$n_i\partial_s^2\phi_{,i} = -\sigma M_n(\phi), \quad \text{on } \partial\Sigma, \tag{1.11}$$

$$\partial_s(t_i\partial_s^2\phi_{,i}) = -\sigma\{\partial_s M_s(\phi) + Q(\phi)\}, \quad \text{on } \partial\Sigma. \tag{1.12}$$

Here $M_n(\phi)$ is the bending moment of the reinforcement Σ given by $M_n(\phi) = n_i n_j \phi_{,ij}$ and $\partial_s M_s(\phi) + Q(\phi)$ is the Kirchhoff shear force where $M_s(\phi) = t_i n_j \phi_{,ij}$ and $Q(\phi) = \partial_n \Delta\phi = 0$. We have denoted normal differentiation by ∂_n and the Laplace operator by Δ. From its definition we see that σ is the largest constant C for which the inequality

$$\int_{\partial\Sigma} \partial_s\varphi_{,i}\, \partial_s\varphi_{,i}\, dl \geq C \int_{\Sigma} \varphi_{,ij}\, \varphi_{,ij}\, dx \tag{1.13}$$

holds for all φ in the space $\mathcal{C}$. For a disk of radius a one has that

$$\sigma = \frac{2}{3a}; \tag{1.14}$$

this is established in Section 4.

For a given reinforcement configuration A_r, we focus our attention on one of the components of the reinforcement phase denoted by Σ. We let $A_r \backslash \Sigma$ denote the reinforcement configuration without the component Σ. The compliance energy for this configuration is given by $E(A_r\backslash\Sigma, g)$. We write $\sigma = \sigma(\Sigma)$ to indicate its dependence on the geometry of the reinforcement Σ. We denote the identity tensor on 2×2 strains by I and introduce the isotropic fourth-order tensor $\sigma(\Sigma)I$. The effect of the reinforcement is given by the following compliance inequality.

THEOREM 1.1. **Compliance energy inequality**. If $\sigma(\Sigma)$ satisfies

$$\mathcal{T}^{-1} \leq \sigma(\Sigma)I, \tag{1.15}$$

then for every load case $g \in L$, the reinforcement does not reduce the compliance energy, i.e.,

$$E(A_r\backslash\Sigma, g) \leq E(A_r, g). \tag{1.16}$$

The inequality holds for any reinforcement shape with Lipschitz continuous boundary. Here the inequality (1.15) holds in the sense of quadratic forms over the space of strains.

We emphasize that this result is independent of the location and geometry of the other components of the reinforcement phase and applies to every load case $g \in L$.

An equivalent but more applicable form of the compliance energy inequality is given in terms of the contrast between the bulk and shear moduli of the matrix and reinforcement. We define the relative compliance γ by

$$\gamma = \min\left\{\frac{\beta^{-1}}{(2\mu_m)^{-1} - (2\mu_r)^{-1}}, \frac{\beta^{-1}}{(2\kappa_m)^{-1} - (2\kappa_r)^{-1}}\right\}. \tag{1.17}$$

Here γ has dimensions of length. In terms of the bulk and shear moduli the compliance energy inequality is given by

COROLLARY 1.1. **Compliance energy inequality (in terms of bulk and shear moduli).**

If $\sigma(\Sigma)$ satisfies

$$\gamma^{-1} \leq \sigma(\Sigma), \tag{1.18}$$

then for every load case $g \in L$, the reinforcement does not reduce the compliance energy.

The corollary follows immediately from the spectral representation of the compliance tensors given by

$$C_r^{-1} = (2\mu_r)^{-1} P_d + (2\kappa_r)^{-1} P_s, \; C_m^{-1} = (2\mu_m)^{-1} P_d + (2\kappa_m)^{-1} P_s,$$

where P_d is the projection onto the space of 2×2 strains with zero trace and P_s is the projection onto the 2×2 identity matrix.

The compliance inequality naturally implies a reinforcement size effect. Here the length scale is set by γ. The effect of reinforcement size is seen clearly when Σ is a disk of radius a. We have the following size effect for a circular reinforcement.

THEOREM 1.2. **Size effect for a circular reinforcement.** If the reinforcement is a disk of radius a, and

$$a \leq \frac{2}{3}\gamma, \tag{1.19}$$

then for every load case $g \in L$, the reinforcement does not reduce the compliance energy.

This Theorem gives a rigorous rule of thumb for the selection of the size of a reinforcement disk, namely: *Only disks of radius greater than* $(2/3)\gamma$ *can provide reinforcement. This statement holds true independently of where the disks are placed in the structure.* This result is in striking contrast to what is seen when there is perfect bonding between structural materials. For that situation, the addition of an infinitesimally small disk of stiffer material always reduces the compliance energy.

Theorems 1.1 and 1.2 are directly applicable to problems of optimal compliance design for plane strain problems. A prototypical problem is the optimal design of a structure reinforced with disks of different radii. Each disk has compliance C_r^{-1} and the matrix has compliance C_m^{-1}. Here we suppose that each disk is made from stiffer material, i.e., $C_r^{-1} < C_m^{-1}$. The class of admissible designs is given by the set of all reinforcements consisting of a finite number of non-intersecting disks. We restrict the joint area of the disks to be less than a prescribed area fraction θ_r of the design domain Ω. However, no lower bound is placed on the size of the disks nor do we place a constraint on the number of disks appearing in any design. We show in Theorem 5.1 that all energy minimizing configurations of disks can be found among those that contain disks of radii greater than or equal to $\frac{2}{3}\gamma$ or no disks at all. It is evident that minimizing sequences made from progressively finer suspensions of disks (i.e., *homogenized designs*) can be excluded.

The paper is organized as follows. In Section 2 the compliance energy inequality is proved. The existence of a minimizer for the variational definition of σ is established in Section 3, and σ is calculated for a disk in Section 4. Section 5 discusses applications of the compliance inequality to problems of optimal design.

2. Derivation of the compliance inequality. In this section we derive Theorem 1.1 using the variational formulation for the compliance energy. The stress associated with the configuration obtained by removing Σ and replacing it with matrix material is written $\tilde{\tau}$. It is the minimizer of the variational principle given by

$$E(A_r\backslash\Sigma, g) = \min\{C(A_r\backslash\Sigma, \tau) \mid \tau_{ji} \in L^2(\Omega), i = 1,2, j = 1,2, \tau_{ij} = \tau_{ji}, \\ \operatorname{div}\tau = 0, -\tau_{ij}n_j = g_i, \quad \text{on } \partial\Omega\} \tag{2.1}$$

and

$$C(A_r\backslash\Sigma, \tau) = \frac{1}{2}\left\{\int_\Omega C^{-1}(\chi_{A_r\backslash\Sigma})\tau : \tau dx + \beta^{-1}\int_{\Gamma\backslash\partial\Sigma} |\tau n|^2 dl\right\}. \tag{2.2}$$

Here $\chi_{A_r\backslash\Sigma}$ is the characteristic function of the set $A_r\backslash\Sigma$. It is evident that τ_{A_r} is an admissible trial for the variational formulation of the compliance $E(A_r\backslash\Sigma, g)$ and we have

$$E(A_r\backslash\Sigma, g) = C(\tilde{\tau}, A_r\backslash\Sigma) \leq C(\tau_{A_r}, A_r\backslash\Sigma). \tag{2.3}$$

Expanding $C(\tau_{A_r}, A_r\backslash\Sigma)$ gives

$$C(\tau_{A_r}, A_r\backslash\Sigma) = E(A_r, g) - \delta, \tag{2.4}$$

where

$$\delta = \frac{1}{2}\beta^{-1}\left\{\int_{\partial\Sigma} |\tau_{A_r} n|^2 dl - \int_\Sigma \beta(C_m^{-1} - C_r^{-1})\tau_{A_r} : \tau_{A_r} dx\right\}. \tag{2.5}$$

From compatibility conditions we have $\Delta(\operatorname{tr}(\tau_{A_r})) = 0$ on Σ and since $\operatorname{div}\tau_{A_r} = 0$ on Σ, there exists a stress potential φ such that $\Delta^2\varphi = 0$ and

$$\tau_{A_r} = \begin{bmatrix} \varphi_{,22} & -\varphi_{,12} \\ -\varphi_{,21} & \varphi_{,11} \end{bmatrix}. \tag{2.6}$$

Substitution of (2.6) into (2.5) gives

$$\delta = \frac{1}{2}\beta^{-1}\left\{\int_{\partial\Sigma} \partial_s\varphi_{,i}\,\partial_s\varphi_{,i}\,dl - \int_\Sigma \beta(C_m^{-1} - C_r^{-1})_{ijkl}\varphi_{,kl}\,\varphi_{,ij}\,dx\right\}. \tag{2.7}$$

In view of the inequality (1.13) it follows that δ is zero or positive if

$$\mathcal{T}^{-1} \leq \sigma(\Sigma)I \tag{2.8}$$

and the theorem follows.

3. The eigenvalue problem. In this section we proceed using direct methods to establish existence of a function ϕ in $\mathcal{C}$ that minimizes the Rayleigh quotient

$$J(\varphi) = \left\{\frac{\int_{\partial\Sigma} \partial_s\varphi_{,i}\,\partial_s\varphi_{,i}\,dl}{\int_\Sigma \varphi_{,ij}\,\varphi_{,ij}\,dx}\right\}. \tag{3.1}$$

We consider a minimizing sequence $\{\varphi^m\}_{m=0}^\infty$ such that

$$\lim_{m\to\infty} J(\varphi^m) = \inf_{\substack{\varphi\in\mathcal{C} \\ \varphi\neq 0}} J(\varphi). \tag{3.2}$$

Normalizing the functions φ^m we may assume that

$$\int_\Sigma \varphi^m_{,ij}\varphi^m_{,ij}dx = 1 \quad \text{for all } m = 1, 2, \dots . \tag{3.3}$$

We make use of the following uniform bounds on the sequence.

THEOREM 3.1. **Uniform bounds on the minimizing sequence**. There exists a constant K independent of m for which

$$\|\partial_n\varphi^m\|_{H^1(\partial\Sigma)} \le K \quad \text{and} \quad \|\varphi^m\|_{H^2(\partial\Sigma)} \le K. \tag{3.4}$$

This theorem will be established in the sequel.

We apply the uniform bounds to extract a subsequence $(\partial_n\varphi^m, \varphi^m)$ converging weakly in $H^1(\partial\Sigma) \times H^2(\partial\Sigma)$ to the pair (h, u). Application of the Sobolev Imbedding Theorem shows that $(\partial_n\varphi^m, \varphi^m)$ converges strongly in $H^{1/2}(\partial\Sigma) \times H^{3/2}(\partial\Sigma)$. We let ϕ be the solution of the boundary value problem

$$\begin{aligned} \Delta^2\phi &= 0 \quad \text{on } \Sigma, \\ \partial_n\phi &= h \quad \text{on } \partial\Sigma, \\ \phi &= u \quad \text{on } \partial\Sigma. \end{aligned} \tag{3.5}$$

Putting $w_m = \varphi^m - \phi$ we apply the regularity theorem for Dirichlet's problem for the biharmonic operator (cf. [3]) to find a constant C independent of m such that

$$\|w_m\|_{H^2(\Sigma)} \le C(\|w_m\|_{H^{3/2}(\partial\Sigma)} + \|\partial_n w_m\|_{H^{1/2}(\partial\Sigma)}). \tag{3.6}$$

This estimate together with the strong convergence of $(\partial_n\varphi^m, \varphi^m)$ shows that the sequence φ^m converges strongly to ϕ in $H^2(\Sigma)$. It now follows that

$$\int_\Sigma \phi_{,ij}\,\phi_{,ij}\,dx = 1. \tag{3.7}$$

From the uniform bounds it is evident that $\phi \in H^2(\partial\Sigma)$ and $\partial_n\phi \in H^1(\partial\Sigma)$. Thus by regularity theory for the Dirichlet problem we find that $\phi \in H^{5/2}(\Sigma)$. It is easily seen that $\int_{\partial\Sigma}\phi\,dl = 0$ and $\int_{\partial\Sigma}\phi_{,i}\,dl = 0$, $i = 1, 2$ and we conclude that $\phi \in \mathcal{C}$. To establish that ϕ is a minimizer we first expand

$$\int_{\partial\Sigma} \partial_s\varphi^m_{,i}\partial_s\varphi^m_{,i}dl$$

to find

$$\begin{aligned} \int_{\partial\Sigma} \partial_s\varphi^m_{,i}\partial_s\varphi^m_{,i}dl = &\int_{\partial\Sigma} |\partial^2_{sn}\varphi^m - \rho^{-1}\partial_s\varphi^m|^2 dl \\ &+ \int_{\partial\Sigma} |\partial^2_{ss}\varphi^m + \rho^{-1}\partial_n\varphi^m|^2 dl, \end{aligned} \tag{3.8}$$

where ρ is the radius of curvature of the interface. It now follows from (3.8) and the weak convergence of $(\partial_n\varphi^m, \varphi^m)$ together with the lower semicontinuity of the $L^2(\partial\Sigma)$ norm, that

$$\liminf_{m\to\infty} \int_{\partial\Sigma} \partial_s\varphi^m_{,i}\partial_s\varphi^m_{,i}dl \ge \int_{\partial\Sigma} \partial_s\phi_{,i}\partial_s\phi_{,i}dl, \tag{3.9}$$

and we conclude that ϕ is the minimizer.

We prove the uniform bounds given by Theorem 3.1. We start by showing that there exists a constant C independent of m such that

$$\|\varphi^m\|_{H^1(\partial\Sigma)} \leq C \quad \text{and} \quad \|\partial_n \varphi^m\|_{L^2(\partial\Sigma)} \leq C. \tag{3.10}$$

In order to proceed we need estimates that follow from the Rayleigh quotient given by

$$\nu_2 = \min_{\int_{\partial\Sigma} u dl = 0} \frac{\int_{\partial\Sigma} |\partial_s u|^2 dl}{\int_{\partial\Sigma} u^2 dl}. \tag{3.11}$$

The value ν_2 is the Wirtinger eigenvalue. For all functions u for which $\int_{\partial\Sigma} u\, dl = 0$ it follows that

$$\int_{\partial\Sigma} u^2 dl \leq (\nu_2)^{-1} \int_{\partial\Sigma} |\partial_s u|^2 dl. \tag{3.12}$$

Since $\{\varphi_m\}_0^\infty$ is a minimizing sequence, it follows that there is a constant D for which

$$\int_{\partial\Sigma} \partial_s \varphi^m_{,i} \partial_s \varphi^m_{,i} dl \leq D \tag{3.13}$$

for all m. Noting that $\int_{\partial\Sigma} \varphi^m_{,i} dl = 0$ we apply (3.12) to obtain

$$\int_{\partial\Sigma} |\nabla \varphi^m|^2 dl \leq (\nu_2)^{-1} \int_{\partial\Sigma} |\partial_s \varphi^m_{,i}|^2 dl \leq (\nu_2)^{-1} D. \tag{3.14}$$

Observing that

$$\int_{\partial\Sigma} |\nabla \varphi^m|^2 dl = \int_{\partial\Sigma} |\partial_s \varphi^m|^2 + |\partial_n \varphi^m|^2 dl,$$

we immediately obtain the estimates $\|\partial_n \varphi^m\|_{L^2(\partial\Sigma)} \leq C_1$ and $\|\partial_s \varphi^m\|_{L^2(\partial\Sigma)} \leq C_1$. Noting that $\int_{\partial\Sigma} \varphi^m dl = 0$ we apply (3.12) again to find $\|\varphi^m\|_{L^2(\partial\Sigma)} \leq C_2$, and the estimates given by (3.10) follow.

To complete the proof of Theorem 3.1 we establish the estimates

$$\|\partial^2_{ss} \varphi^m\|_{L^2(\partial\Sigma)} \leq C \tag{3.15}$$

and

$$\|\partial^2_{sn} \varphi^m\|_{L^2(\partial\Sigma)} \leq C. \tag{3.16}$$

Here C represents a generic constant independent of m. We establish (3.16), noting that (3.15) follows along similar lines. We put

$$F_m = \int_{\partial\Sigma} |\partial^2_{sn} \varphi^m - \rho^{-1} \partial_s \varphi^m|^2 dl. \tag{3.17}$$

From (3.8) and (3.13) we find that there exists a constant C_3 for which $F_m \leq C_3$ for all m. Expanding (3.17) gives

$$\begin{aligned} \int_{\partial\Sigma} |\partial^2_{sn} \varphi^m|^2 dl &= F_m - \int_{\partial\Sigma} \rho^{-2} |\partial_s \varphi^m|^2 dl + 2 \int_{\partial\Sigma} \rho^{-1} \partial^2_{sn} \varphi^m \partial_s \varphi^m dl \\ &= F_m - 2 \int_{\partial\Sigma} \rho^{-1} \partial_s \varphi^m (\rho^{-1} \partial_s \varphi^m - \partial^2_{sn} \varphi^m) dl \\ &\quad + \int_{\partial\Sigma} \rho^{-2} |\partial_s \varphi^m|^2 dl. \end{aligned} \tag{3.18}$$

Application of the arithmetic-geometric mean inequality yields

$$\int_{\partial\Sigma} |\partial_{sn}^2 \varphi^m|^2 dl \leq 2F_m + 2\|\rho^{-2}\|_{L^\infty(\partial\Sigma)} \int_{\partial\Sigma} |\partial_s \varphi^m|^2 dl, \tag{3.19}$$

and (3.16) follows from the uniform bounds on F_m and $\|\partial_s \varphi^m\|_{L^2(\partial\Sigma)}$.

4. The eigenvalue problem for a disk. For a disk of radius a we show that $\sigma = 2/(3a)$. To do this we calculate a countable set $\mathcal{S}$ of stationary values of the Rayleigh quotient (3.1). We then use the completeness of the trigonometric polynomials to show that $\mathcal{S}$ contains all the stationary values. The minimum stationary value in $\mathcal{S}$ corresponds to $2/(3a)$. Along the way we obtain all the eigenfunctions for the eigenvalue problem (1.10)–(1.12).

For the disk we adopt polar coordinates (r, θ) and the eigenvalue problem (1.10)–(1.12) becomes

$$a\partial_\theta^2 \partial_r \phi - 2\partial_\theta^2 \phi - a\partial_r \phi = -ya^2 \partial_r^2 \phi \text{ on } r = a, \tag{4.1}$$

$$\begin{aligned} \partial_\theta^4 \phi + 2a\partial_\theta^2 \partial_r \phi - \partial_\theta^2 \phi = -y\{&2a\partial_\theta^2 \partial_r \phi - 3\partial_\theta^2 \phi - a\partial_r \phi \\ &+ a^2 \partial_r^2 \phi + a^3 \partial_r^3 \phi\} \quad \text{on } r = a, \end{aligned} \tag{4.2}$$

where ϕ is biharmonic in the disk. Here y is the "dimensionless eigenvalue" given by $y = a\sigma$. We put

$$u_k = (A_k r^k + B_k r^{k+2}) \exp ik\theta, \tag{4.3}$$

and look for biharmonic solutions of the form $\mathrm{Re}(u_k), \mathrm{Im}(u_k)$, for $k = 0, 1, 2, \ldots$. Substitution of u_k into (4.1, 4.2) gives a linear system in the unknowns A_k, B_k

$$c_{11}(y,k)A_k + c_{12}(y,k)B_k = 0, \tag{4.4}$$

$$c_{21}(y,k)A_k + c_{22}(y,k)B_k = 0, \tag{4.5}$$

where

$$c_{11}(y,k) = k(k-1)\{(k-1) - y\}, \tag{4.6}$$

$$c_{12}(y,k) = a^2(k+1)\{(k^2 - k + 2) - y(k+2)\}, \tag{4.7}$$

$$c_{21}(y,k) = k^2(k-1)\{(k-1) - y\}, \tag{4.8}$$

$$c_{22}(y,k) = a^2 k(k+1)\{k(k-3) - y(k-4)\}. \tag{4.9}$$

The determinant of the coefficient matrix is given by

$$-6a^2k^2(k-1)(k+1)\{(k-1) - y\}\left\{\frac{k+1}{3} - y\right\}. \tag{4.10}$$

We generate a countable set $\mathcal{S}$ of eigenvalues by choosing y so that the system (4.4, 4.5) has a nontrivial solution for $k = 0, 1, 2, \ldots$. The cases $k = 0, k = 1$ are special in that both left and right-hand sides of (1.11) and (1.12) are identically zero for linear functions. This is reflected in the system (4.4, 4.5). Indeed, for $k = 0$ the system reduces to the single equation $B_0 c_{12}(y, 0) = 0$. The condition $c_{12}(y, 0) = 0$ delivers $y = 1$. For $k = 1$ the system reduces to $B_1 c_{12}(y, 1) = 0$, $B_1 c_{22}(y, 1) = 0$. One finds that $c_{12}(y, 1) = c_{22}(y, 1)$ and the condition $c_{12}(y, 1) = 0$ delivers $y = 2/3$. For $k > 1$ the determinant given by

(4.10) vanishes for $y = k-1$ and $y = \frac{k+1}{3}$ and nontrivial solutions follow for these choices. Collecting these results we have a sequence of eigenvalues given by

$$\mathcal{S} = \left\{ \frac{j}{3a},\ j = 2, 3, 4, \dots \right\}. \tag{4.11}$$

To complete the calculation we show that $\mathcal{S}$ contains all the eigenvalues associated with eigenfunctions belonging to the space $\mathcal{C}$. In order to proceed we list all the eigenfunctions associated with $\mathcal{S}$. For $\sigma = 2/(3a)$ the eigenspace is spanned by

$$r^3 \cos\theta, \quad r^3 \sin\theta.$$

For $\sigma = 1/a$ the eigenspace is spanned by

$$r^4 \cos 2\theta, \quad r^4 \sin 2\theta, \quad r^2 \cos\theta, \quad r^2 \sin\theta, \quad r^2.$$

For $\sigma = l/(a)$, $l = 2, 3, \dots$, the eigenspace is spanned by

$$\begin{aligned}
&\left(r^{(3l-1)} - \frac{(3l-1)}{a^2(3l)} r^{(3l+1)} \right) \cos(3l-1)\theta, \\
&\left(r^{(3l-1)} - \frac{(3l-1)}{a^2(3l)} r^{(3l+1)} \right) \sin(3l-1)\theta, \\
&r^{(l+1)} \cos(l+1)\theta, \qquad r^{(l+1)} \sin(l+1)\theta.
\end{aligned}$$

For $\sigma = l/(3a)$, $l > 2$ such that $l/3$ is not an integer, the eigenspace is spanned by

$$\left(r^{(l-1)} - \frac{(l-1)}{a^2(l)} r^{(l+1)} \right) \cos(l-1)\theta,$$

$$\left(r^{(l-1)} - \frac{(l-1)}{a^2(l)} r^{(l+1)} \right) \sin(l-1)\theta,$$

Last we suppose the existence of an eigenvalue $\sigma_b > 0$ that is not in $\mathcal{S}$. It is shown that the eigenspace for this eigenvalue is given by the span of the linear functions. Since linear functions are excluded from the admissible class $\mathcal{C}$ it follows that $\mathcal{S}$ is the set of all stationary values of the Rayleigh quotient.

Before proceeding we state the following theorem.

THEOREM 4.1. **Orthogonality of eigenspaces**. Consider the eigenvalue problem (1.10)–(1.12). Given two distinct eigenvalues σ_a and σ_b and associated eigenfunctions f^a and f^b then

$$\int_\Sigma f^a_{,ij} f^b_{,ij} dx = 0. \tag{4.12}$$

The theorem follows immediately from the identity

$$\begin{aligned}
\int_{\partial\Sigma} \partial_s f^a_{,i} \partial_s f^b_{,i} dl &= \sigma_a \int_\Sigma f^a_{,ij} f^b_{,ij} dx \\
&= \sigma_b \int_\Sigma f^a_{,ij} f^b_{,ij} dx.
\end{aligned}$$

We suppose σ_b does not lie in $\mathcal{S}$ to show that the associated eigenfunction f^b is a linear function. Consider any function g given by a linear combination of eigenfunctions associated with the set of eigenvalues $\mathcal{S}$. From Theorem 4.1 and integration by parts it follows that

$$\begin{aligned} 0 &= \int_\Sigma f^b_{,ij} g_{,ij} dx \\ &= -\int_{\partial\Sigma} ((\partial_s M_s(f^b) + Q(f^b))g - M_n(f^b)\partial_n g) dl. \end{aligned} \tag{4.13}$$

The goal now is to show that $\partial_s M_s(f^b) + Q(f^b) = M_n(f^b) = 0$. From this we easily see that $\int_\Sigma |f^b_{,ij}|^2 dx = 0$ and conclude that f^b is a linear function, see [2]. The following system of functions can easily be constructed from linear combinations of eigenfunctions associated with $\mathcal{S}$. The system is given by

$$r^2; \qquad r^k \cos k\theta, r^k \sin k\theta, \quad k = 2, 3, \dots; \qquad r^{k+2} \cos k\theta, r^{k+2} \sin k\theta, \quad k = 1, 2, 3, \dots.$$

For $k = 2, 3, \dots$, we form the linear combinations

$$g_k = C_k r^k \exp ik\theta + D_k r^{k+2} \exp ik\theta.$$

For any vector (v_1, v_2) in R^2 we may choose C_k and D_k such that on $r = a$

$$(\partial_r g_k, g_k) = (v_1, v_2) \exp ik\theta. \tag{4.14}$$

That such a choice can be made is evident from the associated linear system

$$ka^{k-1} C_k + a^{k+1}(k+2) D_k = v_1, \tag{4.15}$$

$$a^k C_k + a^{k+2} D_k = v_1, \tag{4.16}$$

since the determinant of the coefficient matrix of the system (4.15), (4.16) is $-2a^{2k-1} \neq 0$. Substitution of g_k into equation (4.13) together with (4.14) shows that $\partial_s M_s(f^b) + Q(f^b)$ and $M_n(f^b)$ are orthogonal to the system $(\cos k\theta, \sin k\theta)$, $k > 1$, with respect to the $L^2(\partial\Sigma)$ inner product. This implies that $\partial_s M_s(f^b) + Q(f^b) = (\zeta_1^1 \cos\theta + \zeta_2^1 \sin\theta) + c_1$ and $M_n(f^b) = (\zeta_1^2 \cos\theta + \zeta_1^2 \sin\theta) + c_2$, for constant vectors (ζ_1^1, ζ_2^1) and (ζ_1^2, ζ_2^2) in R^2 and constants c_1 and c_2. Next we set $g = D_0 r^2$ and $g = D_1 r \exp i\theta$ and apply the identity (4.13) to find that

$$M_n(f^b) = \frac{a}{3}(\zeta_1^1 \cos\theta + \zeta_2^1 \sin\theta) + ac_1.$$

Last we show $\zeta_1^1 = \zeta_2^1 = 0$ and $c_1 = 0$. For linear functions $g = r(\eta_1 \cos\theta + \eta_2 \sin\theta) + b$ we have that $g_{,ij} = 0$. For this choice of g, (4.13) holds and we obtain the identity

$$0 = 2|\Sigma| \frac{2a}{3}(\eta_1 \zeta_1^1 + \eta_2 \zeta_2^1) + |\partial\Sigma| b c_1, \tag{4.17}$$

for every choice of η_1, η_2, and constant b. This shows that $\zeta_1^1 = \zeta_2^1 = 0$ and $c_1 = 0$.

5. Optimal compliance design of imperfectly bonded reinforced materials. We consider the problem of minimizing the compliance energy over the class of designs composed of suspensions of stiff elastically isotropic disks immersed in a more compliant elastically isotropic matrix. Here the suspensions consist of a finite number of nonintersecting disks of different radii. We suppose that the suspension takes up no more than a prescribed area fraction θ_r of the total composite domain Σ. However, we assume no lower bound on the disk radii and place no constraint on the number of disks. Denoting the ith disk by B_i we write $A_r = \bigcup B_i$. We denote this class of suspensions by $\mathcal{Q}_{\theta_r}$. We consider the subclass $\mathcal{SQ}_{\theta_r}$ of $\mathcal{Q}_{\theta_r}$, defined to be all suspensions with disk radii greater than or equal to $\frac{2}{3}\gamma$. For a prescribed traction $g \in L$ on the boundary $\partial\Sigma$, we consider the problem:

$$\min\{E(A_r, g)\colon A_r \in \mathcal{Q}_{\theta_r}\}. \tag{5.1}$$

We have the following optimal design theorem.

THEOREM 5.1. **Optimal compliance design theorem**. The optimal distribution of disks can be either found in the class $\mathcal{SQ}_{\theta_r}$ or the optimal design consists of no disks whatsoever. Moreover, if Ω has dimensions for which $\mathcal{SQ}_{\theta_r}$ is empty, then the best design is obtained by not reinforcing at all.

Proof. We consider any suspension in the class $\mathcal{Q}_{\theta_r}$. If there exist disks of radius less than $\frac{2}{3}\gamma$, then Theorem 1.2 shows that there is no advantage to keeping them in the suspension. Moreover, the configuration obtained by removing these disks is now either in the class $\mathcal{SQ}_{\theta_r}$ or we have removed all the disks. If Ω is "too small" and we can only fit disks of radii less than $\frac{2}{3}\gamma$ inside, then Theorem 1.2 shows it is better not to reinforce. □

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