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THE GENERALIZED QUASILINEARIZATION METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider the nonlinear parabolic integro-differential equation with initial and boundary conditions. We develop the method of generalized quasilinearization to generate linear iterates that converge quadratically to the unique solution of the nonlinear parabolic integro-differential equation. For this purpose, we establish comparison results for the parabolic integro-differential equation. These comparison results are used to develop monotone sequences and to establish quadratic convergence.

1. Introduction. The method of quasilinearization due to Bellman [1] and Bellman and Kalaba [2] yields monotone sequences that converge quadratically to the unique solution of the nonlinear problem on the interval of existence. Further, the sequences are either increasing or decreasing depending on the forcing function being convex or concave. Recently the method of quasilinearization has been extended to a variety of nonlinear problems to obtain simultaneous bounds for the solution when the forcing function is neither convex nor concave. Yet, it has all the advantages of the method of quasilinearization. This method is now referred to as the generalized quasilinearization method. The generalized quasilinearization method is well developed for a variety of ordinary differential equations in [7]. The method of generalized quasilinearization has been developed to ordinary integro-differential equations in [9]. In this paper we develop the method of generalized quasilinearization for the nonlinear parabolic integro-differential equations of Volterra type. The model we consider is a more general form of the Hodgkin-Huxley model for the propagation of the voltage pulse through a nerve axon, which is often referred to as the Fitzhugh-Nagumo equation. See [8] for more details. Here, we prove that the monotone sequences, which are solutions of linear parabolic integro-differential equations, converge quadratically to the unique solution of the nonlinear parabolic integro-differential equations. The quadratic convergence is achieved by developing a comparison theorem of the parabolic integro-differential equations with the

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ordinary integro-differential equations. We note that the method of generalized quasilinearization provides a numerical procedure for the computation of solutions of nonlinear problems. Finally, we present a numerical example to demonstrate the application of our results.

2. Comparison theorem and main results. We consider the following problem:

$$\begin{aligned} Lu = f(t,x,u(t,x)) + \int_0^t g(t,x,s,u(s,x))\,ds, \quad 0 \le t \le T,\ x \in \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u(t,x)|_{x\in\partial\Omega} = h(t,x), \end{aligned} \tag{2.1}$$

where Ω is a bounded open domain in $\mathbb{R}^n$, $L = \frac{\partial}{\partial t} - A$ is a parabolic operator, and

$$A = \sum_{i,j=1}^{N} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x),$$

with $a_{i,j}, b_i, c \in C^{\alpha,\alpha/2}(\overline{Q}_T)$ and $\sigma_0|\xi|^2 \le \sum_{i,j=1}^N a_{i,j}\xi_i\xi_j \le \sigma_1|\xi|^2$, $\sigma_0, \sigma_1 > 0$ for $(t,x) \in \overline{Q}_T$ and $\xi \in \mathbb{R}^n$.

Throughout this paper we assume the initial and boundary conditions are such that $u_0 \in C^{2+\alpha}(\overline{\Omega}), h \in C^{2+\alpha,1+\alpha/2}((0,T)\times\partial\Omega)$ and

$$u_0(x) = h(0,x), \quad h_t = Au_0 + f(0,x,u_0) \quad \text{for } t = 0 \text{ and } x \in \partial\Omega.$$

Next we develop two comparison results, which we need in order to develop our main results. The first lemma will be used to show that the sequences we develop are monotone sequences. The second lemma will be used to prove that sequences converge quadratically to the unique solution of (2.1).

LEMMA 2.1. Assume that

(i) $g(t,x,s,u)$ is monotone nondecreasing in u for each (t,x,s) and

$$\begin{aligned} f(t,x,u) - f(t,x,v) \le M_1(u-v), \\ g(t,x,s,u) - g(t,x,s,v) \le M_2(u-v) \quad \text{for } u \ge v, \end{aligned}$$

where M_i, for $i = 1,2$, are nonnegative constants;

(ii) $v(t,x)$ and $w(t,x)$ satisfy

$$\begin{aligned} Lv \le f(t,x,v(t,x)) + \int_0^t g(t,x,s,v(s,x))\,ds, \\ Lw \ge f(t,x,w(t,x)) + \int_0^t g(t,x,s,w(s,x))\,ds, \end{aligned}$$

and

$$\begin{aligned} v(0,x) \le w(0,x), \\ v(t,x)|_{x\in\partial\Omega} \le w(t,x)|_{x\in\partial\Omega}. \end{aligned}$$

Then we have $w(t,x) \ge v(t,x)$.

Proof. We shall first prove the result for strict inequalities. Assume that the conclusion of the lemma is false. Then there exists a point (t_0, x_0) such that $w(t,x) > v(t,x)$ for $0 \le t < t_0$, and $w(t_0, x_0) = v(t_0, x_0)$. Certainly $(t_0, x_0) \notin \partial\Omega$. Hence

$$\begin{aligned} Lv(t_0,x_0) &< f(t_0,x_0,v(t_0,x_0)) + \int_0^{t_0} g(t,x_0,s,v(s,x_0))\,ds \\ &\le f(t_0,x_0,w(t_0,x_0)) + \int_0^{t_0} g(t,x_0,s,w(s,x_0))\,ds \\ &< Lw(t_0,x_0), \end{aligned}$$

i.e., $L[w(t_0,x_0) - v(t_0,x_0)] > 0$.

If $(t_0, x_0) \in \Omega$, since $w(t,x) - v(t,x)$ has a minimum value at (t_0, x_0), we have

$$L[w(t_0,x_0) - v(t_0,x_0)] \le 0,$$

which leads to a contradiction. Therefore, we have $w(t,x) > v(t,x)$ for $t \ge 0$. In order to prove the result for nonstrict inequality, set

$$w_\varepsilon(t,x) = w(t,x) + \varepsilon \exp(2Mt),$$

where $\varepsilon > 0$ and $M \ge \max\{\sqrt{M_2/2}, M_1\}$. We have $v(0,x) \le w(0,x) < w_\varepsilon(0,x)$. Since

$$f(t,x,w_\varepsilon(t,x)) - f(t,x,w(t,x)) \le M_1(w_\varepsilon(t,x) - w(t,x)) \le M\varepsilon\exp(2Mt),$$

$$\int_0^t g(t,x,s,w_\varepsilon(s,x)) - g(t,x,s,w(s,x))\,ds \le \frac{M_2}{2M}\varepsilon(\exp(2Mt) - 1) < M\varepsilon\exp(2Mt),$$

we have

$$\begin{aligned} Lw_\varepsilon &= Lw + 2M\varepsilon\exp(2Mt) \\ &\ge f(t,x,w(t,x)) + \int_0^t g(t,x,s,w(s,x))\,ds + 2M\varepsilon\exp(2Mt) \\ &> f(t,x,w_\varepsilon(t,x)) + \int_0^t g(t,x,s,w_\varepsilon(s,x))\,ds. \end{aligned}$$

Now using the strict inequality result, we have $w_\varepsilon(t,x) > v(t,x)$ for $t > 0$. Letting $\varepsilon \to 0$, we have $w(t,x) \ge v(t,x)$ for $t > 0$. □

LEMMA 2.2. Assume that

(i) $g(t,x,s,u)$ is monotone nondecreasing in u for each fixed (t,x,s);

(ii) $v(t,x)$ satisfy

$$\begin{aligned} Lv &\le f(t,x,v(t,x)) + \int_0^t g(t,x,s,v(s,x))\,ds, \\ v(0,x) &= u_0(x), \\ v(t,x)|_{x\in\partial\Omega} &= 0, \end{aligned}$$

and

$$\begin{aligned} r' &= h_1(t,r) + \int_0^t h_2(t,s,r)\,ds, \\ r(0) &= \max\left\{\max_{x\in\Omega} u_0(x), 0\right\}, \end{aligned}$$

where $h_1(t,r) \geq \max_{x\in\Omega} f(t,x,r)$ and $h_2(t,s,r) \geq \max_{x\in\Omega} g(t,x,s,r)$. Then we have $r(t) \geq v(t,x)$ on $\overline{Q}_T$.

Proof. Letting $V(t) = \max_{x\in\Omega} v(t,x)$, we have

$$V' \leq f(t,x,V) + \int_0^t g(t,x,s,V(s))\,ds \leq h_1(t,V) + \int_0^t h_2(t,s,V)\,ds,$$
$$V(0) = \max_{x\in\Omega} u_0(x).$$

This means that V is a lower solution of r. Hence $r \geq V \geq v(t,x)$. □

The proof can also be referred to Lemma 2.6.2 on p. 72 of [8].

Next we develop the method of generalized quasilinearization for the parabolic integro-differential equation (2.1). Using upper and lower solutions, we develop two monotone sequences that are solutions of linear parabolic integro-differential equations. We prove that the sequences converge uniformly and monotonically to the unique solution of the nonlinear parabolic integro-differential equation (2.1). Further, we prove that the rate of convergence is quadratic.

THEOREM 2.1. Assume that

(i) $g(t,x,s,u)$ is monotone nondecreasing in u for each fixed $(t,x,s) \in I \times \Omega \times I$, $f_{uu} \geq 0$.

(ii) $v_0(t,x)$ and $w_0(t,x)$ satisfy

$$Lv_0 \leq f(t,x,v_0(t,x)) + \int_0^t g(t,x,s,v_0(s,x))\,ds,$$
$$Lw_0 \geq f(t,x,w_0(t,x)) + \int_0^t g(t,x,s,w_0(s,x))\,ds,$$

and

$$v_0(0,x) \leq w_0(0,x),$$
$$v_0(t,x)|_{x\in\partial\Omega} \leq w_0(t,x)|_{x\in\partial\Omega}.$$

(iii) $f_u, g_u \in C^{\alpha,\frac{\alpha}{2}}[I \times \Omega]$, and f_{uu}, g_{uu} exist and are continuous such that $f_{uu} \geq 0$. There exist functions $\phi(t,x,s,u)$ and $G(t,x,s,u) = \phi(t,x,s,u) + g(t,x,s,u)$ such that $G_{uu} \geq 0, \phi_{uu} > 0$.

(iv) $G_u(t,x,s,u_1) - \phi_u(t,x,s,u_2) \geq 0$, for $v_0 \leq u_1 \leq u_2 \leq w_0$.

Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ that converge monotonically and quadratically to the unique solution of (2.1).

Proof. Initially we construct the sequences $\{v_n\}$ and $\{w_n\}$. For that purpose let v_1 and w_1 be the solutions of the linear parabolic integro-differential equations with initial and boundary conditions as

$$\begin{aligned} Lv_1 = {} & f(t,x,v_0(t,x)) + f_u(t,x,v_0(t,x))(v_1(t,x) - v_0(t,x)) \\ & + \int_0^t \{g(t,x,s,v_0(s,x)) + [G_u(t,x,s,v_0(s,x)) \\ & \qquad - \phi_u(t,x,s,w_0(s,x))](v_1(s,x) - v_0(s,x))\}\,ds, \end{aligned}$$

$$v_1(0,x) = u_0(x), \quad x \in \Omega,$$
$$v_1(t,x)|_{x\in\partial\Omega} = h(t,x),$$

and

$$\begin{aligned} Lw_1 = {} & f(t,x,w_0(t,x)) + f_u(t,x,w_0(t,x))(w_1(t,x) - w_0(t,x)) \\ & + \int_0^t \{g(t,x,s,w_0(s,x)) + [G_u(t,x,s,v_0(s,x)) \\ & \qquad - \phi_u(t,x,s,w_0(s,x))](w_1(s,x) - w_0(s,x))\}\, ds, \end{aligned}$$

$$w_1(0,x) = u_0(x), \quad x \in \Omega,$$
$$w_1(t,x)|_{x\in\partial\Omega} = h(t,x).$$

From Theorem 3.3 in [3], it follows that (v_1, w_1) exist and are unique. We can now prove that $v_0 \le v_1 \le w_1 \le w_0$ on $\overline{Q}_T$.

First we let $\eta = v_0 - v_1$. Since $\eta(0,x) = 0$, $\eta(t,x)|_{x\in\partial\Omega} = 0$, we have

$$\begin{aligned} L\eta &= Lv_0 - Lv_1 \\ &\le f(t,x,v_0(t,x)) - [f(t,x,v_0(t,x)) + f_u(t,x,v_0(t,x))(v_1(t,x) - v_0(t,x))] \\ &\quad + \int_0^t g(t,x,s,v_0(s,x))\, ds - \int_0^t \{g(t,x,s,v_0(s,x)) \\ &\quad + [G_u(t,x,s,v_0(s,x)) - \phi_u(t,x,s,w_0(s,x))](v_1(s,x) - v_0(s,x))\}\, ds \\ &\le f_u(t,x,v_0(t,x))\eta + \int_0^t [G_u(t,x,s,v_0(s,x)) - \phi_u(t,x,s,w_0(s,x))]\eta\, ds. \end{aligned}$$

Now applying Lemma 2.1, it follows that $\eta \le 0$. This proves $v_0 \le v_1$ on $\overline{Q}_T$.

Similarly, we can prove $w_1 \le w_0$ on $\overline{Q}_T$. In order to prove $v_1 \le w_1$ on $\overline{Q}_T$, we shall first prove $v_1 \le w_0$.

Setting $\xi = v_1 - w_0$, we have $\xi(0,x) = 0, \xi(t,x)|_{x\in\partial\Omega} = 0$. Now it follows that

$$\begin{aligned} L\xi \le {} & f(t,x,v_0(t,x)) + f_u(t,x,v_0(t,x))(v_1(t,x) - v_0(t,x)) \\ & + \int_0^t \{g(t,x,s,v_0(s,x)) + [G_u(t,x,s,v_0(s,x)) \\ & \qquad - \phi_u(t,x,s,w_0(s,x))](v_1(s,x) - v_0(s,x))\}\, ds \\ & - f(t,x,w_0(t,x)) - \int_0^t g(t,x,s,w_0(s,x))\, ds. \end{aligned}$$

Since $G_{uu} \ge 0, \phi_{uu} > 0$, and $f_{uu} > 0$, we get

$$\begin{aligned} \int_0^t g(t,x,s,w_0(s,x)) \ge {} & \int_0^t \{g(t,x,s,v_0(s,x)) \\ & + G_u(t,x,s,v_0(s,x))(w_0(s,x) - v_0(s,x)) \\ & \qquad - [\phi(t,x,s,w_0(s,x)) - \phi(t,x,s,v_0(s,x))]\}\, ds, \end{aligned}$$

and

$$\phi(t,x,s,w_0(s,x)) - \phi(t,x,s,v_0(s,x)) \le \phi_u(t,x,s,w_0(s,x))(w_0(s,x) - v_0(s,x)).$$

Also, we have

$$f(t,x,v_0(t,x)) - f(t,x,w_0(t,x)) \le f_u(t,x,v_0(t,x))(v_1 - w_0).$$

This yields

$$L\xi \le f_u(t,x,v_0(t,x))\xi + \int_0^t [G_u(t,x,s,v_0(s,x)) - \phi_u(t,x,s,w_0(s,x))]\xi(x,s)\,ds,$$

and using Lemma 2.1, we have $\xi \le 0$, i.e., $v_1 \le w_0$.

Similarly, we can prove $v_0 \le w_1$.

Now we prove $v_1 \le w_1$ on $\overline{Q}_T$. Setting $\alpha = v_1 - w_1$, we get

$$\begin{aligned} L\alpha = {} & f(t,x,v_0(t,x)) + f_u(t,x,v_0(t,x))(v_1(t,x) - v_0(t,x)) \\ & + \int_0^t \{g(t,x,s,v_0(s,x)) - g(t,x,s,w_0(s,x)) + [G_u(t,x,s,v_0(s,x)) \\ & \qquad - \phi_u(t,x,s,w_0(s,x))](v_1(s,x) - v_0(s,x) - (w_1(s,x) - w_0(s,x)))\}\,ds. \end{aligned}$$

Since $f_{uu} \ge 0$, we get

$$f(t,x,v_0(t,x)) - f(t,x,w_0(t,x)) \le -f_u(t,x,v_0(t,x))(w_0(t,x) - v_0(t,x)).$$

Using this, we obtain

$$\begin{aligned} & f(t,x,v_0(t,x)) + f_u(t,x,v_0(t,x))(v_1(t,x) - v_0(t,x)) \\ & \qquad - [f(t,x,w_0(t,x)) + f_u(t,x,w_0(t,x))(w_1(t,x) - w_0(t,x))] \\ & \qquad\qquad \le -f_u(t,x,v_0(t,x))(w_1(t,x) - v_1(t,x)). \end{aligned}$$

Since $G_{uu} \ge 0$, we also have

$$\begin{aligned} & \int_0^t \{g(t,x,s,v_0(s,x)) - g(t,x,s,w_0(s,x)) \\ & \qquad + [G_u(t,x,s,v_0(s,x)) - \phi_u(t,x,s,w_0(s,x))](w_0(s,x) - v_0(s,x))\}\,ds \le 0. \end{aligned}$$

From this, it easily follows that

$$L\alpha \le f_u(t,x,v_0(t,x))\alpha + \int_0^t [G_u(t,x,s,v_0(s,x)) - \phi_u(t,x,s,w_0(s,x))]\alpha\,ds.$$

By comparison Lemma 2.1, this yields $\alpha \le 0$. This implies $v_1 \le w_1$ on $\overline{Q}_T$.

Assume that for some $n > 0$ we have

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots \le w_1 \le w_0. \tag{2.2}$$

Certainly (2.2) is true for $n = 1$.

Let v_{n+1} and w_{n+1} be the solutions of the following linear equations:

$$\begin{aligned} Lv_{n+1} = {} & f(t,x,v_n(t,x)) + f_u(t,x,v_n(t,x))(v_{n+1}(t,x) - v_n(t,x)) \\ & + \int_0^t \{g(t,x,s,v_n(s,x)) + [G_u(t,x,s,v_n(s,x)) \\ & \qquad - \phi_u(t,x,s,w_n(s,x))](v_{n+1}(s,x) - v_n(s,x))\}\,ds, \end{aligned}$$

$$v_{n+1}(0,x) = u_0(x), \quad x \in \Omega,$$
$$v_{n+1}(t,x)|_{x\in\partial\Omega} = h(t,x),$$

and

$$\begin{aligned} Lw_{n+1} = {} & f(t,x,w_n(t,x)) + f_u(t,x,w_n(t,x))(w_{n+1}(t,x) - w_n(t,x)) \\ & + \int_0^t \{g(t,x,s,w_n(s,x)) + [G_u(t,x,s,v_n(s,x)) \\ & \qquad - \phi_u(t,x,s,w_n(s,x))](w_{n+1}(s,x) - w_n(s,x))\}\, ds, \end{aligned}$$

$$w_{n+1}(0,x) = u_0(x), \quad x \in \Omega,$$
$$w_{n+1}(t,x)|_{x\in\partial\Omega} = h(t,x).$$

We assume that (2.2) is true for some n, and we prove that $v_n \le v_{n+1} \le w_{n+1} \le w_n$. This proves (2.2) is true for $n = n+1$ also.

First we show that $v_n \le v_{n+1}$, by letting $\alpha = v_{n+1} - v_n$. Since

$$\begin{aligned} & \int_0^t g(t,x,s,v_n(s,x))\, ds \\ & \ge \int_0^t \{g(t,x,s,v_{n-1}(s,x)) + G_u(t,x,s,v_{n-1}(s,x)(v_n(s,x) - v_{n-1}(s,x))) \\ & \qquad\qquad - [\phi(t,x,s,v_n(s,x)) - \phi(t,x,s,v_{n-1}(s,x))]\}\, ds \\ & \ge \int_0^t \{g(t,x,s,v_{n-1}(s,x)) \\ & \quad + [G_u(t,x,s,v_{n-1}(s,x)) - \phi_u(t,x,s,v_{n-1}(s,x))](v_n(s,x) - v_{n-1}(s,x))\}\, ds \\ & \ge \int_0^t \{g(t,x,s,v_{n-1}(s,x)) \\ & \quad + [G_u(t,x,s,v_{n-1}(s,x)) - \phi_u(t,x,s,w_{n-1}(s,x))](v_n(s,x) - v_{n-1}(s,x))\}\, ds, \end{aligned}$$

and

$$f(t,x,v_n(t,x)) \ge f(t,x,v_{n-1}(t,x)) + f_u(t,x,v_{n-1}(t,x))(v_n(t,x) - v_{n-1}(t,x)),$$

it easily follows that

$$\begin{aligned} L\alpha = {} & f(t,x,v_n(t,x)) + f_u(t,x,v_n(t,x))(v_{n+1}(t,x) - v_n(t,x)) \\ & + \int_0^t \{g(t,x,s,v_n(s,x)) + [G_u(t,x,s,v_n(s,x)) \\ & \qquad - \phi_u(t,x,s,w_n(s,x))](v_{n+1}(s,x) - v_n(s,x))\}\, ds \\ & - [f(t,x,v_{n-1}(t,x)) + f_u(t,x,v_{n-1}(t,x))(v_n(t,x) - v_{n-1}(t,x)) \\ & + \int_0^t \{g(t,x,s,v_{n-1}(s,x)) + [G_u(t,x,s,v_{n-1}(s,x)) \\ & \qquad - \phi_u(t,x,s,w_{n-1}(s,x))](v_n(s,x) - v_{n-1}(s,x))\}\, ds] \\ \ge {} & f_u(t,x,v_n(t,x))\alpha + \int_0^t [G_u(t,x,s,v_n(s,x)) - \phi_u(t,x,s,w_n(s,x))]\alpha\, ds. \end{aligned}$$

By the comparison Lemma 2.1, it follows that $v_n \le v_{n+1}$ on $\overline{Q}_T$.

Next we show that $w_{n+1} \le w_n$. Letting $\beta = w_{n+1} - w_n$, we have

$$\begin{aligned} L\beta = {} & f(t,x,w_n(t,x)) + f_u(t,x,w_n(t,x))(w_{n+1}(t,x) - w_n(t,x)) \\ & + \int_0^t \{g(t,x,s,w_n(s,x)) + [G_u(t,x,s,v_n(s,x)) \\ & \qquad - \phi_u(t,x,s,w_n(s,x))](w_{n+1}(s,x) - w_n(s,x))\}\, ds \\ & - [f(t,x,w_{n-1}(t,x)) + f_u(t,x,w_{n-1}(t,x))(w_n(t,x) - w_{n-1}(t,x)) \\ & + \int_0^t \{g(t,x,s,w_{n-1}(s,x)) + [G_u(t,x,s,v_{n-1}(s,x)) \\ & \qquad - \phi_u(t,x,s,w_{n-1})(s,x))](w_n(s,x) - w_{n-1}(s,x))\}\, ds. \end{aligned}$$

Since $G_{uu} \ge 0$ and $\phi_{uu} \ge 0$, we also have

$$\begin{aligned} & \int_0^t \{g(t,x,s,w_n(s,x)) - g(t,x,s,w_{n-1}(s,x))\, ds \\ & \quad \le \int_0^t [G_u(t,x,s,v_{n-1}(s,x)) - \phi_u(t,x,s,w_{n-1}(s,x))](w_n(s,x) - w_{n-1}(s,x))\}\, ds. \end{aligned}$$

Since $w_n \le w_{n-1}$ and $f_{uu} \ge 0$, by our assumption we have

$$f(t,x,w_n(t,x)) \le f(t,x,w_{n-1}(t,x)) + f_u(t,x,w_{n-1}(t,x))(w_n(t,x) - w_{n-1}(t,x)).$$

Hence it follows that

$$L\beta \le f_u(t,x,w_n(t,x))\beta + \int_0^t [G_u(t,x,s,v_{n-1}(s,x)) - \phi_u(t,x,s,w_{n-1}(s,x))]\beta\, ds.$$

Again using the comparison Lemma 2.1, we obtain that $w_{n+1} \le w_n$.

On the same lines as we proved $v_1 \le w_1$ on $\overline{Q}_T$, we can also prove $v_{n+1} \le w_{n+1}$.

We have established (2.2) for $n = 1$. Now using mathematical induction, it follows that

$$v_0 \le v_1 \le \cdots \le v_n \le v_{n+1} \le w_{n+1} \le w_n \le \cdots \le w_1 \le w_0, \quad \text{for all } n.$$

Let $\overline{v}(t,x) = \lim_{n\to\infty} v_n(t,x)$ and $\underline{w}(t,x) = \lim_{n\to\infty} w_n(t,x)$. Since $\{v_n\}$ and $\{w_n\}$ possess the monotone property, it ensures that $\overline{v}(t,x)$ and $\underline{w}(t,x)$ exist. By the same method as in [8], it is easy to show that $\overline{v}(t,x) = \underline{w}(t,x)$ and is the unique solution of (2.1).

We claim that the convergence of v_n is uniform. Assume that the claim is false. Then for some $\varepsilon_0 > 0$, and for any i, there exist $n_i > i$ and $(t_{n_i}, x_{n_i}) \in \overline{Q}_T$ such that

$$|v_{n_i}(t_{n_i}, x_{n_i}) - u(t_{n_i}, x_{n_i})| > \varepsilon_0. \tag{2.3}$$

We choose $i > n_{i-1}$ so that we can construct a sequence $\{(t_{n_i}, x_{n_i})\}$. Since $\overline{Q}_T$ is compact, there exists a subsequence $\{(t_{n_j}, x_{n_j})\}$ such that $\{(t_{n_j}, x_{n_j})\}$ has a limit $(\overline{t}, \overline{x})$. Since v_n and u are uniformly continuous, it follows from (2.3) that

$$\varepsilon_0 \le \lim_{n_j\to\infty} |v_{n_j}(t_{n_j}, x_{n_j}) - u(t_{n_j}, x_{n_j})| = |\overline{v}(\overline{t}, \overline{x}) - u(\overline{t}, \overline{x})| = 0,$$

which is a contradiction. Similarly we can prove that w_n converges uniformly to $\underline{w}(t,x) = u(t,x)$, the unique solution of (2.1).

Finally, we show that the convergence is quadratic. For that, let $p_{n+1} = u(t,x) - v_{n+1}$ and $q_{n+1} = w_{n+1} - u(t,x)$. It follows that

$$\begin{aligned}
Lp_{n+1} &= f(t,x,u(t,x)) - [f(t,x,v_n(t,x)) + f_u(t,x,v_n(t,x))(v_{n+1}(t,x) - v_n(t,x))] \\
&\quad + \int_0^t g(t,x,s,u(s,x))\,ds - \int_0^t \{g(t,x,s,v_n(s,x)) + [G_u(t,x,s,v_n(s,x)) \\
&\qquad\qquad - \phi_u(t,x,s,w_n(s,x))](v_{n+1}(s,x) - v_n(s,x))\}\,ds \\
&= f_u(t,x,v_n(t,x))(u(t,x) - v_n(t,x)) + f_{uu}(t,x,\xi)(u(t,x) - v_n(t,x))^2 \\
&\quad - f_u(t,x,v_n(t,x))(v_{n+1}(t,x) - v_n(t,x)) \\
&\quad + \int_0^t [g(t,x,s,u(s,x)) - g(t,x,s,v_n(s,x))]\,ds \\
&\quad - \int_0^t \{[G_u(t,x,s,v_n(s,x)) - \phi_u(t,x,s,w_n(s,x))](v_{n+1}(s,x) - u(t,x) \\
&\qquad\qquad + u(t,x) - v_n(s,x))\}\,ds \\
&= f_u(t,x,v_n(t,x))(u(t,x) - v_{n+1}(t,x)) + f_{uu}(t,x,\xi)(u(t,x) - v_n(t,x))^2 \\
&\quad + \int_0^t g_u(t,x,s,\mu)p_n\,ds + \int_0^t [G_u(t,x,s,v_n(s,x)) - \phi_u(t,x,s,w_n(s,x))]p_{n+1}\,ds \\
&\quad - \int_0^t [G_u(t,x,s,v_n(s,x)) - \phi_u(t,x,s,w_n(s,x))]p_n\,ds \\
&\le f_u(t,x,v_n(t,x))p_{n+1} + f_{uu}(t,x,\xi)p_n^2 \\
&\quad + \int_0^t g_u(t,x,s,w_n(s,x))p_n\,ds \\
&\quad + \int_0^t [G_u(t,x,s,v_n(s,x)) - \phi_u(t,x,s,w_n(s,x))]p_{n+1}\,ds \\
&\quad - \int_0^t [G_u(t,x,s,v_n(s,x)) - \phi_u(t,x,s,w_n(s,x))]p_n\,ds \\
&\le f_u(t,x,v_n(t,x))p_{n+1} + f_{uu}(t,x,\xi)p_n^2 \\
&\quad + \int_0^t G_{uu}(t,x,s,\mu_1)(p_n + q_n)p_n\,ds + \int_0^t g_u(t,x,s,v_n(s,x))p_{n+1}\,ds \\
&\le M_1 p_{n+1} + M_2 p_n^2 + \int_0^t \left[M_3 p_n^2 + \frac{1}{2}M_3(p_n^2 + q_n^2) + M_4 p_{n+1}\right] ds,
\end{aligned}$$

where $|f_u| \le M_1, |f_{uu}| \le M_2, |G_{uu}| \le M_3, |g_u| \le M_4, |g_{uu}| \le M_5$.

Clearly $p_{n+1}(0,x) = 0, p_{n+1}(t,x)|_{x\in\partial\Omega} = 0$.

Let $r(t)$ be the solution of the related IVP for a linear integro-differential equation. Then we have

$$\begin{aligned}
&r'(t) = M_1 r(t) + M_4 \int_0^t r(s)\,ds + \left(M_2 + \frac{3}{2}M_3 T\right) \max_{x\in\Omega, 0<t<T} p_n^2 + \frac{1}{2}M_3 T \max_{x\in\Omega, 0<t<T} q_n^2, \\
&r(0) = 0.
\end{aligned}$$

We obtain the following estimate for $r(t)$:

$$r(t) \le \frac{2\exp(M_1T)}{\sqrt{M_1^2+4M_4}}\left[\left(M_2+\frac{3}{2}M_3T\right)\max_{x\in\Omega,0<t<T}p_n^2+\frac{1}{2}M_3T\max_{x\in\Omega,0<t<T}q_n^2\right].$$

It is easy to see that

$$\int_0^t\left[M_3p_n^2+\frac{1}{2}M_3(p_n^2+q_n^2)\right]ds \le \frac{3}{2}M_3T\max_{x\in\Omega,0<t<T}p_n^2+\frac{1}{2}M_3T\max_{x\in\Omega,0<t<T}q_n^2.$$

Using Lemma 2.2 we have $p_{n+1}(t,x)\le r(t)$. Therefore

$$\max_{x\in\Omega,0<t<T}p_{n+1} \le \frac{2\exp(M_1T)}{\sqrt{M_1^2+4M_4}}\left[\left(M_2+\frac{3}{2}M_3T\right)\max_{x\in\Omega,0<t<T}p_n^2+\frac{1}{2}M_3T\max_{x\in\Omega,0<t<T}q_n^2\right].$$

Similarly,

$$\max_{x\in\Omega,0<t<T}q_{n+1} \le \frac{2\exp(M_1T)}{\sqrt{M_1^2+4M_4}}\left[\left(M_2+\frac{3}{2}TM_3+TM_5\right)\max_{x\in\Omega,0<t<T}q_n^2 + \frac{1}{2}M_3T\max_{x\in\Omega,0<t<T}p_n^2\right].$$

Hence, the sequences $\{v_n\}$ and $\{w_n\}$ converge quadratically to $u(t,x)$, the unique solution of (2.1). The proof is complete. □

REMARK 2.1. Actually, we do not need to assume $f_{uu}\ge 0$. It is enough if there exists a function $\psi(t,x,s,u)$ such that $F(t,x,s,u)=\psi(t,x,s,u)+f(t,x,s,u)$ and $F_{uu}\ge 0, \psi_{uu}>0$. It is easy to see that we can always construct a $\psi(t,x,s,u)$ that satisfies the above inequalities.

3. Numerical examples. In this section, we apply the generalized quasilinearization method to the following problem, which models the propagation of a voltage pulse through a nerve axon, often referred to as the Fitzhugh-Nagumo equation [8],

$$u_t-u_{xx}=u(u-\theta)(1-u)+b\int_0^t u(s,x)\,ds, \quad 0<x,t<1. \tag{3.1}$$

Here, we choose $\theta=0.5$ and $b=0$, with the initial boundary conditions as

$$\begin{aligned} u(0,x)&=\sin(\pi x), && 0<x<1,\\ u(0,t)&=u(1,t)=0, && 0<t<1. \end{aligned}$$

Let $\psi(t,x,u)=u^3+\theta u$. Then it is easy to see that $f+\psi$ is convex.

For each step of the iteration in the generalized quasilinearization method, we have the following linear IBVPs:

$$\begin{aligned} \frac{\partial}{\partial t}v^{(k+1)}-\frac{\partial^2}{\partial x^2}v^{(k+1)} &= v^{(k)}(v^{(k)}-\theta)(1-v^{(k)})\\ &\quad+(2(1+\theta)w^{(k)}-\theta-3(v^{(k)})^2)(v^{(k+1)}-v^{(k)})+\int_0^t v^{(k+1)}\,ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned} v^{(k+1)}(0,x)&=\sin(\pi x), \quad 0<x<1,\\ v^{(k+1)}(t,x)|_{x=0,1}&=0, \end{aligned}$$

and

$$\frac{\partial}{\partial t}w^{(k+1)} - \frac{\partial^2}{\partial x^2}w^{(k+1)} = w^{(k)}(w^{(k)} - \theta)(1 - w^{(k)}) + (2(1+\theta)w^{(k)} - \theta - 3(v^{(k)})^2)(w^{(k+1)} - w^{(k)}) + b\int_0^t w^{(k+1)}\,ds, \quad (3.3)$$

$$w^{(k+1)}(0,x) = \sin(\pi x), \quad 0 < x < 1,$$
$$w^{(k+1)}(t,x)|_{x=0,1} = 0.$$

The solutions of (3.2) and (3.3) can be obtained by the implicit method. The final finite difference equations are

$$\begin{aligned}
&\frac{v^{(k+1)}(l+1,i) - v^{(k+1)}(l,i)}{\Delta t} \\
&\quad - \frac{v^{(k+1)}(l+1,i-1) - 2v^{(k+1)}(l,i) + v^{(k+1)}(l,i+1)}{(\Delta x)^2} \\
&= v^{(k)}(l+1,i)(v^{(k)}(l+1,i) - \theta)(1 - v^{(k)}(l+1,i) \\
&\quad + (2(1+\theta)w^{(k)}(l+1,i) - \theta - 3(v^{(k)}(l+1,i))^2)(v^{(k+1)}(l+1,i) - v^{(k)}(l+1,i)) \\
&\quad + \left(b\sum_{m=1}^{l} v^{(k+1)}(m,i) + \frac{1}{2}v^{(k+1)}(l+1,i) + \frac{1}{2}v^{(k+1)}(0,i)\right)\Delta t,
\end{aligned} \quad (3.4)$$

and

$$\begin{aligned}
&\frac{w^{(k+1)}(l+1,i) - w^{(k+1)}(l,i)}{\Delta t} \\
&\quad - \frac{w^{(k+1)}(l+1,i-1) - 2w^{(k+1)}(l,i) + w^{(k+1)}(l,i+1)}{(\Delta x)^2} \\
&= w^{(k)}(l+1,i)(w^{(k)}(l+1,i) - \theta)(1 - w^{(k)}(l+1,i) \\
&\quad + (2(1+\theta)w^{(k)}(l+1,i) - \theta - 3(v^{(k)}(l+1,i))^2)(w^{(k+1)}(l+1,i) - w^{(k)}(l+1,i)) \\
&\quad + \left(b\sum_{m=1}^{l} w^{(k+1)}(m,i) + \frac{1}{2}w^{(k+1)}(l+1,i) + \frac{1}{2}w^{(k+1)}(0,i)\right)\Delta t,
\end{aligned} \quad (3.5)$$

where $v^{(k)}(l,i) = v^{(k)}(l\Delta t, i\Delta x)$.

We take $L = 400, I = 20, \Delta t = 1/L, \Delta x = 1/I$, and $v_0 = 0, w_0 = \sin(\pi x)$ as lower and upper solutions, respectively. We use the LU decomposition, and we stop the iteration when the difference of lower and upper solutions is small enough. Actually, we take the difference $\varepsilon = 10^{-5}$. Figures 3.1, 3.2, and 3.3 show the changes of lower solution, upper solution, and the difference between them.

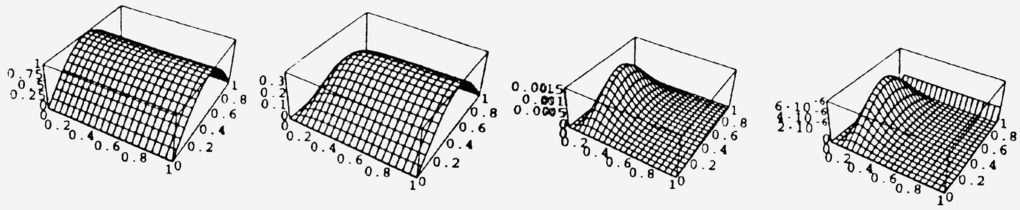

FIG. 3.1. The distance surface between lower and upper solutions

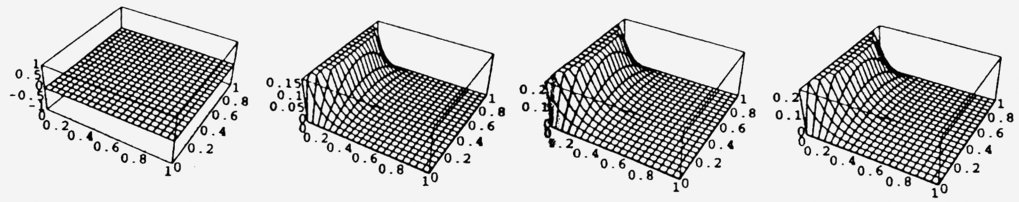

FIG. 3.2. The lower solution for the $0, 1, 3, 5$ iteration

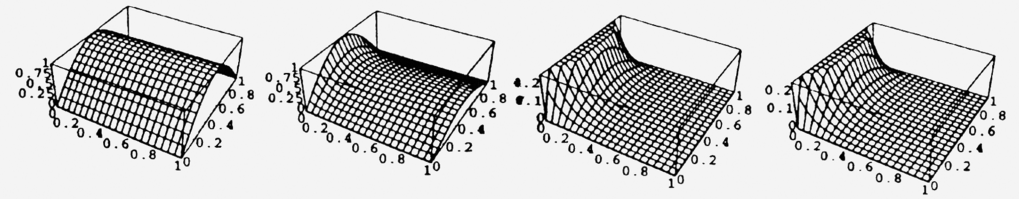

FIG. 3.3. The upper solution for the $0, 1, 3, 5$ iteration

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