QUARTERLY OF APPLIED MATHEMATICS
VOLUME LX, NUMBER 2
JUNE 2002, PAGES 213–244

ASYMPTOTIC AND EXACT FUNDAMENTAL SOLUTIONS IN HEREDITARY MEDIA WITH SINGULAR MEMORY KERNELS

By

ANDRZEJ HANYGA (*Institute of Solid Earth Physics, University of Bergen, Allégaten 41, N5007 Bergen, Norway*)

AND

M. SEREDYŃSKA (*Institute of Fundamental Technological Research, ul. Świętokrzyska 21, Warszawa, Poland*)

Abstract. A method for constructing time-domain asymptotic solutions of hyperbolic partial differential equations with delay, with singular memory kernels, is presented. The asymptotic solutions are expressed in terms of basis functions that are regularizations of a sequence of distributions related by fractional integration.

It is demonstrated that the signal builds up gradually from a zero value after the passage at the wavefront, resulting in signal delay. Attenuation splits into two parts: a frequency-independent amplitude modulating factor and a frequency-dependent part implicit in the basis functions.

Explicit basis functions are obtained for the $t^{-1/2}$ and $t^{-n/3}$ singularities. Asymptotic solutions are derived for systems of PDEs with the $t^{-1/2}$ singularity and for scalar equations with the $t^{-n/3}$ singularities. Asymptotic fundamental solutions are constructed for scalar equations with singular kernels. For two families of scalar equations with singular kernels, asymptotic fundamental solutions are shown to be exact.

1. Introduction. For many hyperbolic pseudo-differential operators, including Maxwellian viscosity, ray asymptotic solutions provide accurate amplitudes of discontinuities propagating at the wavefronts. In the time domain, the solutions can be constructed in terms of wavefront expansions involving a superposition of a sequence of distributions supported by the wavefronts and scaled by the amplitudes provided by the ray theory [9]. The superposition principle implies that the amplitude of a localized signal propagating with the wavefront is correctly estimated by ray asymptotics.

The situation changes radically when the solutions of the hyperbolic system are infinitely smooth, as is the case for hereditary models of viscoelasticity with weakly singular

Received October 27, 1998.
2000 *Mathematics Subject Classification.* Primary 34K06.
E-mail address: andrzej@ifjf.uib.no
E-mail address: msered@ippt.gov.pl

memory kernels [6, 31, 32, 35, 44, 45, 11], and for poroelasticity [20, 23]. In this case an initial discontinuity is immediately smoothed out and delayed with respect to the wavefront. The resulting time-domain asymptotic series is a superposition of an infinite series of basis functions that are regularized distributions and that vanish at the wavefront with all their derivatives. Since the basis functions replace the distributions appearing in wavefront expansions, the corresponding asymptotic expansions are called *generalized wavefront expansions* [22].

Wavefront smoothing for a class of scalar equations with singular convolution kernels applied to the time derivatives was studied in some detail by [31, 30] and the results were later reported in [32, 11]. In these papers, closed-form solutions were obtained in integral form. For two selected cases we have obtained explicit solutions in analytic form. Explicit asymptotic solutions for equations with constant coefficients in one spatial dimension were derived in the paper [6]. Wavefront smoothing for hereditary viscoelasticity with singular kernels is demonstrated in the papers [44, 35].

On the physico-experimental side, kernels with the $t^{-1/2}$ singularity have been linked to diffusion relaxation [25] and Biot's theory of poroelasticity [36, 20]. More general weakly singular kernels appeared in the theory of viscoelasticity developed by the school of Rabotnov [42, 43] and Rzhanitsyn [49] as well as in polymer rheology [51, 46, 3, 26, 27, 10, 52]. Additional cases involve acoustic one-dimensional problems with the lateral boundary conditions incorporated in the differential equations [54, 41] and bubbly liquids (cf. references in the paper [38]). Relevant experimental results in creep tests are discussed in [37, 42, 43].

Pipkin [39, 40, 29] studied problems of a quasi-parabolic type defined in terms of non-analytic creep functions and demonstrated the agreement of the large-time pulse asymptotics with the experiments of Kolsky [28]. His asymptotic solutions are expressed in terms of a basis function defined in our paper without the delay effect associated with finite propagation speed. Universality of pulse shapes for varying propagation times and materials, observed by Kolsky, is related to the representation of the pulse in terms of a single basis function.

This paper extends the time-domain asymptotic solutions of [6] to 3D problems with variable coefficients. An advantage of the method developed in [6] is an explicit analytic representation of the pulse. For the 3D problems with variable coefficients, the amplitude of a signal is not constant and it has to be determined by ray asymptotic methods. The amplitudes are calculated by integrating an infinite sequence of transport equations. It is demonstrated below that transport equations can be put in recurrence form in two cases: for systems of equations with the $t^{-1/2}$ singularity and for scalar equations with the $t^{-n/3}$ singularity with $n = 1$ or 2. The explicit form of basis functions for the $t^{-1/2}$ singularity was found in [6]. In this paper we determine an explicit form of the basis function in the $t^{-n/3}$ case that is more appropriate for numerical applications than the Wright function of [6].

Finally, exact analytic fundamental solutions of scalar equations considered in [31, 30] are found. This is an improvement over the integral solutions of the last references. In addition, the exact fundamental solutions coincide exactly with the asymptotic solutions of the same problems.

A rigorous formulation of the problem is given in Sec. 2. Basic facts about basis functions are collected in Sec. 3. In Secs. 4–5, basis functions and asymptotic solutions are constructed for the $t^{-1/2}$ singularity. Explicit fundamental solutions for a one-parameter family of scalar equations with such a singularity are constructed in Sec. 6.

Explicit basis functions for the $t^{-1/3}$ and $t^{-2/3}$ singularities are constructed in Sec. 7. In this case, transport equations are not in recurrence form (Sec. 8) except for the scalar case (Sec. 9). Fundamental solutions in an explicit closed form are constructed for a two-parameter family of scalar equations.

Both families of exact fundamental solutions mentioned above are superpositions of basis functions with constant coefficients. In these cases, the attenuation is totally contained in the basis functions. In the more general cases, amplitudes of the basis functions involve exponential factors modifying the total attenuation.

Asymptotic solutions involve a sequence of free parameters that can be determined from the initial and boundary-value conditions using the results of Appendices E and F. Regularity of the higher-order derivatives at $t = 0$ completes the set of conditions for determination of the free parameters.

2. Formulation of the problems. We consider here asymptotic solutions of a symmetric hyperbolic system of partial differential equations with time delay

$$(\mathbf{G}_{\alpha\beta}\mathbf{u}_{,\beta})_{,\alpha} + \mathbf{H}_\alpha \mathbf{u}_{,\alpha} = 0, \tag{2.1}$$

$$(\mathbf{G}_{\alpha\beta}\mathbf{w})(\xi) := \int_0^\infty \mathbf{G}_{\alpha\beta}(\tau,\xi)\mathbf{w}(t-\tau,\mathbf{x})\,d\tau, \tag{2.2}$$

with $\alpha, \beta = 0, \ldots, 3$, $\xi_0 = t$, $\xi_k = x_k$ for $k = 1, 2, 3$, and

$$\mathbf{G}_{\alpha\beta}(\tau,\xi) = \mathbf{G}^{(0)}_{\alpha\beta}(\xi)\delta(\tau) + \mathbf{G}^{(\mathrm{M})}_{\alpha\beta}(\tau,\xi). \tag{2.3}$$

The coefficient matrix $\mathbf{H}_\alpha$ is also a convolution operator with the $N \times N$ matrix-valued kernel $\mathbf{H}_\alpha(\tau,\xi)$.

For simplicity of the exposition it is assumed that $\mathbf{G}_{0k} = \mathbf{G}_{k0} = 0$. The assumption of symmetric hyperbolicity amounts to

$$\mathbf{G}^{(0)}_{00} = \mathbf{G}^{(0)^{\mathrm{T}}}_{00} \text{ is positive definite;} \tag{2.4}$$

$$\mathbf{G}^{(0)}_{\alpha\beta} = \mathbf{G}^{(0)^{\mathrm{T}}}_{\beta\alpha}. \tag{2.5}$$

In particular, we shall consider the scalar equation

$$(1 + K*)u_{,tt} - \nabla \cdot (n(\mathbf{x})^2 \nabla u) = 0. \tag{2.6}$$

In applications to continuum mechanics the coefficients $\mathbf{G}^{(0)}_{\alpha\beta}$ represent the elastic response of the medium while the integral kernels $\mathbf{G}^{(\mathrm{M})}_{\alpha\beta}$ account for viscous effects. The coefficients $\mathbf{G}^{(\mathrm{M})}_{ij}$ account for the viscous effects in the stress-strain relations while $\mathbf{G}^{(\mathrm{M})}_{00}$ accounts for the effect of viscosity on effective inertia, e.g., the viscodynamic operator of Biot's poroelasticity. In many viscoelastic models, the stress-strain relation involves a singular kernel. A good example is the well-established Bagley-Torvik model of uniaxial

extension of a polymer [46, 3] with the Young modulus in the frequency domain given by the formula

$$\hat{E}(\omega) = \frac{E_{\mathrm{R}} + E_{\mathrm{G}}(-i\omega/\omega_c)^{\alpha}}{1 + (-i\omega/\omega_c)^{\alpha}}, \quad 0 < \alpha < 1,\ E_{\mathrm{G}} \geq E_{\mathrm{R}} > 0,\ \omega_c > 0, \tag{2.7}$$

which implies that in the time domain, the Young modulus E is a time convolution operator with a singular kernel $E(\tau) \sim t^{-(1-\alpha)}$ for $\tau \to 0$. The coefficients E_{G} and E_{R} are known as the glassy and rubbery moduli while the characteristic frequency ω_c defines the transition region. The formula (2.7) covers a very wide range of frequencies.

In Biot's poroelasticity [5],

$$\mathbf{G}_{00}(\tau, \mathbf{x}) = \begin{vmatrix} \rho\mathbf{I} & \rho_f\mathbf{I} \\ \rho_f\mathbf{I} & \mathbf{N}(\tau, \mathbf{x}) \end{vmatrix}, \tag{2.8}$$

where $\mathbf{I}$ denotes the 3×3 unit matrix and the *viscodynamic operator* $\mathbf{N}(\tau, \mathbf{x})$, explicitly known for some idealized pore geometries, has a singularity $\sim \tau^{-1/2}$ [5, 36, 4, 53, 7, 2, 56]. The singularity of the viscodynamic operator is important for frequencies $\omega > \omega_{\mathrm{B}} \sim 10^5$Hz, where ω_{B} denotes Biot's frequency.

Biot's theory is not adequate for rocks and hydrocarbon reservoir models in view of microinhomogeneity leading to scattering and fast-to-slow wave conversion [19, 16]. In this case an effective dispersion relation for the fast longitudinal wave accounting for the fast-to-slow conversion has been derived for the fast P wave for frequencies in the range between a characteristic frequency ω_0 and ω_{B} [19]. The resulting dispersion law can be closely matched by a causal power law [21] leading to the memory singularity $t^{-0.66}$ if the Biot frequency ω_{B} can be considered infinite for the problem at hand.

It is therefore assumed here that the convolution kernels $\mathbf{G}^{(\mathrm{M})}_{\alpha\beta}(\tau, \xi)$ have an integrable singularity $\tau^{-\gamma}$ at $\tau \to 0$ with $0 < \gamma < 1$. A singularity $\tau^{-\gamma}$, $0 < \gamma < 1$, in the convolution kernels has important qualitative effects on wave propagation:

- wavefronts do not carry any discontinuities [20]; this property extends to nonlinear equations [22];
- the wavefield is $\mathcal{C}^{\infty}$-smooth at the wavefront [6, 32, 22] (cf. also [48]);
- the signal gradually builds up after the passage of the wavefront and the peak arrives with a delay with respect to the wavefront [22, 23].

In the extreme cases, the signal is detached from the wavefront and exhibits a nearly diffusive behavior.

For scalar linear equations (2.6), a precise criterion for wavefront smoothing is given in [32]: the solutions are smooth at the wavefronts if $K(\tau)/\ln\tau \to -\infty$ for $\tau \to 0+$. By a Tauberian theorem ([12], Thm. 16.1.3), this is equivalent to the Laplace image $\widetilde{K}(s)$ decreasing for $s \to \infty$ slower than $s^{-1}\ln s$.

The limiting case $\gamma \to 0+$ is the logarithmic singularity in the kernel $K(\tau)$, studied in detail in [32]. An example of such a kernel can be found in [45]. At the opposite end, for non-integrable kernels ($\gamma \geq 1$), the solutions of Eq. (2.6) are analytic [44] and, consequently, Eq. (2.6) is not hyperbolic.

The fundamental solution of Eq. (2.1) is a matrix-valued solution of Eq. (2.1) satisfying the initial condition

$$\begin{aligned} \mathbf{u}(0,\mathbf{x}) &= 0, \\ \mathbf{u}_{,0}(0,\mathbf{x}) &= \delta(\mathbf{x})\mathbf{I}, \end{aligned} \tag{2.9}$$

where $\mathbf{I}$ denotes the unit matrix.

Exact fundamental solutions will be constructed in analytic form for two classes of the scalar equations (2.6) in $1+3$ dimensions, where the convolution kernel $K(\tau,\xi)$ has the singularity $\tau^{-1/2}$ (Sec. 6), $\tau^{-2/3}$ or $\tau^{-1/3}$ (Sec. 9, Appendix D) for $\tau \to 0$. Some exact solutions in integral form were constructed in [31] (cf. also [11], vol. 5, 16.5).

Asymptotic solutions are sought in the form of *generalized wavefront expansions*

$$\mathbf{u}(t,\mathbf{x}) = \sum_{\mu=0}^{\infty} F_\mu(t - S^{(0)}(\mathbf{x}), S'(\mathbf{x}))\mathbf{u}^{(\mu)}(\mathbf{x}), \tag{2.10}$$

where the functions F_μ are defined on $\mathbb{R}\times\mathbb{R}_+^N$, vanish on $\mathbb{R}_-\times\mathbb{R}_+^N$ and satisfy a set of recurrence equations (Sec. 3). The sequence can be chosen in such a way that $F_0(t,\sigma)\to\delta(t)$ for $\sigma\to 0$ in the distributions sense. The symbol $S'(\mathbf{x})$ stands for a collection of functions $S^{(\nu)}(\mathbf{x})$, $\nu = 1,\dots,N$, with $S^{(1)}\geq 0$.

For $\gamma = 1/2$ and $\gamma = 1/3$, $2/3$, the basis functions can be expressed in terms of the complementary error function (Sec. 4) and the Airy function (Sec. 7). For $\gamma = 1/2$, the amplitudes can be calculated by integrating a sequence of ordinary differential equations in recurrence form (Sec. 5). For $\gamma = 1/3$, $2/3$, this is true for a scalar equation (Sec. 9).

3. Basis functions: General results. We consider asymptotic solutions of equations (2.1) of the form

$$\tilde{\mathbf{u}}(s,\mathbf{x}) = e^{-sS^{(0)}(\mathbf{x})-\sum_{r=1}^{N} s^{\gamma_r}S^{(r)}(\mathbf{x})}\sum_{r=0}^{\infty} s^{-\alpha_r}\mathbf{u}^{(r)}(\mathbf{x}), \tag{3.1}$$

where $\tilde{\mathbf{u}}(s,\mathbf{x})$ denotes the Laplace transform of $\mathbf{u}(\mathbf{x},t)$ and the variable s is identified with $-i\omega$,

$$\begin{gathered} \gamma_0 = 1, \quad 0<\gamma_r<\gamma_1<1 \quad \text{for } r>1, \\ 0<\alpha_r<\alpha_{r+1} \quad \text{for } r>0, \\ S^{(1)}(\mathbf{x})\geq 0. \end{gathered}$$

Asymptotic expansions with a similar phase function were introduced for the first time in [6] in a study of one-dimensional wave motion in a homogeneous viscoelastic medium. The right-hand side of (3.1) can be regarded as the result of expanding the exponential in the apparently more general expression

$$\tilde{\mathbf{u}}(s,\mathbf{x}) = e^{-sS^{(0)}(\mathbf{x})-\sum_{r=-N}^{\infty} s^{\gamma_r}S^{(-r)}(\mathbf{x})}\mathbf{w}(s,\mathbf{x}), \quad \gamma_r<0 \quad \text{for } r>0$$

with respect to the negative powers of s.

The inverse Laplace transform of Eq. (3.1) has the form

$$\mathbf{u}(t,\mathbf{x}) = \sum_{r=0}^{\infty} f_{\alpha_r}(t - S^{(0)}(\mathbf{x}), S'(\mathbf{x}))\mathbf{u}^{(r)}(\mathbf{x}), \tag{3.2}$$

where $S'(\mathbf{x})$ stands for $S^{(r)}(\mathbf{x})$, $r = 1, \ldots, N$, collectively, and the basis functions f_α are defined by the formula

$$f_\alpha(t, \lambda) = \frac{1}{2\pi i} \int_{\mathcal{B}} e^{st} s^{-\alpha} \exp\left(-\sum_{r=1}^{N} \lambda_r s^{\gamma_r} \right) ds, \tag{3.3}$$

where $\mathcal{B}$ denotes the Bromwich contour $\operatorname{Re} s = \varepsilon > 0$. Since $\operatorname{Re} s^{\gamma_1} > 0$ on $\mathcal{B}$, the functions f_r are defined for $\lambda_1 > 0$.

For $t = 0$, the Bromwich contour can be closed in the right half of the complex plane, where the integrand of

$$\frac{\partial^n f_\alpha}{\partial t^n} = \frac{1}{2\pi i} \int_{\mathcal{B}} e^{st} s^{n-\alpha} \exp\left(-\sum_{r=1}^{N} \lambda_r s^{\gamma_r} \right) ds \tag{3.4}$$

does not have singularities. Consequently,

$$\frac{\partial^n f_\alpha}{\partial t^n}(0, \lambda') = 0 \quad \text{for } n = 0, 1, \ldots, \tag{3.5}$$

where $\lambda' = \{\lambda_1, \ldots, \lambda_N\}$.

From the definition of the functions f_α, it follows immediately that

$$\int_0^\infty f_0(t, \lambda')\, dt = 1 \tag{3.6}$$

and

$$f_0(t, \lambda') \to \delta(t) \tag{3.7}$$

in the sense of distributions for $\lambda_\nu \to 0$, $\nu = 1, \ldots, N$, $\lambda_1 > 0$.

A rigorous proof of Eq. (3.7) is given in Appendices B and C.

We now assume that $\lambda_\nu \geq 0$, $\nu = 1, \ldots, N$. Let

$$\phi(t) := \begin{cases} \sum_{\nu=1}^{N} \lambda_\nu \frac{t^{-\gamma_\nu}}{\Gamma(1-\gamma_\nu)} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases} \tag{3.8}$$

The Laplace transform $\tilde{\phi}(s)$ of $\phi(t)$ satisfies the equation

$$s\tilde{\phi}(s) = \sum_{\nu=1}^{N} \lambda_\nu s^{\gamma_\nu}. \tag{3.9}$$

Applying Bernstein's theorem [55], it can be proved that for $\lambda_\nu \geq 0$, $\nu = 0, \ldots, N$,

$$f_\alpha(t, \lambda') \geq 0 \tag{3.10}$$

(Appendix A). Thermodynamics (and existence of inverse Laplace transforms) requires that $\lambda_1 > 0$ but places no restrictions on λ_ν, $\nu > 1$. For sufficiently large negative values λ_ν, $1 < \nu < N$, the signal exhibits some initial oscillations and then tends to a positive signal.

For $\lambda_\nu \geq 0$, $\nu = 1, \ldots, N$, the function f_0 is an infinitely divisible probability density function and a convolution of N stable probability density functions [15, 17], as noted in [29, 33]. This implies that the signal can be viewed as the result of N successive fractional diffusion processes [14, 17]. The asymmetry of the signal is due to the total

skewness of the probability density functions. In particular, for $N = 1$ and $\gamma_1 = 1/2$, the function f_0 is the Lévy probability density function [15].

In general, the rate of decay of the pulse at the wavefront can be estimated from the following upper bound [32]:

$$f_0(t, \lambda') \leq t^{-1} e^{-C} e^{-\phi(t)} \tag{3.11}$$

for $0 < t < t_m$, where t_m is the unique solution of the equation

$$\sum_{\nu=1}^{N} \gamma_\nu \lambda_\nu \frac{t^{-\gamma_\nu}}{\Gamma(1-\gamma_\nu)} = 1$$

and $C = \Gamma'(1) = 0.577216\ldots$ is the Euler-Mascheroni constant. Similarly,

$$f_1(t, \lambda') \leq e^{-\phi(t)} \tag{3.12}$$

for arbitrary $t > 0$. Much sharper local estimates for the functions f_α for $t \to 0+$ can be obtained in specific cases from asymptotic expansions of the basis functions. The estimates (3.12) can be used to estimate higher-order terms for which the basis functions cannot be evaluated explicitly.

The following recurrence relations can be used to calculate the basis functions:

$$\frac{\partial f_\alpha}{\partial \lambda_\nu} = -f_{\alpha-\gamma_\nu} \tag{3.13}$$

and

$$t f_\alpha(t, \lambda') = \alpha f_{\alpha+1}(t, \lambda') + \sum_{\nu=1}^{N} \lambda_\nu \gamma_\nu f_{\alpha-\gamma_\nu+1}(t, \lambda'). \tag{3.14}$$

Equation (3.14) is a generalization of a recurrence relation from [6]. It can be proved by applying integration by parts to Eq. (3.3).

In the following sections it is assumed that $\gamma_\nu = \nu/m, \nu = 0, \ldots, m-1$, for $m = 2, 3, \ldots; \alpha = \nu/m$, $\nu = 0, 1, \ldots$. Accordingly, we shall use the notation $f_\nu^{(m)}(t, \lambda') := f_{\nu/m}(t, \lambda')$.

4. Basis time-domain functions for half-integer power expansions. For memory kernels given by half-integer power expansions,

$$\begin{aligned} \widetilde{\mathbf{G}}_{\alpha\beta}(s, \mathbf{x}) &= \sum_{r=0}^{\infty} s^{-r/2} \mathbf{G}_{\alpha\beta}^{(r)}(\mathbf{x}), \\ \widetilde{\mathbf{H}}_{\alpha}(s, \mathbf{x}) &= \sum_{r=0}^{\infty} s^{-r/2} \mathbf{H}_{\alpha}^{(r)}(\mathbf{x}), \end{aligned} \tag{4.1}$$

the asymptotic solution is assumed in the form

$$\tilde{\mathbf{u}} = \exp(-sS^{(0)}(\mathbf{x}) - s^{1/2} S^{(1)}(\mathbf{x})) \sum_{r=0}^{\infty} s^{-r/2} \mathbf{u}^{(r)}(\mathbf{x}). \tag{4.2}$$

The basis functions $f_r^{(2)}(t,\lambda)$ can be expressed in terms of elementary functions and the complementary error function

$$f_0^{(2)}(t,\lambda) = \frac{\lambda}{2\sqrt{\pi}} t^{-3/2} e^{-\lambda^2/(4t)}, \tag{4.3}$$

$$f_1^{(2)}(t,\lambda) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-\lambda^2/(4t)}, \tag{4.4}$$

$$f_2^{(2)}(t,\lambda) = \operatorname{erfc}(\lambda/(2\sqrt{t})), \tag{4.5}$$

$$f_3^{(2)}(t,\lambda) = \frac{2}{\sqrt{\pi}} t^{1/2} e^{-\lambda^2/(4t)} + \lambda \operatorname{erfc}(\lambda/(2\sqrt{t})) \tag{4.6}$$

for $t > 0$, with $f_r^{(2)}(t,\lambda) = 0$ for $t < 0$ (see Figures 4.1, 4.2) [1, 12].

The functions (4.3)–(4.6) can be calculated from the recurrence relations

$$\frac{\partial f_{(r+1)/2}}{\partial \lambda} = -f_{r/2}(t,\lambda). \tag{4.7}$$

The functions $f_0^{(2)}(t,\lambda), f_1^{(2)}(t,\lambda), f_2^{(2)}(t,\lambda)$ are regularizations of the distributions $\delta(t)$, $t_+^{-1/2}/\sqrt{\pi}, \theta(t)$ (Appendix B). $\theta(t)$ denotes the Heaviside step function. $f_0^{(2)}(t,\lambda)$ peaks at $t = \lambda^2/6$,$f_1^{(2)}(t,\lambda)$ has a less pronounced peak and $f_r^{(2)}(t,\lambda)$ are monotonically increasing.

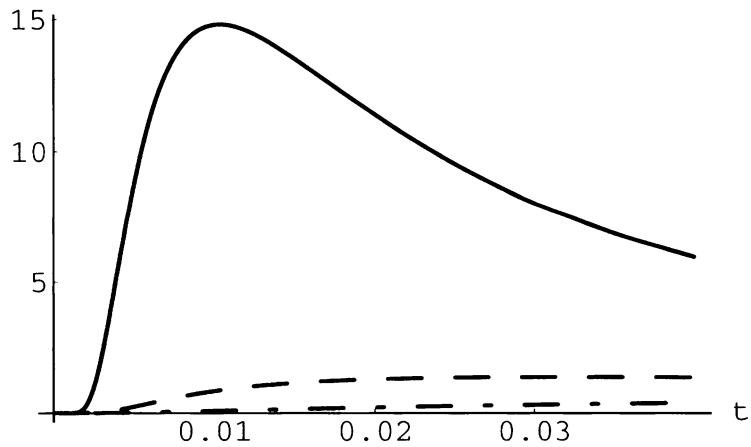

FIG. 4.1. The basis functions $f_0^{(2)}$, $f_1^{(2)}$ and $f_2^{(2)}$; $\lambda = 0.1$

5. Transport equations for memory kernels with the $\tau_+^{-1/2}$ singularity. In this section the eikonal equation, auxiliary eikonal equation, and transport equations are constructed for memory functions with $\tau_+^{-1/2}$ singularities. To this effect the expansion (4.2) is substituted into the Laplace transform of Eq. (2.1), namely,

$$\widetilde{\mathbf{G}_{00}}[s^2\tilde{\mathbf{u}} - \delta(\mathbf{x})\mathbf{I}] + \partial_j[\widetilde{\mathbf{G}_{jl}}\partial_l\tilde{\mathbf{u}}] + s\widetilde{\mathbf{H}_0}\tilde{\mathbf{u}} + \widetilde{\mathbf{H}_j}\partial_j\tilde{\mathbf{u}} = 0, \tag{5.1}$$

taking into account the initial data (2.9). The initial conditions (2.9) will be satisfied a posteriori by adjusting a set of free constants appearing in the general asymptotic

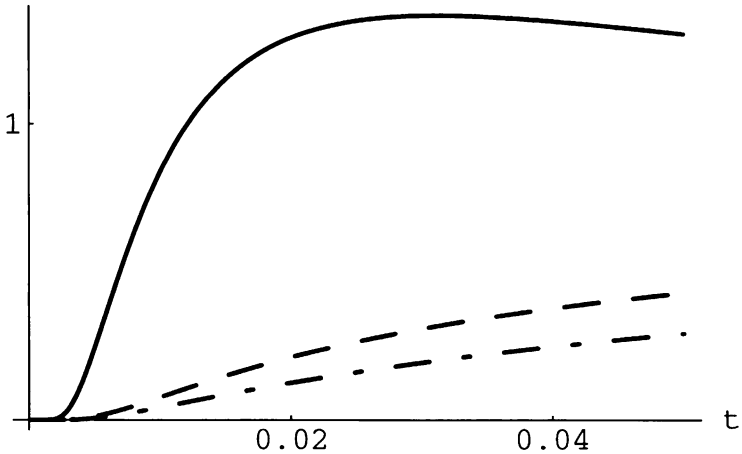

FIG. 4.2. Functions $f_1^{(2)}$, $f_2^{(2)}$ and $f_3^{(2)}$ for $s^{1/2}$-power expansions; $\lambda = 0.1$

solution of the differential equations (2.1). Consequently, the $\delta(\mathbf{x})$ term in (5.1) will be ignored and the solution $\mathbf{u}$ will be considered a vector.

The amplitudes $\mathbf{u}^{(\nu)}$ satisfy the following system of linear equations in recurrence form:

$$\sum_{\mu+\nu=r} \mathbf{A}^{(\mu)}\mathbf{u}^{(\nu)} = 0 \tag{5.2$_r$}$$

for $r = 0, 1, \ldots,$ where

$$\mathbf{A}^{(0)} = \mathbf{G}_{00}^{(0)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(0)}, \tag{5.3}$$

$$\mathbf{A}^{(1)} = 2k_j^{(0)}k_l^{(1)}\mathbf{G}_{jl}^{(0)} + \mathbf{C}^{(1)}, \tag{5.4}$$

$$\mathbf{C}^{(1)} = \mathbf{G}_{00}^{(1)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(1)}, \tag{5.5}$$

$$\mathbf{A}^{(r)} = \mathbf{B}^{(r)} + \mathbf{C}^{(r)} \quad \text{for } r \geq 2, \tag{5.6}$$

$$\mathbf{B}^{(2)} = -2k_j^{(0)}\mathbf{G}_{(jl)}^{(0)}\partial_l - S_{jl}^{(0)}\mathbf{G}_{jl}^{(0)} - k_l^{(0)}[\partial_j\mathbf{G}_{jl}^{(0)}], \tag{5.7}$$

$$\mathbf{C}^{(2)} = \mathbf{G}_{00}^{(2)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(2)} + k_j^{(1)}k_l^{(1)}\mathbf{G}_{jl}^{(0)} + 2k_j^{(0)}k_l^{(1)}\mathbf{G}_{(jl)}^{(1)} + \mathbf{H}_0^{(0)} - k_l^{(0)}\mathbf{H}_l^{(0)}, \tag{5.8}$$

$$\begin{aligned}\mathbf{B}^{(3)} = &-2[k_j^{(0)}\mathbf{G}_{(jl)}^{(1)} + k_j^{(1)}\mathbf{G}_{(jl)}^{(0)}]\partial_l - 2k_j^{(1)}\mathbf{G}_{(jl)}^{(0)}\partial_l - 2k_j^{(0)}\mathbf{G}_{(jl)}^{(1)}\partial_l \\ &- S_{jl}^{(0)}\mathbf{G}_{jl}^{(1)} - S_{jl}^{(1)}\mathbf{G}_{jl}^{(0)},\end{aligned} \tag{5.9}$$

$$\mathbf{C}^{(3)} = \mathbf{G}_{00}^{(3)} + k_j^{(0)}k_j^{(0)}\mathbf{G}_{jl}^{(3)} + 2k_j^{(0)}k_l^{(1)}\mathbf{G}_{jl}^{(2)} + 2k_j^{(0)}k_l^{(2)}\mathbf{G}_{jl}^{(1)} + k_j^{(1)}k_j^{(1)}\mathbf{G}_{jl}^{(1)} \tag{5.10}$$

with

$$\mathbf{k}^{(r)} = \nabla S^{(r)} \tag{5.11}$$

and $\mathbf{G}_{\alpha\beta}^{(r)}(\mathbf{x})$ defined by Eq. (4.1). While $\mathbf{A}^{(0)}$, $\mathbf{A}^{(1)}$, and $\mathbf{C}^{(r)}$ are matrix operators, $\mathbf{B}^{(r)}$ are differential operators.

Equation (5.2_0) has a nontrivial solution provided that

$$\det \mathbf{A}^{(0)} = 0. \tag{5.12}$$

Equation (5.12) has several solutions corresponding to different wave species. The wave species can be defined in terms of the solutions of the eigenvalue problem

$$\mathbf{B}\mathbf{p} = H\mathbf{G}_{00}^{(0)}\mathbf{p}, \qquad \mathbf{p}^{\mathrm{T}}\mathbf{G}_{00}^{(0)}\mathbf{p} = 1 \tag{5.13}$$

with a symmetric matrix

$$\mathbf{B}(\mathbf{x},\mathbf{k}) = -k_j k_l \mathbf{G}_{jl}^{(0)}(\mathbf{x}). \tag{5.14}$$

For simplicity, it is assumed that the eigenvalue H of (5.13) is simple. In this case, $H(\mathbf{x},\mathbf{k})$ are homogeneous functions of second order of the second argument.

For every solution $(H(\mathbf{x},\mathbf{k}), \mathbf{p}(\mathbf{x},\mathbf{k}))$ of Eq. (5.13),

$$\mathbf{u}^{(0)} = \alpha^{(0)}\mathbf{p}(\mathbf{x},\mathbf{k}^{(0)}), \tag{5.15}$$

$$H(\mathbf{x},\mathbf{k}^{(0)}) = 1 \tag{5.16}$$

provides a solution of Eq. (5.2_0).

The Hamilton-Jacobi equation (5.16) can be solved by the method of bicharacteristics:

$$\begin{aligned} \frac{d\mathbf{x}}{d\sigma} &= \frac{\partial H}{\partial \mathbf{k}}, \\ \frac{d\mathbf{k}}{d\sigma} &= -\frac{\partial H}{\partial \mathbf{x}}. \end{aligned} \tag{5.17}$$

In view of the homogeneity of the function $H(\mathbf{x},\cdot)$, Eqs. (5.11), (5.17), and (5.16) yield

$$\frac{dS^{(0)}}{d\sigma} = 2. \tag{5.18}$$

Equation (5.2_r) has a nontrivial solution provided the compatibility condition

$$\mathbf{p}^{\mathrm{T}} \sum_{\mu+\nu=r,\mu>0} \mathbf{A}^{(\mu)}\mathbf{u}^{(\nu)} = 0 \tag{5.19}$$

is satisfied.

Assuming that the compatibility condition (5.19) is satisfied, Eq. (5.2_r) has a solution depending on a parameter $\alpha^{(r)}$,

$$\mathbf{u}^{(r)} = \sum_{\nu=0}^{r-1} \beta^{(\nu)}\mathbf{v}^{(r,\nu)} + \alpha^{(r)}\mathbf{p}(\mathbf{x},\mathbf{k}^{(0)}), \tag{5.20}$$

where $\mathbf{v}^{(r,\nu)}$ is the unique solution of

$$\mathbf{A}^{(0)}\mathbf{w} = -\mathbf{A}^{(r)}\mathbf{v}^{(\nu)}, \qquad \mathbf{p}^{\mathrm{T}}\mathbf{w} = 0 \tag{5.21}$$

for a multi-index $\nu = \{\nu_1, \ldots, \nu_s\}$, $\mathbf{v}^{(\mu)}$ is the unique solution of

$$\mathbf{A}^{(0)}\mathbf{v} = -\mathbf{A}^{(\mu)}\mathbf{p}, \qquad \mathbf{p}^{\mathrm{T}}\mathbf{v} = 0, \tag{5.22}$$

and $\beta^{(\nu)}$ is a linear combination of lower-order scalar amplitudes.

A differential equation for the amplitude $\alpha^{(r)}$ is provided by the compatibility condition for Eq. (5.2_{r+2}). More generally, the rth compatibility condition is obtained by taking the scalar product of Eq. (5.2_r) with the polarization vector $\mathbf{p}$ and noting that

$\mathbf{p}^{\mathrm{T}}\mathbf{A}^{(0)} = 0$. The compatibility condition for $r = 0$ is the eikonal equation (5.16). The compatibility condition for $r = 1$ provides an equation for the function $S^{(1)}(\mathbf{x})$:

$$\frac{dS^{(1)}}{d\sigma} = \mathbf{p}^{\mathrm{T}}\mathbf{C}^{(1)}\mathbf{p} :\equiv h(\mathbf{x}, \mathbf{k}^{(0)}). \tag{5.23}$$

The function $S^{(1)}$ can be determined by integrating Eq. (5.23) along the rays of the eikonal $S^{(0)}$, with appropriate initial conditions at the source. For a source emitting a sharp signal such as $\delta(t)$, the initial condition for Eq. (5.23) is $S^{(1)} = 0$.

It is vital for the consistency of the construction that $S^{(1)}$ is nonnegative. This condition is satisfied if the inequality

$$h(\mathbf{x}, \mathbf{k}^{(0)}) \geq 0 \tag{5.24}$$

holds for arbitrary vectors $\mathbf{k}^{(0)}$. Inequality (5.24) follows directly from the assumption of nonnegative time-averaged dissipation rate (the generalized Graffi inequality, Eq. (31) in [24]).

In the following we shall also need the gradient $\mathbf{k}^{(1)}$ and the second-order derivatives $S^{(1)}_{jl}$ of $S^{(1)}$.

The vector $\mathbf{k}^{(1)}$ can be determined by integrating the equation

$$\frac{dk^{(1)}_\gamma}{d\sigma} = \frac{\partial h}{\partial x_j} Q_{j\gamma} + \frac{\partial h}{\partial k_j} K_{j\gamma}, \tag{5.25}$$

where

$$k^{(1)}_\gamma = k^{(1)}_j Q_{j\gamma} \tag{5.26}$$

and the matrices $Q_{j\gamma}$, $K_{j\gamma}$ are determined by the 1-jet prolongation of the bicharacteristic equations

$$\begin{aligned} \frac{dQ_{j\gamma}}{d\sigma} &= \frac{\partial^2 H}{\partial k_j \partial x_l} Q_{l\gamma} + \frac{\partial^2 H}{\partial k_j \partial k_l} K_{l\gamma}, \\ \frac{dK_{j\gamma}}{d\sigma} &= -\frac{\partial^2 H}{\partial x_j \partial x_l} Q_{l\gamma} - \frac{\partial^2 H}{\partial x_j \partial k_l} K_{l\gamma} \end{aligned} \tag{5.27}$$

with appropriate initial conditions at the source. The Greek indices denote partial derivatives with respect to the ray field parameters μ_γ, where $u_1 = \sigma$ and u_2, u_3 parametrize the initial manifold for the ray field.

For a point source, the variables u_2, u_3 denote the colatitude and longitude specifying the initial direction of the vector $\mathbf{k}^{(0)} = k\mathbf{n}(u_2, u_3)$, where

$$\mathbf{n}(u_2, u_3) = [\cos u_2 \cos u_3, \cos u_2 \sin u_3, \sin u_2]^{\mathrm{T}}$$

and the parameter k is given by the equation $H(\mathbf{x}^{(0)}, k\mathbf{n}) = 1$. The initial conditions for a point source $\mathbf{x}_0$ emitting a sharp signal ($S^{(1)}(\mathbf{x}_0) = 0$) are

$$\mathbf{Q}(0) = \left[\frac{\partial H}{\partial \mathbf{k}}(\mathbf{x}^{(0)}, k\mathbf{n}), \mathbf{0}, \mathbf{0}\right], \tag{5.28}$$

$$\mathbf{K}(0) = \frac{1}{\sin u_2}\left[-\frac{\partial H}{\partial \mathbf{x}}, k\frac{\partial \mathbf{n}}{\partial u_2}, k\frac{\partial \mathbf{n}}{\partial u_3}\right], \tag{5.29}$$

$$k^{(1)}_\gamma = h \quad \text{for } \gamma = 1; \qquad k^{(1)}_\gamma = 0 \quad \text{for } \gamma > 1. \tag{5.30}$$

The vector $\mathbf{k}^{(1)}$ is then given by the inverse of (5.26),

$$k_j^{(1)} = k_\gamma^{(1)} Q_{\gamma j}^{-1}, \tag{5.31}$$

except at the caustics $\det \mathbf{Q} = 0$.

Calculation of the second-order derivatives of $S^{(1)}$ is discussed in Appendix G.

Higher-order compatibility conditions turn out to be transport equations for the amplitudes. The differential operator of each transport equation is contained in the term $\mathbf{p}^{\mathrm{T}}\mathbf{B}^{(2)}(\zeta\mathbf{p})$:

$$\begin{aligned}
&\mathbf{p}^{\mathrm{T}}\mathbf{B}^{(2)}(\zeta\mathbf{p}) \\
&\quad = \frac{d\zeta}{d\sigma} - 2\mathbf{p}^{\mathrm{T}}\mathbf{G}_{(jl)}^{(0)} k_j^{(0)} \partial_l \mathbf{p}\zeta - \mathbf{p}^{\mathrm{T}} k_l^{(0)} [\partial_j \mathbf{G}_{jl}^{(0)}]\mathbf{p}\zeta - \mathbf{p}^{\mathrm{T}} S_{jl}^{(0)} \mathbf{G}_{jl}^{(0)} \mathbf{p}\zeta \\
&\quad \equiv \frac{d\zeta}{d\sigma} + \frac{1}{2}\frac{d\ln J}{d\sigma}\zeta,
\end{aligned} \tag{5.32}$$

where

$$\frac{d\zeta}{d\sigma} = \frac{\partial H}{\partial k_j^{(0)}} \frac{\partial \zeta}{\partial x_j}, \tag{5.33}$$

and J is the ray spreading $J = \det \mathbf{Q}$. The proof of Eq. (5.32) is literally identical with the complex ray case considered in [20]. Transport equations are obtained by substituting (5.32) as well as Eqs. (5.20) into Eq. (5.19). The first two transport equations are

$$\frac{d\alpha^{(0)}}{d\sigma} + \frac{1}{2}\frac{d\ln|J|}{d\sigma}\alpha^{(0)} + [\mathbf{p}^{\mathrm{T}}\mathbf{C}^{(2)}\mathbf{p} + \mathbf{p}^{\mathrm{T}}\mathbf{A}^{(1)}\mathbf{v}^{(1)}]\alpha^{(0)} = 0, \tag{5.34}$$

$$\begin{aligned}
&\frac{d\alpha^{(1)}}{d\sigma} + \frac{1}{2}\frac{d\ln|J|}{d\sigma}\alpha^{(1)} + \mathbf{p}^{\mathrm{T}}\mathbf{C}^{(2)}\mathbf{p}\alpha^{(1)} + \mathbf{p}^{\mathrm{T}}\mathbf{C}^{(2)}\mathbf{v}^{(1)}\alpha^{(0)} + \mathbf{p}^{\mathrm{T}}\mathbf{C}^{(3)}\mathbf{p}\alpha^{(0)} \\
&\quad + \mathbf{p}^{\mathrm{T}}\mathbf{A}^{(1)}[\alpha^{(1)}\mathbf{v}^{(1)} + \alpha^{(0)}(\mathbf{v}^{(2)} + \mathbf{v}^{(1,1)})] \\
&\quad + \mathbf{p}^{\mathrm{T}}\mathbf{B}^{(2)}(\alpha^{(0)}\mathbf{v}^{(1)}) + \mathbf{p}^{\mathrm{T}}\mathbf{B}^{(3)}(\alpha^{(0)}\mathbf{p}) = 0.
\end{aligned} \tag{5.35}$$

Equation (5.35) also involves partial derivatives of $\alpha^{(0)}$. These can be calculated by a 1-jet prolongation of Eq. (5.34). Calculation of $S_{jl}^{(1)}$ requires a 1-jet prolongation of Eqs. (5.17) and (5.23) (Appendix G).

6. An explicit solution for memory kernels with a $\tau_+^{-1/2}$ singularity. A closed-form fundamental solution of Eq. (2.6) with the kernel

$$K(\tau) = (2a/\sqrt{\pi})\tau_+^{-1/2} + a^2\theta(\tau) \tag{6.1}$$

is derived in Appendix D. It has the form of a truncated generalized wavefront expansion. We shall present here an asymptotic solution of Eq. (2.6) for a somewhat more general kernel K with the fading memory property:

$$K(\tau) = 2a\phi(\tau) + b\phi(\tau) * \phi(\tau) \equiv [2a\phi(\tau) + b\theta(\tau)]e^{-\beta\tau}, \tag{6.2}$$

where $\phi(\tau) = (1/\sqrt{\pi})\tau_+^{-1/2}e^{-\beta\tau}$. The Laplace image of $K(\tau)$ is

$$\widetilde{K}(s) = 2a(s+\beta)^{-1/2} + b(s+\beta)^{-1} \cong 2as^{-1/2} + bs^{-1} - a\beta s^{-3/2} + \cdots. \tag{6.3}$$

We consider a point source at the origin of the coordinate system. The functions $S^{(0)}$ and $S^{(1)}$ can be calculated from the equations $|\nabla S^{(0)}| = 1$, $dS^{(1)}/dr = a$ with zero initial conditions at the source $\mathbf{x} = 0$: $S^{(0)}(\mathbf{x}) = r$, $S^{(1)}(\mathbf{x}) = ar$. The ray spreading is now $J = (4\pi r)^{-1}$ and the amplitudes are given by the formula

$$\alpha^{(0)} = C_0 e^{-(b-a^2)r/2}/(4\pi r), \tag{6.4}$$

$$\alpha^{(1)} = \left[\tfrac{1}{2}C_0 a(b - a^2 + \beta)r + C_1\right] e^{-(b-a^2)r/2}/(4\pi r), \tag{6.5}$$

$$\begin{aligned}\alpha^{(2)} = \{&[C_0(b-a^2)^2/4 - C_0 b\beta + C_0 a^2(\beta + b - a^2) - C_1 a(\beta + b - a^2)]r \\ &- C_0 a^2(\beta + b - a^2)^2 r^2/4 + C_2\}e^{-(b-a^2)r/2}/(4\pi r),\end{aligned} \tag{6.6}$$

where $C_0, C_1, C_2, \ldots$ are arbitrary constants. The first three terms of the expansion give

$$\begin{aligned}u(t,\mathbf{x}) = a^{(0)}(\mathbf{x})f_0^{(2)}(t-r,ar) + \alpha^{(1)}(\mathbf{x})f_1^{(2)}(t-r,ar) \\ + \alpha^{(2)}(\mathbf{x})f_2^{(2)}(t-r,ar) + \text{less pronounced signals}\end{aligned} \tag{6.7}$$

for $t > r$. The peak of $f_0^{(2)}(t-r,ar)$ lies at $t = r(1 + a^2 r/6)$.

The constants $C_0, C_1, C_2, \ldots$ can be determined from initial or boundary-value data using the results of Appendix F. In particular, the fundamental solution corresponds to $C_0 = 1$, $C_1 = 2a, C_2 = a^2$; $C_k = 0$ for $k > 2$. For $b = a^2$, $\beta = 0$, the asymptotic fundamental solution is exact (Appendix D).

The rate of decrease of the signal at the wavefront depends on the coefficient a while signal attenuation along the ray path is additionally controlled by the coefficient b (the Darcy coefficient in a porous medium). The exponential factor $\exp[-(b-a^2)r/2]$ represents a correction to the attenuation already implicit in the basis functions.

The coefficient β responsible for memory fading plays a secondary role at the wavefront. It, however, controls the gradual decay of the signal at the source [47].

7. Basis time functions for expansions in inverse powers of $s^{1/3}$. For memory kernels with $\tau^{-1/3}$ and $\tau^{-2/3}$ singularities,

$$\widetilde{\mathbf{G}}(s,\mathbf{x}) \sim \sum_{\mu=0}^{\infty} s^{-\mu/3}\mathbf{G}^{(\mu)}(\mathbf{x}) \tag{7.1}$$

or

$$\widetilde{K}(s,\mathbf{x}) \sim \sum_{\mu=0}^{\infty} s^{-\mu/3}K^{(\mu)}(\mathbf{x}), \tag{7.2}$$

the solution is constructed in the form of the asymptotic expansion (3.2) with

$$f_\mu^{(3)}(t,\lambda_1,\lambda_2) = \frac{1}{2\pi i}\int_{\mathcal{B}} s^{-\mu/3}e^{st}e^{-\lambda_1 s^{2/3} - \lambda_2 s^{1/3}}\,ds. \tag{7.3}$$

The functions $f_0^{(3)}$, $f_1^{(3)}$, $f_2^{(3)}$ can be calculated in an explicit form. Substituting $\mu = 2$, $\tau = s^{1/3} - \lambda_1/(3t)$ in Eq. (7.3), we have

$$\begin{aligned} f_2^{(3)}(t,\lambda_1,\lambda_2) &= \frac{1}{2\pi i}\int_{\mathcal{B}} s^{-2/3} e^{st} e^{-\lambda_1 s^{2/3} - \lambda_2 s^{1/3}}\, ds \\ &= \frac{3^{2/3}}{2\pi t^{1/3}} e^{-\beta} \int_{\Gamma} e^{i(\tau^3/3+\gamma\tau)}\, d\tau \end{aligned} \tag{7.4}$$

with the contour Γ shown in Fig. 7.1:

$$\gamma = (3t)^{-1/3}\left(\lambda_2 + \frac{1}{3}\frac{\lambda_1^2}{t}\right), \qquad \beta = \frac{\lambda_1}{3t}\left[\lambda_2 + \frac{2\lambda_1^2}{9t}\right].$$

The contour Γ is equivalent to the real axis, whence

$$f_2^{(3)}(t,\lambda_1,\lambda_2) = \frac{3^{2/3}}{t^{1/3}}\left[\mathrm{Ai}\left((3t)^{-1/3}\left(\lambda_2 + \frac{\lambda_1^2}{3t}\right)\right)\exp\left(-\frac{\lambda_1}{3t}\left(\lambda_2 + \frac{2\lambda_1^2}{9t}\right)\right)\right]. \tag{7.5}$$

Furthermore,

$$f_0^{(3)}(t,\lambda_1,\lambda_2) = -\frac{df_2^{(3)}}{d\lambda_2}, \tag{7.6}$$

$$f_1^{(3)}(t,\lambda_1,\lambda_2) = -\frac{df_2^{(3)}}{d\lambda_1}. \tag{7.7}$$

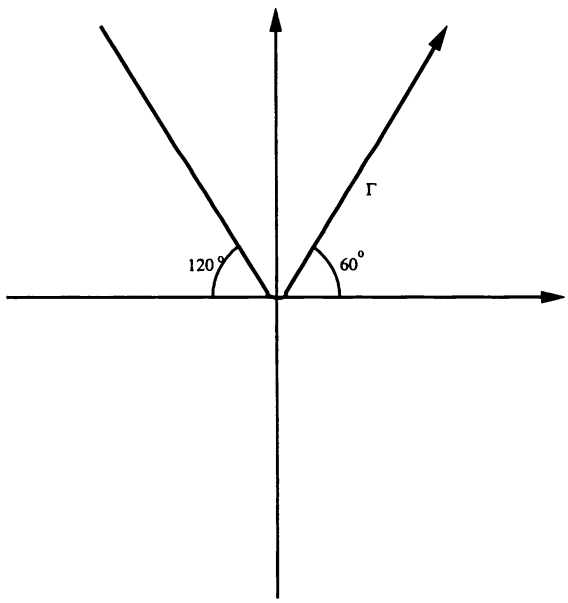

FIG. 7.1. Contour Γ in the complex plane

The function $f_3^{(3)}(t,\lambda_1,\lambda_2)$ can be calculated asymptotically for $\lambda_1^2/(3t)^{4/3} + \lambda_2/(3t)^{1/3} > \varepsilon > 0$ by applying the same variable transformations followed by the steepest descent

evaluation:

$$f_3^{(3)}(t,\lambda_1,\lambda_2) = 3\,\mathrm{Ais}\left((3t)^{-1/3}\left(\lambda_2+\frac{\lambda_1^2}{3t}\right)\right)\exp\left(-\frac{\lambda_1}{3t}\left(\lambda_2+\frac{2\lambda_1^2}{9t}\right)\right)$$
$$\times\left[\sqrt{(3t)^{-1/3}}\left(\lambda_2+\frac{\lambda_1^2}{3t}\right)+\frac{\lambda_1}{(3t)^{2/3}}\right]^{-1}, \quad (7.8)$$

where Ais denotes the asymptotic expression for the Airy function for large positive arguments [1]. The basis functions for $r > 3$ can be calculated from the identity

$$\mu f_{\mu+3}^{(3)} = 3t f_\mu^{(3)} - 2\lambda_2 f_{\mu+1}^{(3)} - \lambda_1 f_{\mu+2}^{(3)}, \quad (7.9)$$

which can be derived by integrating Eq. (7.3) by parts.

The functions $f_\mu^{(3)}(t, 0.1, 0.1)$ are shown in Figs. 7.2 and 7.3. The error associated with the asymptotic evaluation of the function $f_4^{(3)}$ is visible for $t > 0.05$. Functions $f_0^{(3)}$, $f_1^{(3)}$ and $f_2^{(3)}$ have a single maximum, while the functions $f_\mu^{(3)}$ for $\mu > 2$ are monotonically increasing.

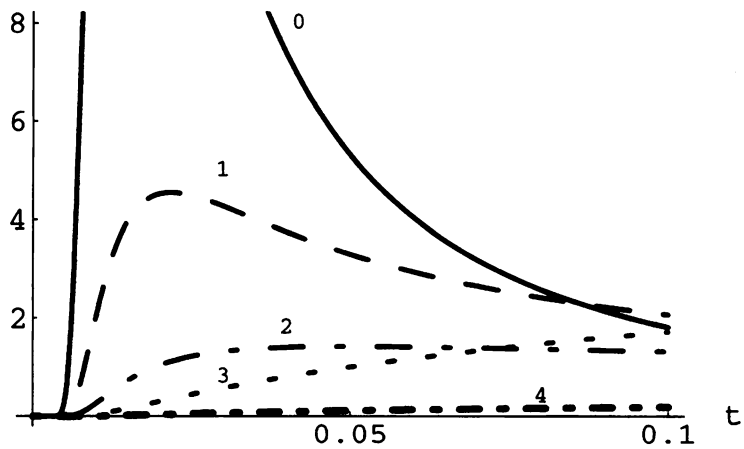

FIG. 7.2. The functions $f_\mu^{(3)}(t, 0.1, 0.1)$ for $0 < t < 0.1$, $\mu = 0, 1, 2, 3, 4$

The functions $f_r^{(3)}(t, \lambda_1, \lambda_2)$ are nonnegative for $\lambda_2 \geq 0$. For sufficiently large negative λ_2, the signal exhibits initial oscillations (see Fig. 7.4).

In Appendix C it is shown that

$$\begin{aligned} f_0^{(3)}(t,\lambda_1,\lambda_2) &\to \delta(t), \\ f_1^{(3)}(t,\lambda_1,\lambda_2) &\to t_+^{-1/3}/\Gamma(2/3), \\ f_2^{(3)}(t,\lambda_1,\lambda_2) &\to t_+^{-2/3}/\Gamma(1/3), \\ f_3^{(3)}(t,\lambda_1,\lambda_2) &\to \theta(t), \\ f_4^{(3)}(t,\lambda_1,\lambda_2) &\to t_+^{1/3}/\Gamma(4/3), \end{aligned} \quad (7.10)$$

etc., for $\lambda_1, \lambda_2 \to 0$, $\lambda_1 > 0$, in the sense of distributions.

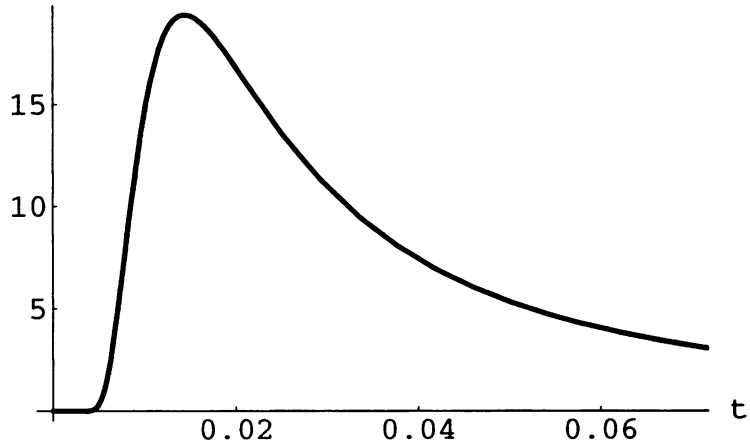

FIG. 7.3. The basis function $f_0^{(3)}(t, \lambda, \mu)$ ($\lambda = \mu = 0.1$)

Equations (7.10) establish a relation between generalized wavefront expansions and ordinary wavefront expansions. The limits of basis functions $t \to 0$ obtained in Appendix F can be used to verify the initial conditions.

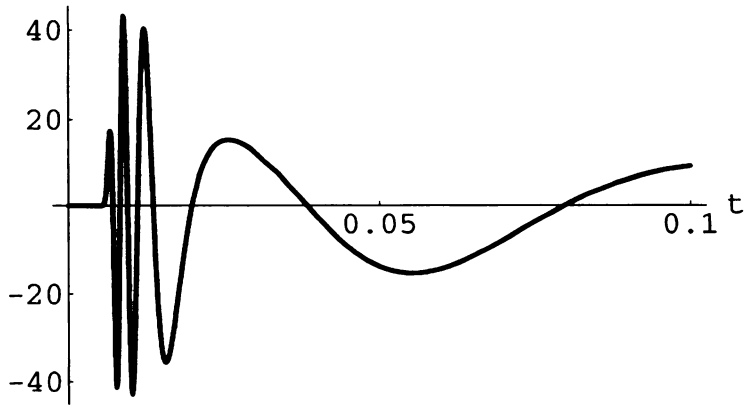

FIG. 7.4. The basis function $f_0^{(3)}(t, \lambda, \mu)$ for $\lambda = 0.1, \mu = -5.0$

The asymptotic behavior of $f_0^{(3)}$ for $t \to 0$ can be obtained from the asymptotic properties of the Airy function [1]: if $\lambda_2 > 0$, then $f_0^{(3)}(t, \lambda_1, \lambda_2) = O[e^{-c_1/t^2}]$ for some constant $c_1 > 0$, while for $\lambda_2 = 0$ the estimate is $f_0^{(3)}(t, \lambda_1, \lambda_2) = O[e^{-c_2/t}]$ for some constant $c_2 > 0$.

8. Limitations of asymptotic expansions in inverse powers of $s^{1/3}$. We now consider Eqs. (2.1) with the memory kernels

$$\begin{aligned}\widetilde{\mathbf{G}}_{\alpha\beta}(s,\mathbf{x}) &= \sum_{\mu=0}^{\infty} s^{-\mu/3}\mathbf{G}_{\alpha\beta}^{(\mu)}(\mathbf{x}),\\ \widetilde{\mathbf{H}}_{\alpha}(s,\mathbf{x}) &= \sum_{\mu=0}^{\infty} s^{-\mu/3}\mathbf{H}_{\alpha}^{(\mu)}(\mathbf{x}).\end{aligned} \tag{8.1}$$

It will be shown that the resulting transport equations are not in recurrence form except for a scalar equation. The asymptotic expansion in the frequency domain is assumed in the form

$$\tilde{\mathbf{u}} = \exp[-(sS^{(0)} + s^{2/3}S^{(1)} + s^{1/3}S^{(2)})]\mathbf{w}, \tag{8.2}$$

$$\mathbf{w} = \sum_{\mu=0}^{\infty} s^{-\mu/3}\mathbf{u}^{(\mu)}. \tag{8.3}$$

We define

$$\mathbf{A}^{(0)} = \mathbf{G}_{00}^{(0)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(0)}, \tag{8.4}$$

$$\mathbf{A}^{(1)} = \mathbf{G}_{00}^{(1)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(1)} + 2k_j^{(0)}k_l^{(1)}\mathbf{G}_{(jl)}^{(0)}, \tag{8.5}$$

$$\mathbf{A}^{(2)} = \mathbf{G}_{00}^{(2)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(2)} + 2k_j^{(0)}k_l^{(1)}\mathbf{G}_{(jl)}^{(1)} + k_j^{(1)}k_l^{(1)}\mathbf{G}_{(jl)}^{(0)} + 2k_j^{(0)}k_l^{(2)}\mathbf{G}_{(jl)}^{(0)}, \tag{8.6}$$

$$\mathbf{A}^{(3)} = \mathbf{B}^{(3)} + \mathbf{C}^{(3)}, \tag{8.7}$$

$$\begin{aligned}\mathbf{C}^{(3)} = {}& \mathbf{G}_{00}^{(3)} + k_j^{(0)}k_l^{(0)}\mathbf{G}_{jl}^{(3)} + 2k_j^{(0)}k_l^{(1)}\mathbf{G}_{(jl)}^{(2)} + k_j^{(1)}k_l^{(1)}\mathbf{G}_{(jl)}^{(1)} + 2k_j^{(0)}k_l^{(2)}\mathbf{G}_{(jl)}^{(1)}\\ &+ 2k_j^{(2)}k_l^{(1)}\mathbf{G}_{(jl)}^{(0)} + \mathbf{H}_0^{(0)} - k_l^{(0)}\mathbf{H}_l^{(0)},\end{aligned} \tag{8.8}$$

$$\mathbf{B}^{(3)} = -2k_j^{(0)}\mathbf{G}_{(jl)}^{(0)}\partial_l - S_{jl}^{(0)}\mathbf{G}_{jl}^{(0)} - [\partial_j\mathbf{G}_{jl}^{(0)}]. \tag{8.9}$$

Following the method of Sec. 5, one obtains Eq. (5.12). Equation (5.13) defines the polarization vector $\mathbf{p}$ and the associated Hamiltonian $H(\mathbf{x},\mathbf{k})$. The function $S^{(1)}$ satisfies Eq. (5.23). This ensures existence of the vector amplitude $\mathbf{a}^{(1)} = \alpha^{(1)}\mathbf{p} + \alpha^{(0)}\mathbf{v}^{(1)}$. The next compatibility condition provides the differential equation for the function $S^{(2)}$:

$$\frac{dS^{(2)}}{d\sigma} = \mathbf{p}^{\mathrm{T}}\mathbf{A}^{(1)}\mathbf{v}^{(1)} + \mathbf{p}^{\mathrm{T}}\mathbf{C}^{(2)}\mathbf{p} \tag{8.10}$$

and ensures the existence of a vector amplitude $\mathbf{a}^{(2)} = \alpha^{(2)}\mathbf{p} + \alpha^{(0)}\mathbf{v}^{(2)} + \alpha^{(1)}\mathbf{v}^{(1)} + \alpha^{(0)}\mathbf{v}^{(1,1)}$, with $\mathbf{v}^{(1)}$, $\mathbf{v}^{(1,1)}$ defined by Eqs. (5.22) and (5.21). The next compatibility conditions yields an equation

$$\begin{aligned}\frac{d\alpha^{(0)}}{d\sigma} + \frac{1}{2}\frac{d\ln|J|}{d\sigma}\alpha^{(0)} + \mathbf{p}^{\mathrm{T}}\mathbf{C}^{(3)}\mathbf{p}\alpha^{(0)} + \mathbf{p}^{\mathrm{T}}\mathbf{A}^{(2)}\mathbf{v}^{(1)}\alpha^{(0)}\\ + \mathbf{p}^{\mathrm{T}}\mathbf{A}^{(1)}[\alpha^{(0)}\mathbf{v}^{(2)} + \alpha^{(1)}\mathbf{v}^{(1)} + \alpha^{(0)}\mathbf{v}^{(1,1)}] = 0.\end{aligned} \tag{8.11}$$

Equation (8.11) involves $\alpha^{(1)}$. Consequently, the transport equations are not in recurrence form unless $\mathbf{p}^{\mathrm{T}}\mathbf{A}^{(1)}\mathbf{v}^{(1)} = 0$. The last condition is certainly satisfied in the scalar case where $\mathbf{p} = 1$, the matrix $\mathbf{A}^{(1)}$ reduces to a scalar $A^{(1)}$ and the second compatibility

condition reduces to $A^{(1)} = 0$. In the next section, a generalized wavefront expansion for the $t^{-2/3}$ singularity is constructed for a special scalar model.

9. $s^{1/3}$-asymptotic expansions for a scalar equation. We now apply generalized wavefront asymptotics to the solution of the initial value problem,

$$[1 + K(t, \mathbf{x}))*]u_{,tt} - \nabla \cdot [n(\mathbf{x})^2 \nabla u] = 0, \tag{9.1}$$

$$u(0, \mathbf{x}) = 0, \tag{9.2}$$

$$u_{,t}(0, \mathbf{x}) = \delta(\mathbf{x}), \tag{9.3}$$

with a kernel $K(t, \mathbf{x})$ whose Laplace transform can be asymptotically expressed in the following form:

$$\widetilde{K}(s, \mathbf{x}) = A(\mathbf{x})s^{-1/3} + B(\mathbf{x})s^{-2/3} + C(\mathbf{x})s^{-1} + D(\mathbf{x})s^{-4/3} + \cdots . \tag{9.4}$$

It is additionally assumed that the higher-order time derivatives $\partial_t^n u(t, \mathbf{x})$ are bounded functions of t for $t \to 0$ in a sense made precise later.

The function $K(t, \mathbf{x})$ has the form

$$K(t, \mathbf{x}) = At_+^{-2/3}/\Gamma(1/3) + Bt_+^{-1/3}/\Gamma(2/3) + C\theta(t) + Dt_+^{1/3}/\Gamma(4/3) + t_+^{2/3}K_1(t, \mathbf{x}). \tag{9.5}$$

For $A = 2a^2$, $B = 2b + a^2$, $C = 2ab$, $D = b^2$, $a > 0$, $b \geq 0$, $n(\mathbf{x}) = 1$ and $K_1 = 0$, the equation has closed-form solutions (Appendix D).

Eikonal and transport equations are obtained by substituting the expansion

$$\tilde{u} = [u_0 + s^{-1/3}u_1 + s^{-2/3}u_2 + s^{-1}u_3 + s^{-4/3}u_4 + \cdots]e^{-sS^{(0)} - s^{2/3}S^{(1)} - s^{1/3}S^{(2)}} \tag{9.6}$$

into the Laplace-transformed equation

$$[1 + \widetilde{K}(s, \mathbf{x})][s^2\tilde{u} - \delta(\mathbf{x})] - \nabla \cdot [n(\mathbf{x})^2 \nabla \tilde{u}] = 0 \tag{9.7}$$

and assuming that $u_0 \neq 0$. The eikonal equation assumes the form

$$1 - n^2(\nabla S^{(0)})^2 = 0. \tag{9.8}$$

Taking Eq. (9.8) into account, the next two equations reduce to the equations for the rates of $S^{(1)}$ and $S^{(2)}$ along the rays

$$\frac{dS^{(1)}}{d\sigma} = A/2, \tag{9.9}$$

$$\frac{dS^{(2)}}{d\sigma} = [B - n^2(\nabla S^{(1)})^2]/2, \tag{9.10}$$

where

$$\frac{d}{d\sigma} = n^2 \mathbf{k}^{(0)} \cdot \nabla \tag{9.11}$$

denotes the derivative along the rays associated with Eq. (9.8). The remaining equations yield the transport equations for $u_0, u_1, u_2, \dots$:

$$\frac{du_0}{d\sigma} + \frac{1}{2}[n^2\nabla^2 S^{(0)} + \mathbf{k}^{(0)} \cdot \nabla n^2]u_0 + \left[\frac{C}{2} - n^2 \nabla S^{(1)} \cdot \nabla S^{(2)}\right] u_0 = 0, \tag{9.12}$$

$$\begin{aligned}\frac{du_1}{d\sigma} &+ \frac{1}{2}[n^2\nabla^2 S^{(0)} + \mathbf{k}^{(0)} \cdot \nabla n^2]u_1 + \left[\frac{C}{2} - n^2 \nabla S^{(1)} \cdot \nabla S^{(2)}\right] u_1 \\ &+ \frac{1}{2}[n^2\nabla^2 S^{(1)} + D - n^2(\nabla S^{(2)})^2 + \nabla n^2 \cdot \nabla S^{(1)}]u_0 + n^2 \nabla S^{(1)} \cdot \nabla u_0 = 0,\end{aligned} \tag{9.13}$$

$$\begin{aligned}\frac{du_2}{d\sigma} &+ \frac{1}{2}[n^2\nabla^2 S^{(0)} + \mathbf{k}^{(0)} \cdot \nabla n^2]u_2 + \left[\frac{C}{2} - n^2 \nabla S^{(1)} \cdot \nabla S^{(2)}\right] u_2 \\ &+ \frac{1}{2}[E + n^2\nabla^2 S^{(2)} + \nabla n^2 \cdot \nabla S^{(2)}]u_0 + n^2 \nabla S^{(2)} \cdot \nabla u_0 \\ &+ \frac{1}{2}[D - n^2(\nabla S^{(2)})^2 + n^2\nabla^2 S^{(1)} + \nabla n^2 \cdot \nabla S^{(1)}]u_1 + n^2 \nabla S^{(1)} \cdot \nabla u_1 = 0.\end{aligned} \tag{9.14}$$

Applying the Smirnoff lemma $dJ/dt = J\,\mathrm{tr}[(d\mathbf{Q}/dt)\mathbf{Q}^{-1}]$, the first equation can be explicitly integrated:

$$u_0 = U_0 |J|^{-1/2} \exp\left[-\int_0^\sigma (C/2 - n^2 \nabla S^{(1)} \cdot \nabla S^{(2)})\, d\sigma'\right]. \tag{9.15}$$

The definition of the basis functions imposes the constraint that the function $S^{(1)}$ is nonnegative. In view of Eq. (9.9), the inequality $A > 0$ ensures this property at every point of the ray if it is satisfied at its origin. The inequality is derived from the principle of nonnegative time-averaged dissipation in Appendix H.

Equation (9.13) involves the first- and second-order gradients of $S^{(1)}$ and $S^{(2)}$. The gradients of $S^{(1)}$ and $S^{(2)}$ can be calculated by integrating a 2-jet prolongation of Eqs. (9.9) and (9.10). This can be done along the lines of Appendix G. The gradient of u_0 in (9.13) can be calculated by integrating a 1-jet prolongation of Eq. (9.12).

Summarizing, the solution of the initial value problem has been obtained in the form

$$u(t, \mathbf{x}) = \sum_{\mu=0}^{\infty} u_\mu(\mathbf{x}) f_\mu^{(3)}(t - S^{(0)}(\mathbf{x}), S^{(1)}(\mathbf{x}), S^{(2)}(\mathbf{x})). \tag{9.16}$$

For a point source in a homogeneous medium (constant coefficients $A, B, C, \dots$),

$$S^{(0)} = r/n, \tag{9.17}$$

$$S^{(1)} = Ar/(2n), \tag{9.18}$$

$$S^{(2)} = (B - A^2/4)r/(2n). \tag{9.19}$$

Since $d/d\sigma = nd/dr$ and $\nabla^2 r = 2/r$, the first three transport equations assume the following form:

$$\frac{du_0}{dr} + \frac{u_0}{r} + \frac{1}{2n}\left[C - \frac{A}{2}\left(B - \frac{A^2}{4}\right)\right] u_0 = 0, \tag{9.20}$$

$$\frac{du_1}{dr} + \frac{u_1}{r} + \frac{1}{2n}\left[C - \frac{A}{2}\left(B - \frac{A^2}{4}\right)\right] u_1 + \left[\frac{An}{r} - \frac{1}{4}A^2\right] u_0 + An\frac{\partial u_0}{\partial r} = 0, \tag{9.21}$$

$$\frac{du_2}{dr} + \frac{u_2}{r} + \frac{1}{2n}\left[C - \frac{A}{2}\left(B - \frac{A^2}{4}\right)\right] u_2 \tag{9.22}$$
$$+ \frac{1}{2n}\left[E + \frac{n}{r}\left(B - \frac{A^2}{4}\right)\right] u_0 + \frac{1}{2}\left(B - \frac{A^2}{4}\right)\frac{\partial u_0}{\partial r}$$
$$+ \frac{1}{2n}\left[D - \frac{1}{4}\left(B - \frac{A^2}{4}\right)^2 + \frac{n}{r}A\right] u_1 + \frac{1}{2}A\frac{\partial u_1}{\partial r} = 0.$$

The last two terms in Eq. (9.21) add up to $-\{A^2/2 + A[C - A(B - A^2/4)/2]\}u_0/2$. Integration of the transport equations results in the closed-form expression

$$u_\mu = \frac{c_\mu}{4\pi r} e^{-[C - A(B - A^2/4)/2]r/(2n)}, \quad \mu = -0, 1, 2, \tag{9.23}$$

with

$$c_0 = \text{const}, \tag{9.24}$$
$$c_1 = c_1^0 + c_1^1 r,$$
$$c_1^1 = \frac{1}{2n}\left\{\frac{1}{4}\left(B - \frac{A^2}{4}\right)^2 - D + \frac{A}{2}\left[C - \frac{A}{2}\left(B - \frac{A^2}{4}\right)\right]\right\} c_0, \tag{9.25}$$
$$c_2 = c_2^0 + c_2^1 r,$$
$$c_2^1 = \frac{1}{2n}\left\{\frac{1}{4n}\left(B - \frac{A^2}{4}\right)\left[C - \frac{A}{2}\left(B - \frac{A^2}{4}\right)\right] - E\right\} c_0 \tag{9.26}$$
$$+ \frac{1}{2n}\left\{\frac{A}{4n}\left[C - \frac{A}{2}\left(B - \frac{A^2}{4}\right)\right] + \frac{1}{4}\left(B - \frac{A^2}{4}\right)^2 - D\right\}\left(c_1^0 + \frac{1}{2}c_1^1 r\right) + \frac{Ac_1^1}{2},$$

where $c_0^0, c_1^0, c_2^0, \ldots$ are arbitrary constants to be determined from initial or boundary-value data using the identities of Appendix F.

Expression (9.16) must also satisfy the initial conditions (9.2)–(9.3). The infinite sequence of free parameters c_μ^0 is determined by Eqs. (9.2)–(9.3) and the condition that the time derivatives of $u(t, \mathbf{x})$ of order ≥ 2 should converge to $t \to 0$ to expressions $w(t)\delta(\mathbf{x})$, where $w(t)$ is bounded for $t \to 0$. Taking into account the results of Appendix F, the last condition provides a sequence of equations that allow expressing $c_\mu^0, \mu > 0$, in terms of c_0:

$$c_1^0 = 2a\, c_0, \tag{9.27}$$
$$c_2^\circ = (a^2 + 2b)c_0, \tag{9.28}$$
$$c_3^0 = 2abc_0, \tag{9.29}$$
$$c_4^0 = b^2 c_0, \tag{9.30}$$

with $a = A/(2n)$, $b = (B - A^2/4)/2$.

In the special case of a two-parameter kernel (D.7), Eq. (2.6) has an exact fundamental solution (D.9) in the form of a generalized wavefront expansion (Appendix D). On the

other hand, for the same case we have

$$C - \frac{A}{2}\left(B - \frac{A^2}{4}\right) = 0, \tag{9.31}$$

$$D = \frac{1}{4}\left(B - \frac{A^2}{4}\right)^2, \tag{9.32}$$

$$c_\mu^1 = 0, \tag{9.33}$$

and, consequently, $c_\mu = c_\mu^0$ for $\mu = 0, 1, 2, \ldots$, which yields (D.9).

The attenuation of the exact fundamental solution (D.9) is totally included in the basis functions. For $C - A(B - A^2/4)/2 \neq 0$, an additional frequency-independent attenuation factor appears in the asymptotic solution.

The point source problem can also be solved by applying the identities proved in Appendix E.

Comparison with linear creep theory [39] shows that the kernel (9.5) corresponds to the creep function

$$\mathcal{J}(t) = \frac{1}{\rho n^2}[\theta(t) + At_+^{1/3}/\Gamma(4/3) + Bt_+^{2/3}/\Gamma(5/3) + Abt_+ + b^2 t_+^{4/3}/\Gamma(7/3) + \cdots], \tag{9.34}$$

which has a dominant singularity fairly closely matching experimental data for some materials.

10. Conclusions. Equations (2.1) with singular memory kernels do not support discontinuities at wavefronts. The signal vanishes at the wavefront with all its derivatives and its peak arrives with some delay after the wavefront. As shown in [23], this behavior radically differs from equations with other models of attenuation.

The singular part of the memory kernel has a significant effect on the pulse shape and propagation, in contrast to the regular part, which has a minor influence on the tail and total attenuation. In fact, the low frequency effects of the memory can be represented by a single additional parameter, as shown for some poroelastic models [2, 56].

Wavefront behavior of waves in singular hereditary models is expressed in terms of basis functions. Basis functions can be derived in explicit form for the singularities $t^{-1/2}$, $t^{-1/3}$, $t^{-2/3}$. For other singularities they can be expressed in terms of functions of two or more arguments (such as more exotic versions of the Pearcey function for the singularities $t^{-r/4}$). Asymptotic expressions for a class of basis functions for $t \to 0+$, $t \to \infty$ and the transition between elastic and fractional viscoelastic creep compliance functions can be found in [29].

Transport equations in recurrence form have been obtained for arbitrary systems of equations in the $\tau^{-1/2}$ case and for scalar equations in the $\tau^{-1/3}$, $\tau^{-2/3}$ case. In the other cases, transport equations are no longer in recurrence form.

Generalized wavefront expansions fail at caustics due to vanishing ray spreading and to divergence of gradients of the functions $S^{(r)}(\mathbf{x}), r > 0$. Uniformly asymptotic generalized wavefront expansions for a simple caustic are derived in a separate paper.

For two important classes of scalar equations, exact fundamental solutions are obtained in the form of linear combinations of basis functions.

A singular hereditary model cannot be approximated by a superposition of a finite number of elementary relaxation mechanisms. Indeed, the relaxation spectrum [8] of the function $\tau_+^{-\gamma}/\Gamma(1-\gamma)$, $0<\gamma<1$, has a singularity at 0:

$$\int_0^\infty \frac{\vartheta^{-1-\gamma}}{\pi\csc(\pi\gamma)} e^{-\tau/\vartheta}\, d\vartheta = \frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} \tag{10.1}$$

for $\tau>0$, as can be deduced from Eq. 3.3.81.4 in [18]. As a result, standard numerical FD schemes for viscoelasticity based on relaxation of "hidden variables" are not applicable to singular hereditary models. On the other hand, discarding the regular part of the memory and applying the methods of fractional calculus and the Grünwald-Letnikow derivatives [34, 50] leads to rigorous FD schemes.

Notation. $G_{(jl)} = (G_{jl}+G_{lj})/2$.

Acknowledgments. The research of M. S. was carried out in the framework of the Joule III project "Reservoir-Oriented Delineation Technology" funded by the CEE under the contract JOF3-CT95-0019. Financial support from the consortium project "Seismic Reservoir Characterization" for the first author is acknowledged.

The authors are indebted to an anonymous referee for critical comments and to Andrew Norris for attracting their attention to a paper by Pipkin.

Appendix A. Nonnegativity of the functions $f_\alpha(t,\lambda')$ for $\lambda_\nu \ge 0; \nu=0,\dots,N$. We prove by recursion that

$$(-1)^k \frac{d^{k+1}(s\tilde\phi(s))}{ds^{k+1}} \ge 0 \quad \text{for } k=0,1,\dots. \tag{A.1}$$

For $k=0$ we have

$$(s\tilde\phi(s))' = -\int_0^\infty t\phi'(t)e^{-st}\,dt \ge 0. \tag{A.2}$$

Indeed, integrating by parts,

$$s\tilde\phi(s) = \lim_{\varepsilon\to 0} s\int_\varepsilon^\infty \phi(t)e^{-st}\,dt = \lim_{\varepsilon\to 0}\left[\phi(\varepsilon)e^{-s\varepsilon} + \int_\varepsilon^\infty \phi'(t)e^{-st}\,dt\right].$$

Differentiating with respect to s and noting that $\varepsilon\phi(\varepsilon)=o[1]$ we get (A.2). Equation (A.1) follows trivially.

If we prove that for $\lambda_\nu \ge 0$, $\nu=1,\dots,N$,

$$I_k := (-1)^k \frac{d^k e^{-s\tilde\phi(s)}}{ds^k} \ge 0 \quad \text{for } k=0,1,\dots, \tag{A.3}$$

then the inequalities $f_1(t,\lambda')\ge 0$ and $f_0=\partial f_1/\partial t \ge 0$ follow from Bernstein's theorem, [55]. Hence, for arbitrary $\alpha>0$,

$$f_\alpha(t,\lambda') \equiv \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f_0(t,\lambda') \ge 0.$$

For $k = 0$, Eq. (A.3) is obvious. The kth derivative of $e^{-s\tilde{\phi}(s)}$ has the form $e^{-s\tilde{\phi}(s)}X_k$, where X_k is the sum of products of derivatives of $-s\tilde{\phi}(s)$ whose orders add up to k. Equation (A.1) implies Eq. (A.3).

Appendix B. $f_0^{(2)}(t,\lambda) \to \delta(t)$ **for** $\lambda \to 0+$**,** We shall prove that for $\lambda \to 0+$,

$$f_2^{(2)}(t,\lambda) \to \theta(t) \tag{B.1}$$

in the distribution sense, whence $f_0^{(2)}(t,\lambda) \to \delta(t)$ follows by differentiation.

In order to prove Eq. (B.1), it is enough to show that

$$\int_0^\infty \left[1 - \operatorname{erfc}\left(\frac{\lambda}{2\sqrt{t}}\right)\right] \varphi(t)\,dt \equiv \int_0^\infty \operatorname{erf}\left(\frac{\lambda}{2\sqrt{t}}\right) \varphi(t)\,dt \to 0 \tag{B.2}$$

for an arbitrary smooth test function $\varphi \in L^1(\mathbb{R})$.

The definition of erf implies the following upper bound:

$$\operatorname{erf}\left(\frac{\lambda}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda/(2\sqrt{t})} e^{-\tau^2}\,d\tau \le \frac{\lambda}{\sqrt{\pi t}}. \tag{B.3}$$

The integral can be split into two parts. The first part is bounded by

$$\left|\int_1^\infty \operatorname{erf}\left(\frac{\lambda}{2\sqrt{t}}\right) \varphi(t)\,dt\right| \le \frac{\lambda}{\sqrt{\pi}} \int_1^\infty \frac{|\varphi(t)|}{\sqrt{t}}\,dt \to 0 \tag{B.4}$$

for $\lambda \to 0$, while the second part is

$$\int_0^1 \operatorname{erf}\left(\frac{\lambda}{2\sqrt{t}}\right) \varphi(t)\,dt = \varphi(0) \int_0^1 \operatorname{erf}\left(\frac{\lambda}{2\sqrt{t}}\right) dt + \int_0^1 [\varphi(t) - \varphi(0)] \operatorname{erf}\left(\frac{\lambda}{2\sqrt{t}}\right) dt. \tag{B.5}$$

In view of (B.3), the first term in Eq. (B.5) is bounded by $2\lambda/\sqrt{\pi}$, while the integrand of the second term is nonsingular at $t = 0$ and $O[\lambda]$. Consequently, both terms tend to 0.

Appendix C. $f_0^{(3)}(t,\lambda_1,\lambda_2) \to \delta(t)$ **for** $\lambda_1, \lambda_2 \to 0, \lambda_1 > 0$**.** The proof will be based on a different idea than in the previous appendix. Let $\varphi(t)$ be a smooth test function in the Schwartz space. We shall show that

$$\int_0^\infty f_0^{(3)}(t-\tau,\lambda_1,\lambda_2)\varphi(\tau)\,d\tau \to \varphi(t) \tag{C.1}$$

for $\lambda_1, \lambda_2 \to 0$, $\lambda_1 > 0$. In the Laplace domain this is equivalent to the statement

$$\frac{1}{2\pi i} \int_{\mathcal{B}} \tilde{\varphi}(s) e^{st} e^{-\lambda_1 s^{2/3} - \lambda_2 s^{1/3}}\,ds \to \varphi(t). \tag{C.2}$$

The integral in (C.2) is absolutely convergent for $\lambda_1 = \lambda_2 = 0$. Since $\operatorname{Re} s^{2/3} > 0$ on the contour $\mathcal{B}$, the Lebesgue dominated convergence theorem implies Eq. (C.2).

The Laplace convolution theorem implies Eqs. (7.10).

Appendix D. Exact fundamental solutions for scalar equations. Explicit fundamental solutions of Eqs. (2.6) for odd spatial dimensions can be constructed provided the coefficients of the convolution kernel satisfy appropriate constraints.

The solution is constructed with the following ansatz:

$$\tilde{u}(s,\mathbf{x}) = w(s)U(s,r), \tag{D.1}$$

where

$$U(s,r) = e^{sr\varphi(s)}/(4\pi r). \tag{D.2}$$

Substituting (D.1) in the Laplace-transformed equation (2.6) and noting that

$$\nabla^2 U = -\delta(\mathbf{x}) + s^2\varphi(s)^2 U \tag{D.3}$$

yields the equation

$$s^2[1+\widetilde{K}-\varphi(s)^2]\tilde{u} + [w(s)-1-\widetilde{K}(s)]\delta(\mathbf{x}) = 0 \tag{D.4}$$

whence

$$\varphi(s)^2 = 1+\widetilde{K}(s) = w(s). \tag{D.5}$$

The following ansätze lead to explicitly computable original functions:

$$\varphi(s) = 1 + as^{-1/2}, \qquad \widetilde{K}(s) = 2a\,s^{-1/2} + a^2 s^{-1}, \tag{D.6}$$

$$\varphi(s) = 1 + a\,s^{-1/3} + b\,s^{-2/3}, \qquad \widetilde{K}(s) = 2a\,s^{-1/3} + (2b+a^2)s^{-2/3} + 2abs^{-1} + b^2 s^{-4/3}. \tag{D.7}$$

Substitution of Eqs. (D.6) or (D.7) into Eq. (D.1) followed by an inverse Laplace transformation yields the fundamental solution [23]:

$$u(t,\mathbf{x}) = \frac{1}{4\pi r}[f_0^{(2)}(t-r,ar) + 2af_1^{(2)}(t-r,ar) + a^2 f_2^{(2)}(t-r,ar)] \tag{D.8}$$

for the convolution kernel (D.6) and

$$\begin{aligned} u(t,\mathbf{x}) = \frac{1}{4\pi r}[&f_0^{(3)}(t-r,ar,br) + 2af_1^{(3)}(t-r,ar,br) + (2b+a^2)f_2^{(3)}(t-r,ar,br) \\ &+ 2abf_3^{(3)}(t-r,ar,br) + b^2 f_4^{(3)}(t-r,ar,br)] \end{aligned} \tag{D.9}$$

for the kernel (D.7). A snapshot of the fundamental solution for $a = b = 0.1$ at time $t = 0.1$ is shown in Fig. D.1. The evolution of the wavefront signal for $t = 0.1$, 1.0, and 10.0 is shown in Fig. D.2 in terms of the scaled coordinate $x = r/t$.

Remarkably, the exact solutions (D.8) and (D.9) are identical with asymptotic solutions of the same problem.

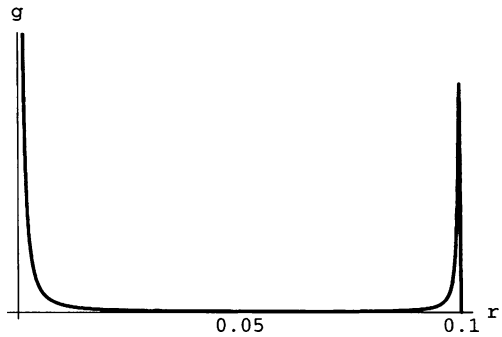

FIG. D.1. A snapshot of the fundamental solution (D.9) at $t = 0.1$; $a, b = 0.1$

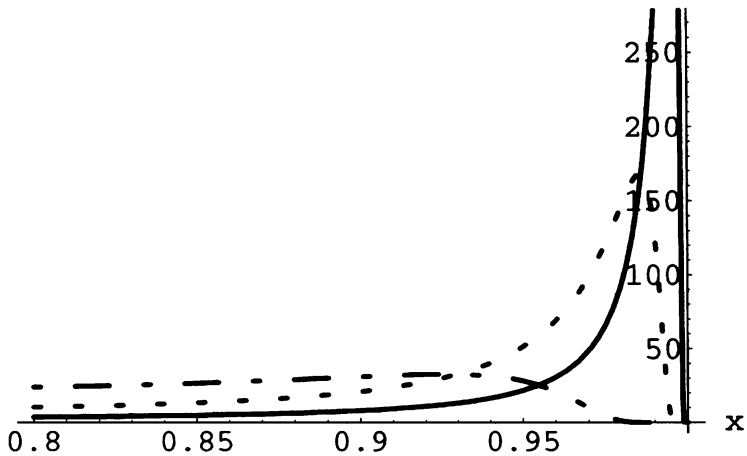

FIG. D.2. A snapshot of the wavefront signal (D.9) at $t = 0.1$ (solid line), 1.0 (dotted line), 10.0 (dash-dotted line) in terms of the variable $x = r/t$; $a, b = 0.1$

Appendix E. Behavior of the solution at the point source. We prove that

$$\lim_{r\to 0} f_0^{(3)}(t-r, ar, br) = \delta(t), \tag{E.1}$$

$$\lim_{r\to 0} f_1^{(3)}(t-r, ar, br) = t_+^{-1/3}/\Gamma(2/3), \tag{E.2}$$

etc. For analogous relations for $f_r^{(2)}$, see [23].

Indeed,

$$f_0^{(3)}(t-r, ar, br) * \varphi(t) = \frac{1}{2\pi i}\int_{\mathcal{B}} \tilde{\varphi}(s) e^{st} e^{-[s+as^{2/3}+bs^{1/3}]r}\, ds. \tag{E.3}$$

By the Lebesgue theorem, the limit $r \to 0$ exists and equals $\varphi(t)$, which proves (E.1). The remaining equations follow trivially.

The above results can be used to solve signaling problems (point source problems).

Appendix F. Initial values of $f_r^{(m)}(t-r,ar,br)$ and their time derivatives. In [23] it is proved that in 3D,

$$\lim_{t\to 0+}\frac{1}{4\pi r}f_0^{(2)}(t-r,ar)=0,\qquad \lim_{t\to 0+}\frac{1}{4\pi r}\frac{\partial f_0^{(2)}(t-r,ar)}{\partial t}=\delta(\mathbf{x}), \tag{F.1}$$

$$\lim_{t\to 0+}\frac{1}{4\pi r}f_1^{(2)}(t-r,ar)=\lim_{t\to 0+}\frac{1}{4\pi r}\frac{\partial f_1^{(2)}(t-r,ar)}{\partial t}=0, \tag{F.2}$$

$$\lim_{t\to 0+}\frac{1}{4\pi r}f_2^{(2)}(t-r,ar)=\lim_{t\to 0+}\frac{1}{4\pi r}\frac{\partial f_2^{(2)}(t-r,ar)}{\partial t}=0, \tag{F.3}$$

$$\lim_{t\to 0+}\frac{1}{4\pi r}\frac{\partial^2 f_2^{(2)}(t-r,ar)}{\partial t^2}=\delta(\mathbf{x}). \tag{F.4}$$

The method applied in [23] does not extend to the $s^{1/3}$ case. We shall show here that

$$\lim_{t\to 0+}\frac{1}{4\pi r}\frac{\partial f_0^{(3)}(t-r,ar,br)}{\partial t}=\delta(\mathbf{x}) \tag{F.5}$$

while the limits of $f_\mu^{(3)}(t-r,ar,br)/(4\pi r)$ for $\mu\geq 0$ and $(\partial f_\mu^{(3)}/\partial t)(t-r,ar,br)/(4\pi r)$ for $\mu>0$ vanish.

In all the cases, we consider the integrals of $f_\mu^{(3)}(t-r,ar,br)/(4\pi r)$ and their time derivatives multiplied by a spherically symmetric smooth test function $\varphi(r)$. In the special case (F.5),

$$\begin{aligned}
I&=\int_0^\infty dr\, r\frac{\partial f_0^{(3)}}{\partial t}(t-r,ar,br)\varphi(r)\\
&=\frac{1}{2\pi i}\int_{\mathcal{B}} ds\, se^{st}\int_0^\infty dr\, re^{-(s+bs^{1/3}+as^{2/3})r}\varphi(r)\\
&=\frac{1}{2\pi i}\int_{\mathcal{B}} ds\, s^{1/3}e^{st}\int_0^\infty dy\, ye^{-(b+as^{1/3}+s^{2/3})y}\Big[\varphi(0)+ys^{-1/3}\varphi'(0)\\
&\qquad\qquad +\frac{1}{2}y^2s^{-2/3}\varphi''(0)+\cdots\Big]\\
&=\frac{1}{2\pi i}\int_{\mathcal{B}} ds\, e^{st}\Big[\varphi(0)\left(-\frac{\partial}{\partial b}\right)\frac{s^{1/3}}{b+as^{1/3}+s^{2/3}}+\varphi'(0)\left(\frac{\partial^2}{\partial b^2}\right)\frac{1}{b+as^{1/3}+s^{2/3}}\\
&\qquad\qquad +\frac{1}{2}\varphi''(0)\left(-\frac{\partial^3}{\partial b^3}\right)\frac{s^{-1/3}}{b+as^{1/3}+s^{2/3}}+\cdots\Big]\\
&=\frac{3}{2\pi i}\int_{\mathcal{C}} d\sigma e^{\sigma^3 t}\Big[\varphi(0)\left(-\frac{\partial}{\partial b}\right)\frac{\sigma^3}{b+a\sigma+\sigma^2}+\varphi'(0)\left(\frac{\partial^2}{\partial b^2}\right)\frac{\sigma^2}{b+a\sigma+\sigma^2}\\
&\qquad\qquad +\frac{1}{2}\varphi''(0)\left(-\frac{\partial^3}{\partial b^3}\right)\frac{\sigma}{b+a\sigma+\sigma^2}+\cdots\Big],
\end{aligned} \tag{F.6}$$

where $\mathcal{B}$ denotes the imaginary axis, $\mathcal{C}$ is shown in Fig. F.1 and $y=r/s^{1/3}$, $\sigma=s^{1/3}$. For $a,b>0$ the integrand has poles in the left half of the complex plane. The integrals of the coefficients of $\varphi'(0)$, $\varphi''(0)$ and higher-order derivatives of φ converge for $t=0$. The Lebesgue dominated convergence theorem is applicable. After substituting $t=0$,

the contour $\mathcal{C}$ can be closed yielding 0. (The same argument can be used to prove that the limits of $f_\mu/(4\pi r)$ and $(\partial f_\mu/\partial t)/(4\pi r)$, $\mu > 0$ vanish.)

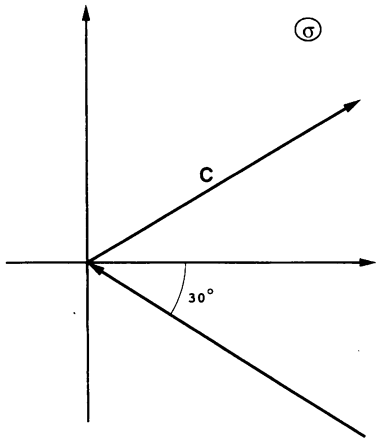

FIG. F.1. Contour $\mathcal{C}$

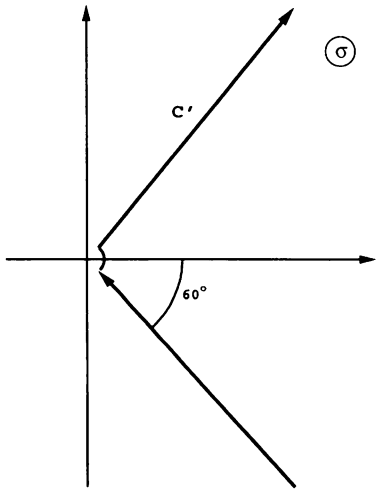

FIG. F.2. Deformation $\mathcal{C}'$ of contour $\mathcal{C}$

Similarly, differentiating the coefficient of $\varphi(0)$ with respect to a, b and setting $t = 0$ yields convergent integrals with algebraic integrands and, as a result, both derivatives vanish. Consequently, the coefficient of $\varphi(0)$ does not depend on a, b provided it is finite. For $a = b = 0$, the coefficient of $\varphi(0)$,

$$J = \frac{3}{2\pi i} \int_{\mathcal{C}} e^{\sigma^3 t} \sigma^{-1} \, d\sigma,$$

does not depend on t for $t > 0$. We set $t = 1$ and deform the contour $\mathcal{C}$ to run along the steepest descent path $\sigma = R^{1/3}e^{\pm i\pi/3}$ while avoiding the pole at 0 (Fig. F.2). The contributions from the straight line parts of the contour cancel, while the arc of the circle gives $J = 1$. This proves Eq. (F.5).

The time derivatives of $f_\mu^{(3)}(t-r, ar, br)/(4\pi r)$ of second and higher order are singular at $t = 0$. These singularities can be expressed in terms of a fractional-order expansion of $(\partial^2 f_\mu^{(3)}(t-r, ar, br)/\partial t^2)/(4\pi r)$ for $\mu = 0, 1, 2, \ldots$ with respect to t. Such expansions can be obtained by deforming the contour $\mathcal{C}$ to the steepest descent path $\arg\tau = \pm 2\pi/3$, substituting $\tau = \sigma t^{1/3}$, expanding the algebraic functions in the integrand in fractional powers of t and evaluating the coefficients of the expansion with the help of the integral formulae

$$\frac{3}{2\pi i}\int_{\mathcal{S}} e^{\tau^3}\tau^2\,d\tau = 0, \tag{F.7}$$

$$\frac{3}{2\pi i}\int_{\mathcal{S}} e^{\tau^3}\tau\,d\tau = \frac{\sqrt{3}}{2\pi}\Gamma(2/3), \tag{F.8}$$

$$\frac{3}{2\pi i}\int_{\mathcal{S}} e^{\tau^3}\,d\tau = \frac{\sqrt{3}}{2\pi}\Gamma(1/3), \tag{F.9}$$

$$\frac{3}{2\pi i}\int_{\mathcal{C}'} e^{\tau^3}\tau^{-1}\,d\tau = 1, \tag{F.10}$$

$$\frac{3}{2\pi i}\int_{\mathcal{S}} e^{\tau^3}\tau^{-2}\,d\tau = -3^{4/3}\int_0^\infty dz\int_z^\infty \mathrm{Ai}(y)\,dy, \tag{F.11}$$

$$\frac{3}{2\pi i}\int_{\mathcal{S}} e^{\tau^3}\tau^{-2}\,d\tau = 3^{4/3}\int_0^\infty dw\int_w^\infty dz\int_z^\infty \mathrm{Ai}(y)\,dy, \tag{F.12}$$

where the contour $\mathcal{S}$ consists of two lines $\tau = Re^{\pm i\pi/3}$ and $\mathcal{C}'$ is shown in Fig. F.2. We list here the leading terms of the small time expansions:

$$\begin{aligned}\frac{1}{4\pi r}\frac{\partial^2 f_0^{(3)}}{\partial t^2}(t-r, ar, br) = \frac{1}{\pi}\sqrt{3}\Big[&-a\Gamma(2/3)t_+^{-2/3} + \frac{1}{2}(3a^2-2b)\Gamma(1/3)t_+^{-1/3}\\ &+ 2a(3b-2a^2) - (3b^2-12a^2b+5a^4)Vt_+^{1/3}\\ &+ (-12ab^2+7a^3b-6b^4)Wt_+^{2/3} + \cdots\Big]\,\delta(\mathbf{x}),\end{aligned} \tag{F.13}$$

$$\begin{aligned}\frac{1}{4\pi r}\frac{\partial^2 f_1^{(3)}}{\partial t^2}(t-r, ar, br) = \frac{1}{\pi}\sqrt{3}\Big[&\frac{1}{2}\Gamma(2/3)t_+^{-2/3} - a\Gamma(1/3)t_+^{-1/3}\\ &+ 3a^2 - 2b - 2a(3b-2a^2)Vt_+^{1/3} + (3b^2-12a^2b+5a^4)Wt_+^{2/3} + \cdots\Big]\,\delta(\mathbf{x}),\end{aligned} \tag{F.14}$$

$$\frac{1}{4\pi r}\frac{\partial^2 f_2^{(3)}}{\partial t^2}(t-r,ar,br) = \frac{1}{\pi}\sqrt{3}\left[\frac{1}{2}\Gamma(1/3)t_+^{-1/3} - 2a - (3a^2-2b)Vt_+^{1/3} + \cdots\right]\delta(\mathbf{x}), \tag{F.15}$$

$$\frac{1}{4\pi r}\frac{\partial^2 f_3^{(3)}}{\partial t^2}(t-r,ar,br) = \left[1 + \frac{2}{\pi}\sqrt{3}aVt_+^{1/3} + (3a^2-2b)Wt_+^{2/3} + \cdots\right]\delta(\mathbf{x}), \tag{F.16}$$

$$\frac{1}{4\pi r}\frac{\partial^2 f_4^{(3)}}{\partial t^2}(t-r,ar,br) = \frac{1}{\pi}\sqrt{3}[-Vt_+^{1/3} - 2aWt_+^{2/3} + \cdots]\delta(\mathbf{x}), \tag{F.17}$$

where

$$V := 3^{5/6}\pi\int_0^\infty dz\int_z^\infty \mathrm{Ai}(y)\,dy, \tag{F.18}$$

$$W := 3^{5/6}\pi\int_0^\infty dw\int_w^\infty dz\int_z^\infty \mathrm{Ai}(y)\,dy. \tag{F.19}$$

Appendix G. 2-jet prolongation of Eqs. (5.17) **and** (5.23)**.** The transport equations involve second-order gradients of the function $S^{(\nu)}$, $\nu \geq 1$, as well as third-order gradients of the eikonal. These can be obtained by integration of the suitable prolongations of the ray equations and the equations for the auxiliary functions $S^{(\nu)}$.

The 2-jet prolongation of Eqs. (5.17) consists of Eqs. (5.17), (5.27), and the additional equations

$$\begin{aligned}
\frac{dQ_{j\gamma\lambda}}{d\sigma} &= \frac{\partial^2 H}{\partial k_j\partial x_l}Q_{l\gamma\lambda} + \frac{\partial^2 H}{\partial k_j\partial k_l}K_{l\gamma\lambda} + \frac{\partial^3 H}{\partial k_j\partial x_l\partial x_m}Q_{l\gamma}Q_{m\lambda} \\
&\quad + \frac{\partial^3 H}{\partial k_j\partial x_l\partial k_m}Q_{l\gamma}K_{m\lambda} + \frac{\partial^3 H}{\partial k_j\partial k_l\partial x_m}K_{l\gamma}Q_{m\lambda} + \frac{\partial^3 H}{\partial k_j\partial k_l\partial k_m}K_{l\gamma}K_{m\lambda}, \\
\frac{dK_{j\gamma\lambda}}{d\sigma} &= -\frac{\partial^2 H}{\partial x_j\partial x_l}Q_{l\gamma\lambda} - \frac{\partial^2 H}{\partial x_j\partial k_l}K_{l\gamma\lambda} - \frac{\partial^3 H}{\partial x_j\partial x_l\partial x_m}Q_{l\gamma}Q_{m\lambda} \\
&\quad - \frac{\partial^3 H}{\partial x_j\partial x_l\partial k_m}Q_{l\gamma}K_{m\lambda} - \frac{\partial^3 H}{\partial x_j\partial k_l\partial x_m}K_{l\gamma}Q_{m\lambda} - \frac{\partial^3 H}{\partial x_j\partial k_l\partial k_m}K_{l\gamma}K_{m\lambda}.
\end{aligned} \tag{G.1}$$

Similarly, the 2-jet prolongation of Eq. (5.23) consists of Eqs. (5.23), (5.25), and

$$\frac{dS_{\gamma\lambda}^{(1)}}{d\sigma} = \frac{\partial^2 h}{\partial x_j\partial x_l}Q_{j\lambda}Q_{l\lambda} + 2\frac{\partial^2 h}{\partial k_j\partial x_l}Q_{j\gamma}K_{l\lambda} + \frac{\partial^2 h}{\partial k_j\partial k_l}K_{j\gamma}K_{l\gamma} \tag{G.2}$$

with the initial condition

$$S_{11}^{(1)} = -\frac{\partial^2 h}{\partial k_j^{(0)2}}\frac{\partial^2 H}{\partial x_j^2} - \frac{\partial^2 H}{\partial k_j^{(0)2}}\frac{\partial^2 h}{\partial x_j^2} \tag{G.3}$$

and $S_{\gamma\lambda}^{(1)}(0) = 0$ for $\gamma \neq 1$ or $\lambda \neq 1$, where the Greek indices refer to the derivatives with respect to the ray field parameters.

The tensor $S_{jl}^{(1)}$ is now given by the formula:

$$S_{jl}^{(1)} = (S_{\gamma\lambda}^{(1)} - k_j^{(1)}Q_{j\gamma\lambda})Q_{\gamma j}^{-1}Q_{\lambda l}^{-1}. \tag{G.4}$$

Appendix H. A thermodynamic inequality for Eq. (2.6). The energy balance for Eq. (2.6) with $K(t) = K_0(t) + C\theta(t)$ is obtained by multiplying both sides by $u_{,t}$ and rearranging the resulting identity

$$\frac{1}{2}\frac{\partial}{\partial t}[u_{,t}^2 + n^2(\nabla u)^2] + D + F_{j,j} = 0, \tag{H.1}$$

where $F_j := -n^2 u_{,t} u_{,j}$ represents the energy flux and

$$D = u_{,t} K_0 * u_{,tt} + C u_{,t}^2 \tag{H.2}$$

can be interpreted as the energy dissipation rate.

Equation (2.6) admits asymptotic time-periodic solutions

$$u(t, \mathbf{x}) = U(\mathbf{x}) \cos[\omega(t - T(\mathbf{x})) + \alpha(\mathbf{x})]. \tag{H.3}$$

For such solutions we define the time average

$$\langle D \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} D(t)\, dt. \tag{H.4}$$

We now deduce some constitutive inequalities from the constraint

$$\langle D \rangle \geq 0 \tag{H.5}$$

for an arbitrary frequency ω. The constraint (H.5) replaces the usual assumption about the energy dissipation in a closed cycle [13].

Denoting by $\widehat{K_0}(\omega)$ the Fourier transform of $K_0(t)$ we have

$$\langle D \rangle = \tfrac{1}{2} U^2 [\omega^3 \operatorname{Im} \widehat{K_0}(\omega) + C\omega^2]. \tag{H.6}$$

For the kernel (6.1),

$$\widehat{K_0} = 2a|\omega|^{-1/2} e^{i \operatorname{sgn} \omega \pi/4}, \qquad C = b, \tag{H.7}$$

and the dissipation inequality (H.4) implies the inequalities

$$a, b \geq 0. \tag{H.8}$$

The second of these inequalities coincides with the Graffi inequality [13].

For Eq. (9.1),

$$\widehat{K_0}(\omega) = A|\omega|^{-2/3} e^{i \operatorname{sgn} \omega \pi/3} + B|\omega|^{1/3} e^{i \operatorname{sgn} \omega \pi/6} \tag{H.9}$$

and the dissipation inequality (H.4) implies the inequalities

$$C \geq 0, \quad A \geq 0, \quad B^2 \leq 4AC/3. \tag{H.10}$$

References

[1] M. Abramowitz and I. Stegun, *Mathematical Tables*, Dover, New York, 1970

[2] J.-F. Allard and Y. Champoux, *New empirical equations for sound propagation in rigid frame fibrous materials*, J. Acoust. Soc. Amer. **91**, 3346–3353 (1992)

[3] R. L. Bagley and P. J. Torvik, *On the fractional calculus model of viscoelastic behavior*, J. of Rheology, 133–155 (1986)

[4] J. G. Berryman, L. Thigpen, and R. C. Y. Chin, *Bulk elastic wave propagation in partially saturated porous solids*, J. Acoust. Soc. Amer. **84**, 360–373 (1988)

[5] M. A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid. II-Higher frequency range*, J. Acoust. Soc. Amer. **28**, 179–191 (1956)

[6] P. W. Buchen and F. Mainardi, *Asymptotic expansions for transient viscoelastic waves*, J. de Mécanique **14**, 597–608 (1975)

[7] Y. Champoux and M. R. Stinson, *On acoustical models for sound propagation in rigid frame porous materials and the influence of shape factors*, J. Acoust. Soc. Amer. **92**, 1120–1131 (1992)

[8] R. M. Christensen, *Theory of Viscoelasticity: An Introduction*, Academic Press, New York, 1971

[9] R. Courant, *Partial Differential Equations*, Wiley Interscience Publishers, New York, 1962

[10] C. F. Curtiss and R. B. Bird, *A kinetic theory for polymer melts, I. The equation for the single-link orientational distribution function*, J. Chem. Phys. **74**, 2016–2025 (1981)

[11] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 5, Springer-Verlag, Berlin, 1992

[12] G. Doetsch, *Einführung in Theorie and Anwendung der Laplace Transformation*, Birkhäuser-Verlag, Basel, 1958

[13] M. Fabrizio and A. Morro, *Mathematical Problems in Linear Viscoelasticity*, SIAM, Philadelphia, 1992

[14] W. Feller, *On a generalization of Marcel Riesz' potentials and the semigroups generated by them*, in Medd. Lund Univ. Matematiska Seminaret, 1952, pp. 73–81. Volume dedicated to M. Riesz.

[15] W. Feller, *Introduction to Probability Theory and its Applications*, vol. 2, John Wiley and Sons, New York, 1971

[16] S. Gelinsky, S. A. Shapiro, T. Müller, and B. Gurevich, *Dynamic poroelasticity of thinly layered structures*, Internat. J. Solids and Structures **35**, 1739–4751 (1998)

[17] R. Gorenflo and F. Mainardi, *Random walk models for space-fractional diffusion processes*, Fractonal Calculus and Applied Analysis **1**, 161–191 (1998)

[18] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 5th edition, Academic Press, New York, 1994

[19] B. Gurevich and S. L. Lopatnikov, *Velocity and attenuation of elastic waves in finely layered porous rocks*, Geophys. J. Internat. **121**, 933–947 (1995)

[20] A. Hanyga, *Asymptotic theory of wave propagation in viscoporoelastic media*, in "Theoretical and Computational Acoustics '97", Y.-C. Teng, E.-C. Shang, Y.-H. Pao, M. H. Shultz, and A. D. Pierce, eds., World Scientific, Singapore, 1999

[21] A. Hanyga and V. E. Rok, *An integro-differential wave equation and applications to wave propagation in micro-inhomogeneous porous media*, J. Acoust. Soc. Amer. **107** (6), 2965–2972 (2000)

[22] A. Hanyga and M. Seredyńska, *Asymptotic ray theory in poro- and viscoelastic media*, Wave Motion **30**, 175–195 (1999)

[23] A. Hanyga and M. Seredyńska, *Some effects of the memory kernel singularity on wave propagation and inversion in poroelastic media, I: Forward modeling*, Geophys. J. Internat. **137**, 319–335 (1994)

[24] A. Hanyga and M. Seredyńska, *Thermodynamics and asymptotic theory of wave propagation in viscoporous media*, Proc. 3rd Internat. Conf. on Computational and Theoretical Acoustics, Newark, NJ, July 14–18, 1997, 1999

[25] M. Ya. Kelbert and I. Ya. Chaban, *Relaxation and propagation of pulses in fluids*, Izv. Ak. Nauk. Ser. Mechanics of Fluids and Gases **5**, 153–160 (1986)

[26] R. C. Koeller, *Applications of fractional calculus to the theory of viscoelasticity*, J. Appl. Mechanics **51**, 299–307 (1984)

[27] R. C. Koeller, *Polynomial operators, Stieltjes convolution, and fractional calculus in hereditary mechanics*, Acta Mechanica **58**, 251–264 (1986)

[28] H. Kolsky, *The propagation of stress pulses in viscoelastic solids*, Philos. Mag. **8**, 693–710 (1956)

[29] A. Kreis and A. C. Pipkin, *Viscoelastic pulse propagation and stable probability distributions*, Quart. Appl. Math. **44**, 353–360 (1986)

[30] A. A. Lokshin and V. E. Rok, *Automodel solutions of wave equations with time lag*, Russian Math. Surveys **33**, 243–244 (1978)

[31] A. A. Lokshin and V. E. Rok, *Fundamental solutions of the wave equation with delayed time*, Doklady AN SSSR, **239**, 1305–1308 (1978)

[32] A. A. Lokshin and Yu. V. Suvorova, *Mathematical Theory of Wave Propagation in Media with Memory*, Moscow University Publ., Moscow, 1982 (in Russian)

[33] F. Mainardi and M. Tomirotti, *Seismic pulse propagation with constant Q and stable probability distributions*, Annali de Geofisica **40**, 1311–1328 (1997)

[34] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, 1993

[35] A. Narain and D. D. Joseph, *Linearized dynamics for step jumps of velocity and displacement of shearing flows for a simple fluid*, Rheol. Acta, **21**, 228–250 (1982)

[36] A. N. Norris, *On the viscodynamic operator in Biot's theory*, J. Wave-Material Interaction **1**, 365–380 (1986)

[37] P. G. Nutting, *Deformation in relation to time, pressure and temperature*, J. Franklin Institute **242**, 449–458 (1946)

[38] M. Ochmann and S. Makarov, *Representation of absorption of non-linear waves by fractional derivatives*, J. Acoust. Soc. Amer. **94**, 3392–3399 (1993)

[39] A. C. Pipkin, *Lectures on Viscoelasticity Theory*, 2nd edition, Springer-Verlag, New York, 1986

[40] A. C. Pipkin, *Asymptotic behaviour of viscoelastic waves*, Quart. J. Mech. Appl. Math. **41**, 51–64 (1988)

[41] J.-D. Polack, *Time domain solution of Kirchhoff's equation for sound propagation in viscothermal gases: a diffusion process*, J. Acoustique **4**, 17–67 (1991)

[42] Yu. N. Rabotnov, *Creep Problems in Structural Elements*, North-Holland, Amsterdam, 1969

[43] Yu. N. Rabotnov, *Elements of Hereditary Solid Mechanics*, Mir Publ., Moscow, 1980

[44] M. Renardy, *Some remarks on the propagation and non-propagation of discontinuities in linearly viscoelastic liquids*, Rheol. Acta **21**, 251–254 (1982)

[45] M. Renardy, W. J. Hrusa, and J. A. Nohel, *Mathematical Problems in Viscoelasticity*, Longman Scientific and Technical, Essex, 1987

[46] Lynn Rogers, *Operators and fractional derivatives for viscoelastic constitutive equations*, J. Rheology **27**, 351–372 (1983)

[47] V. E. Rok, *Time-domain representation of waves in media with frequency power law of dispersion*, in Extended Abstracts 58th EAGE Conference and Technical Exhibition, Amsterdam, 3–7 June 1996, p. C008, 1996

[48] Yu. A. Rossikhin and M. V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanisms of solids*, Appl. Mech. Rev. **50**, 15–67 (1997)

[49] A. R. Rzhanitsyn, *Some problems of the mechanics of systems that are deformed in time*, Gostekhizdat, Moscow, 1949

[50] S. G. Samko, A.A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam, 1993

[51] G. L. Slonimsky, *Laws of mechanical relaxation processes in polymers*, J. Polymer Science **C16**, 1667–1672 (1967)

[52] W. Smit and H. de Vries, *Rheological models containing fractional derivatives*, Rheol. Acta, **6** 525–534 (1970)

[53] M. R. Stinson and Y. Champoux, *Propagation of sound and the assignment of shape factors in model porous materials having simple pore geometries*, J. Acoust. Soc. Amer. **91**, 685–695 (1992)

[54] P. J. Torvik and R. L. Bagley, *On the appearance of the fractional derivative in the behavior of real material*, J. Appl. Mechanics **51**, 294–298 (1983)

[55] D. V. Widder, *An Introduction to Transformation Theory*, Academic Press, New York, 1971

[56] D. K. Wilson, *Relaxation-matched modelling of propagation through porous media, including fractal pore structure*, J. Acoust. Soc. Amer. **94**, 1136–1145 (1992)