A REMARK ON THE EXISTENCE OF GLOBAL BV SOLUTIONS FOR A NONLINEAR HYPERBOLIC WAVE EQUATION

JOÃO-PAULO DIAS and MÁRIO FIGUEIRA

CMAF/Univ. Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

Abstract. By means of a suitable change of variables we obtain, by application of a general result by Dafermos and Hsiao, cf. [2], an existence theorem in $L^\infty \cap BV_{loc}$ of a weak solution of the system corresponding to the quasilinear hyperbolic equation

$$\phi_{tt} - p'(\phi_x) \phi_{xx} + \phi_t + F(\phi) = 0 \quad \text{in } \mathbb{R} \times [0, +\infty[, \quad \text{for small initial data in } BV.$$ 

This theorem is a partial extension of Dafermos’s result for the case with $F(\phi) \equiv 0$, proved in [1].

1. The auxiliary system. Let us consider the following Cauchy problem:

$$-p'(\phi_x) \phi_{xx} + \phi_t + F(\phi) = 0, \quad (x,t) \in \mathbb{R} \times [0, +\infty[, \quad (1.1)$$

$$\phi(x,0) = \phi_0(x), \quad x \in \mathbb{R}, \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}. \quad (1.2)$$

where $p$ is a given smooth function such that $p'(\xi) > 0$, $\forall \xi \in \mathbb{R}$, and $\overline{F}: \mathbb{R} \to \mathbb{R}$ is a smooth function verifying $\overline{F}(0) = 0$. We assume, to simplify, the condition $p'(0) = 1$. Putting $u = \phi_x$, $v = \phi_t$ we can write (1.1), (1.2) as a Cauchy problem for a hyperbolic system:

$$\begin{cases}
\overline{\phi}_t = \overline{v} \\
\overline{u}_t = \overline{v}_x \\
\overline{v}_t - p'(\overline{u}) \overline{u}_x + \overline{v} + \overline{F}(\overline{\phi}) = 0
\end{cases} \quad (x,t) \in \mathbb{R} \times [0, +\infty[, \quad (1.3)$$

$$\overline{\phi}(x,0) = \phi_0(x), \quad \overline{u}(x,0) = u_0(x) = \overline{\phi}_0(x), \quad \overline{v}(x,0) = v_0(x), \quad x \in \mathbb{R}. \quad (1.4)$$

For technical reasons, if we choose $k > 1$, we can, by putting $u(x,t) = \overline{u}(kx,kt)$, $v(x,t) = \overline{v}(kx,kt)$, $\phi(x,t) = \frac{1}{k} \overline{\phi}(kx,kt)$, replace (1.3), (1.4) by

$$\begin{cases}
\phi_t = v \\
u_t = v_x \\
v_t - p'(u) u_x + k v + F(\phi) = 0
\end{cases} \quad (x,t) \in \mathbb{R} \times [0, +\infty[, \quad (1.3')$$

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where $F(\phi) = k \overline{F}(k\phi)$,

$$
\phi(x, 0) = \phi_0(x) = \frac{1}{k} \phi_0(kx) ,
$$

$$
u(x, 0) = u_0(x) = u_0(kx) = \phi_{0x}(x) ,
$$

$$
v(x, 0) = v_0(x) = v_0(kx) , \quad x \in \mathbb{R} .
$$

For $m \in [1, +\infty[$, let us introduce

$$
a = a_m = (m^2 - 2m + 2)^{-1/2} ,
$$

$$
b = b_m = (m^2 + 2m + 2)^{-1/2} ,
$$

$$
c_m = 2 - \left(\frac{a}{b} + \frac{b}{a}\right)/2 ,
$$

$$
k_m = 2c_m^{-1} \left(a(m - 1) + b(m + 1)\right) ,
$$

$$
f_m = c_m^2 \left[2\left(\frac{1}{a} + \frac{1}{b}\right) \left(a(m - 1) + b(m + 1)\right)\right]^{-1}
$$

(notice that $c_1 > 0$ and $k_1 > 1$) and assume

$$
|F'(0)| < f_1 .
$$

We fix $m > 1$ such that $|F'(0)| < f_m$, $c_m > 0$, and $k_m > 1$ and we put $k = k_m$. We obtain

$$
(2 - \left(\frac{a}{b} + \frac{b}{a}\right)/2) k - \left(\frac{1}{2a} + \frac{1}{2b}\right) |F'(0)| - a(m - 1) - b(m + 1)
$$

$$
= c_m k_m - \left(\frac{1}{2a} + \frac{1}{2b}\right) k_m^2 |F'(0)| - a(m - 1) - b(m + 1) > 0 .
$$

Now, we consider the following nonsingular linear transformation $(\phi, u, v) \rightarrow (\phi, u, w)$, where $w = v + mu + k\phi$, $m$ and $k$ as above. The Cauchy problem (1.3'), (1.4') takes the form

$$
\begin{cases}
\phi_t = w - m u - k \phi \\
u_t + k\phi_x + mu_x - wx = 0 \quad (x, t) \in \mathbb{R} \times [0, +\infty[ ,
\end{cases}
$$

$$
\begin{cases}
\phi(x, 0) = \phi_0(x) , \\
u(x, 0) = u_0(x) = \phi_{0x}(x) ,
\end{cases}
$$

$$
\begin{cases}
w(x, 0) = w_0(x) = v_0(x) + mu_0(x) + k\phi_0(x) , \quad x \in \mathbb{R} .
\end{cases}
$$
Now we introduce the following auxiliary Cauchy problem in \((\phi, u, w)\),

\[
\begin{cases}
\phi_t + m \phi_x - w + k \phi = 0 \\
u_t + m u_x - w_x + k u = 0 \\
w_t + (m^2 - p'(u)) u_x - m w_x + m k u + F(\phi) = 0
\end{cases}
\text{ in } \mathbb{R} \times [0, +\infty[ , \tag{1.3'''}
\]

with the initial data \((1.4'')\).

If \((\phi, u, w)\) is a \(C^1\) solution of \((1.3'''), (1.4'')\) we easily derive, for \(\varphi \in C^\infty_c(\mathbb{R} \times [0, +\infty[),\]

\[
- \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} \phi \varphi_{xt} \, dx \, dt - \int_\mathbb{R} \varphi_0 \varphi_x(\cdot, 0) \, dx - m \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} \phi \varphi_{xx} \, dx \, dt \\
- \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} w \varphi_x \, dx \, dt + k \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} \phi \varphi_x \, dx \, dt = 0
\]

and

\[
- \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} u \varphi_t \, dx \, dt - \int_\mathbb{R} u_0 \varphi(\cdot, 0) \, dx - m \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} u \varphi_x \, dx \, dt \\
+ \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} w \varphi_x \, dx \, dt + k \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} u \varphi \, dx \, dt = 0 ,
\]  

and so, by addition, we obtain

\[
\int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} \phi \left[ \varphi_t + m \varphi_x - k \varphi \right] \, dx \, dt + \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} u \left[ \varphi_t + m \varphi_x - k \varphi \right] \, dx \, dt = 0 .
\]

Now, given \(\psi \in \mathcal{D}(\mathbb{R} \times ]0, +\infty[)\), it is easy to find \(\varphi \in C^\infty_c(\mathbb{R} \times [0, +\infty[)\) such that \(\varphi_t + m \varphi_x - k \varphi = \psi\): first, with \(\psi_1 = e^{-kt} \psi, \varphi_1 = e^{-k\varphi} \psi\), we reduce to \(\varphi_{1t} + m \varphi_{1x} = \psi_1\).

We put

\[
\varphi_1(m t + c, t) = \int_0^t \psi_1(m \tau + c, \tau) \, d\tau - \varphi_1(c) ,
\]

where

\[
\varphi_1(c) = \int_0^{+\infty} \psi_1(m t + c, t) \, dt , \quad \forall c \in \mathbb{R} .
\]

Hence,

\[
\int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} \phi \psi_x \, dx \, dt + \int_{\mathbb{R}+\mathbb{R}} \int_\mathbb{R} u \psi \, dx \, dt = 0 , \quad \forall \psi \in \mathcal{D}(\mathbb{R} \times ]0, +\infty[) .
\]

We derive \(u = \phi_x\). It is now easy to prove

**Proposition 1.1.** For given initial data \((\phi_0, u_0 = \phi_{0x}, w_0)\) in \(C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})\) the systems \((1.3''')\) and \((1.3'''')\) have the same (local in time) \(C^1\) solutions.
Given \((\phi_0, u_0, w_0)\) in \(L^\infty(\mathbb{R})\), we say, as usually, that \((\phi, u, w) \in (L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3\) is a weak (global) solution of the Cauchy problem \((1.3''), (1.4'')\) if we have

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \phi \varphi_t \, dx \, dt + \int_{\mathbb{R}} \phi_0 \varphi(x, 0) \, dx + \int_{\mathbb{R}^+} \int_{\mathbb{R}} (w - mu - k \phi) \varphi \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}} u \psi_t \, dx \, dt + \int_{\mathbb{R}} u_0 \psi(x, 0) \, dx + \int_{\mathbb{R}^+} \int_{\mathbb{R}} (k \phi + mu - w) \psi_x \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}} w \theta_t \, dx \, dt + \int_{\mathbb{R}} w_0 \theta(x, 0) \, dx
\]

\[
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[ m^2 u - p(u) - mw \right] \theta_x \, dx \, dt - \int_{\mathbb{R}^+} \int_{\mathbb{R}} F(\phi) \theta \, dx \, dt = 0,
\]

\[
\forall \varphi, \psi, \theta \in C_c^\infty(\mathbb{R} \times [0, +\infty[),
\]

and a similar definition for the Cauchy problem \((1.3''''), (1.4'''')\).

We can repeat the calculations made to prove Proposition 1.1 for a \((\phi, u, w) \in (L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3\) weak solution of \((1.3''''), (1.4'''')\) and we obtain \(\phi_x = u\) in \(D'((\mathbb{R} \times]0, +\infty[)\). We derive \(\phi \in W^{1, \infty}_{\text{loc}}(\mathbb{R} \times [0, +\infty[)\) and it is now easy to prove that \((\phi, u, w)\) is a weak solution of \((1.3''''), (1.4'''').\) The converse is also true, by similar considerations. Hence, we have

**Proposition 1.2.** For a given initial data \((\phi_0, u_0 = \phi_{0x}, w_0)\) in \(L^\infty(\mathbb{R})\), the systems \((1.3'''')\) and \((1.3'''')\) have the same weak solutions.

Now let \((\eta, q)\) be a pair of smooth convex entropy/entropy flux for the system \((1.3'''')\) (cf. [4]).

\[
\begin{align*}
\eta(\phi, u, w) &= \frac{1}{2} \phi^2 + \int_0^u p(\xi) \, d\xi + \frac{1}{2} (w - mu - k \phi)^2, \\
q(\phi, u, w) &= -(w - mu - k \phi) p(u).
\end{align*}
\]

A weak solution \((\phi, u, w)\) of \((1.3''''), (1.4'''')\) is called an entropy weak solution if, in \(D'((\mathbb{R} \times]0, +\infty[)\),

\[
\eta(\phi, u, w)_t + q(\phi, u, w)_x + \nabla \eta \cdot \left(-w + mu + k \phi, 0, F(\phi)\right) \leq 0
\]

for all pairs \((\eta, q), \eta\) convex.

The system \((1.3'''')\) admits the entropy/entropy flux pair \((\tilde{\eta}_1, \tilde{q}_1)\), \(\tilde{\eta}_1\) strictly convex, defined by

\[
\begin{align*}
\tilde{\eta}_1(\phi, u, w) &= \frac{1}{2} \phi^2 + \int_0^u p(\xi) \, d\xi + \frac{1}{2} (w - mu)^2, \\
\tilde{q}_1(\phi, u, w) &= \frac{1}{2} m \phi^2 - (w - mu) p(u).
\end{align*}
\]

If \((\phi, u, w) \in (L^\infty_{\text{loc}} \cap BV_{\text{loc}})^3\) is a weak solution of \((1.3''''), (1.4'''')\) with initial data in \(BV(\mathbb{R})\), we can prove (with some tedious computations, taking in mind that \(\phi_x =
\]
u, cf. Proposition 1.2, and applying the theorem in section 13.2 of [5] concerning the differentiation of the composition) that we have, in $\mathcal{D}'(\mathbb{R} \times [0, +\infty[)$,

$$
\tilde{\eta}_1(\phi, u, w)_t + \tilde{q}_1(\phi, u, w)_x + \nabla \tilde{\eta}_1 \cdot \left(-w + k\phi, k u, m ku + F(\phi)\right)
$$

$$
= \eta_1(\phi, u, w)_t + q_1(\phi, u, w)_x + \nabla \eta_1 \cdot \left(-w + m u + k\phi, 0, F(\phi)\right)
$$

Hence, by Proposition 1.2, we conclude

**Theorem 1.3.** Assume that $(\phi_0, u_0 = \phi_{0x}, w_0) \in BV(\mathbb{R})^3$ and let $(\phi, u, w) \in (L_\infty \cap BV_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ be an entropy weak solution of (1.3''), (1.4''). Then, $(\phi, u, w)$ is also a weak solution of (1.3''), (1.4'') verifying (1.8) for the pair $(\eta_1, q_1)$ defined above.

2. **Application of Theorem 2 in [2].** Now, in order to apply to the Cauchy problem (1.3''), (1.4'') Theorem 2 in [2], we give initial data $(\phi_0, u_0 = \phi_{0x}, w_0) \in BV(\mathbb{R})$. The system (1.3'') can be written as follows (recall that $p'(0) = 1$ and $m > 1$):

$$
\frac{\partial}{\partial t} \begin{pmatrix} \phi \\ u \\ w \end{pmatrix} + A(u) \frac{\partial}{\partial x} \begin{pmatrix} \phi \\ u \\ w \end{pmatrix} + g(\phi, u, w) = 0,
$$

where

$$
A(u) = \begin{pmatrix} m & 0 & 0 \\ 0 & m & -1 \\ 0 & m^2 - p'(u) & -m \end{pmatrix}, \quad g(\phi, u, w) = \begin{pmatrix} -w + k\phi \\ k u \\ m ku + F(\phi) \end{pmatrix}.
$$

The eigenvalues of $A(u)$ are $(m, \sqrt{p'(u)}, -\sqrt{p'(u)})$. The matrix of the corresponding (independent) normalized right eigenvectors for $u = 0$ is

$$
B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & a(m-1) & b(m+1) \end{pmatrix}
$$

and

$$
\nabla g(0,0,0) = \begin{pmatrix} k & 0 & -1 \\ 0 & k & 0 \\ F'(0) & m k & 0 \end{pmatrix},
$$

where $a = (m^2 - 2m + 2)^{-1/2}$ and $b = (m^2 + 2m + 2)^{-1/2}$.

Hence, $R = \{ r_{ij} \} = B^{-1} \nabla g(0,0,0) B$ is given by

$$
R = \begin{pmatrix} k & a(-m+1) & b(-m-1) \\ -\frac{1}{2a} F'(0) & k & b/k \\ \frac{1}{2b} F'(0) & a/k & k/2 \end{pmatrix}
$$

and verifies

$$
\sum_{i} r_{ii} - \sum_{i \neq j} |r_{ij}| = \left(2 - \left(\frac{a}{b} + \frac{b}{a}\right)/2\right) k - \left(\frac{1}{2a} + \frac{1}{2b}\right) |F'(0)|
$$

$$
= -a(m-1) - b(m+1) > 0 \quad \text{by (1.6)}
$$

and so $R$ is diagonal dominant.

By applying Theorem 2 in [2] we derive
Theorem 2.1. Let us assume (1.5). Then, there exist two positive constants $a_0, b_0 > 0$ such that, if

$$(\phi_0, u_0 = \phi_{0x}, w_0) \in (BV(\mathbb{R}))^3$$

and

$$\|(\phi_0, u_0, w_0)\|_{L^\infty(\mathbb{R})} \leq a_0 , \quad TV_x(\phi_0, u_0, w_0) \leq b_0 ,$$

then there exists a weak entropy solution $(\phi, u, w) \in (L^\infty \cap BV_{loc}(\mathbb{R} \times [0, +\infty[))^3$ of (1.3''), (1.4''). Moreover $(\phi(\cdot, t), u(\cdot, t), w(\cdot, t)) \in (BV(\mathbb{R}))^3$ for each $t \geq 0$, with a uniformly bounded (in $t$) total variation $TV_x$.

Hence, by Theorem 1.3, we can derive a similar result for the Cauchy problem (1.3), (1.4) if we replace the general entropy condition (1.8) by the following particular one:

$$\bar{\eta}(\bar{\phi}, \bar{u}, \bar{v})_t + \bar{q}(\bar{\phi}, \bar{u}, \bar{v})_x + \nabla \bar{\eta} \cdot \left(-\bar{v}, 0, \bar{v} + \bar{F}(\bar{\phi})\right) \leq 0 \quad \text{in} \quad D'(\mathbb{R} \times ]0, +\infty[) ,$$

where

$$(\bar{\eta}, \bar{q}) = \left(\frac{1}{2} \bar{\phi}^2 + \int_0^{\bar{u}} p(\xi) d\xi + \frac{1}{2} \bar{v}^2, -\bar{v} p(\bar{u})\right) .$$

See [1] for the case with $\bar{F} \equiv 0$ and [3] for related results.

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References