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INITIAL-BOUNDARY VALUE PROBLEM TO SYSTEMS OF CONSERVATION LAWS WITH RELAXATION

By

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Abstract. In this paper we consider the initial-boundary value problem (IBVP) for the one-dimensional Jin-Xin relaxation model. The main interest is to study the boundary layer behaviors in the solutions to the IBVP of the relaxation system and their asymptotic convergence to solutions of the corresponding hyperbolic conservation laws in the limit of small relaxation rate. First we develop a general expansion theory for the relaxation IBVP using a matched asymptotic analysis. This formal procedure determines a unique equilibrium limit, and also reveals rich initial and boundary layer structures in the solutions of the relaxation system. Arbitrarily accurate solutions to the IBVP of the relaxation system are then constructed by combining the various orders of the equilibrium solutions, the initial and boundary layer solutions. The validity of the initial and boundary layers and the asymptotic convergence results are rigorously justified through a stability analysis for a broad class of boundary conditions in the case when the relaxation system is 2×2.

1. Introduction. In [3], Jin and Xin proposed the following relaxation model:

$$\begin{aligned} &\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0, \\ &\partial_t v^\varepsilon + a\partial_x u^\varepsilon = -\frac{1}{\varepsilon}(v^\varepsilon f(u^\varepsilon)), \end{aligned} \qquad u^\varepsilon, v^\varepsilon \in \mathbf{R}^n, \tag{1.1}$$

to approximate a general system of hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \quad u \in \mathbf{R}^n, \tag{1.2}$$

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where $\varepsilon > 0$ is the rate of relaxation and a is a positive constant satisfying the following sub-characteristic condition:

$$a > \max |\lambda_i(u)|^2, \quad 1 \le i \le n \quad \text{(for all } u \text{ under consideration)} \tag{1.3}$$

where $\lambda_1(u), \ldots, \lambda_n(u)$ are the (real) eigenvalues of the Jacobian matrix $f'(u)$.

We assume that (1.2) is strictly hyperbolic and the boundary $x = 0$ is noncharacteristic, i.e., the eigenvalues $\lambda_i(u)$ of $f'(u)$ are such that

$$\lambda_1(u) > \cdots > \lambda_p(u) > 0 > \lambda_{p+1}(u) > \cdots > \lambda_n(u). \tag{1.4}$$

Let $r_1(u), \ldots, r_n(u)$ be the corresponding right eigenvectors of $f'(u)$, and let $R(u)$ be the matrix whose columns consist of $r_1(u), \ldots, r_n(u)$. Then

$$L(u)f'(u)R(u) = \Lambda(u) \equiv \operatorname{diag}\{\lambda_1(u), \ldots, \lambda_n(u)\}, \quad L(u) = R(u)^{-1}. \tag{1.5}$$

We consider the initial boundary value problem of (1.1) in the quarter plane $x > 0, t > 0$ and supplement (1.1) with initial data

$$u^\varepsilon(x,0) = u_0(x), \quad v^\varepsilon(x,0) = v_0(x), \quad x > 0, \tag{1.6}$$

and the boundary condition

$$B_u u^\varepsilon(0,t) + B_v v^\varepsilon(0,t) = b(t), \quad t > 0. \tag{1.7}$$

For simplicity, we consider only linear boundary conditions with B_u, B_v being constant $n \times n$ real matrices and assume that the given initial and boundary data $u_0(x), v_0(x)$, and $b(t)$ are independent of ε. In addition, we assume that the boundary condition (1.7) satisfies the Uniform Kreiss Condition

$$\det(B_u + \sqrt{a}B_v) \neq 0 \tag{1.8}$$

so that on the boundary $x = 0$, the incoming flow $\sqrt{a}u^\varepsilon + v^\varepsilon$ can be expressed in terms of the outgoing flow $\sqrt{a}u^\varepsilon - v^\varepsilon$ and the boundary data $b(t)$ and, therefore, the IBVP (1.1), (1.6)–(1.7) is well-posed [4], [10] for each fixed ε.

It is generally expected that, under suitable stability conditions (for example, the sub-characteristic condition (1.3)), the solution $(u^\varepsilon, v^\varepsilon)$ of the relaxation system (1.1) tends to an equilibrium limit (u, v) of (1.2) with $v = f(u)$ as $\varepsilon \to 0$. Indeed, in the case of the Cauchy problem, various asymptotic convergence results have been obtained under the sub-characteristic assumptions; see, for example, [1], [3], [6], [7], [8], [11], [15]. See also [9] for a survey on some recent results.

The purpose of this paper is to establish the asymptotic equivalence of (1.1) and (1.2) in the limit of small relaxation rate when boundary effects are present. The initial boundary value problem is physically more relevant and mathematically more challenging than the pure initial value problem. Of particular interest are the boundary layer behavior in the solution of (1.1) and the precise stability requirements on the boundary conditions such that the desired asymptotic convergence holds for the corresponding IBVP of (1.1).

We remark that, for the same model in the linear case, these questions have been successfully answered in [14]. In that case, the necessary and sufficient condition for the stiff well-posedness of the IBVP (1.1), (1.6)–(1.7) and the asymptotic convergence of

(1.1) to (1.2) in the limit of $\varepsilon \to 0$ is the following determinant condition (Stiff Kreiss Condition):

$$\det(B_u R + B_v R G(\zeta)) \neq 0 \tag{1.9}$$

for all complex ζ with $\operatorname{Re}\zeta \geq 0$, where

$$G(\zeta) = \operatorname{diag}\{g_1(\zeta), g_2(\zeta), \ldots, g_n(\zeta)\} \tag{1.10}$$

and

$$g_j(\zeta) = \frac{\lambda_j + \sqrt{\lambda_j^2 + 4a\zeta(1+\zeta)}}{2(1+\zeta)}, \quad 1 \leq j \leq n. \tag{1.11}$$

We note that the Uniform Kreiss Condition (1.8) requires (1.9) to hold only for $\zeta = +\infty$. Also in the case $n = 1$, the Stiff Kreiss Condition (1.9) can be simplified and is equivalent to

$$B_v = 0 \quad \text{or} \quad \frac{B_u}{B_v} \notin \left[-\sqrt{a}, -\frac{\lambda + |\lambda|}{2}\right], \tag{1.12}$$

which is indeed more restrictive than the Uniform Kreiss Condition (1.8).

We consider solutions of (1.1) and (1.2) close to a constant state (u^*, v^*) with $v^* = f(u^*)$. Without loss of generality, we assume

$$\begin{gathered} f(0) = 0, \quad u^* = 0, \quad v^* = 0, \\ f'(0) = \Lambda(0) = \operatorname{diag}\{\lambda_1(0), \ldots, \lambda_n(0)\}, \\ R(0) = L(0) = I_n. \end{gathered} \tag{1.13}$$

The initial data $U_0(x) = (u_0(x), v_0(x))$ and the boundary data $b(t)$ are then required to be suitably small and sufficiently compatible, say,

$$\begin{gathered} v_0(0) = f(u_0(0)), \qquad B_u u_0(0) + B_v v_0(0) = b(0), \\ U_0'(0) = U_0''(0) = 0, \qquad b'(0) = b''(0) = 0. \end{gathered} \tag{1.14}$$

We will first develop a general expansion theory for the relaxation IBVP (1.1), (1.6)–(1.7) using a matched asymptotic analysis. This formal procedure uniquely determines an equilibrium limit that satisfies (1.2). It also reveals rich initial and boundary layer structures in the solution of (1.1), (1.6)–(1.7). The matched asymptotic expansions can be carried out up to any order. As a result, arbitrarily accurate approximate solutions to the IBVP (1.1), (1.6)–(1.7) can be constructed by combining various orders of equilibrium solutions, the initial layer and boundary layer solutions.

Under suitable (sufficient) stability conditions, these formal convergence results can be rigorously justified by using a nonlinear stability analysis. This is done in the simplest case of $n = 1$. However, it is worth noting that the validity of the formal expansion results requires stronger stability conditions than what merely guarantees the expansion procedure itself ($\det(B_u R(u^*) + B_v R(u^*)\Lambda_+(u^*)) \neq 0$). This is already clear in the linear case [14] where necessary and sufficient conditions have been obtained.

We will see that the expansion procedure is greatly simplified in the case $n = 1$ since only one boundary condition is involved. It is clear that in the case $f'(u) < 0$,

the boundary layer $u^{\text{b.l.}}(x/\varepsilon, t)$ (in the u-component only) solves the following nonlinear ODE (regarding t as a parameter):

$$\partial_\xi u^{\text{b.l.}} = f(u(0,t) + u^{\text{b.l.}}) - f(u(0,t)) \tag{1.15}$$

with boundary condition

$$u^{\text{b.l.}}(0,t) = (b(t) - B_u u(0,t) - B_v f(u(0,t)))/B_u \tag{1.16}$$

while the equilibrium limit $u(x,t)$ is determined by solving (1.2) with the initial condition

$$u(x,0) = u_0(x), \quad x \geq 0. \tag{1.17}$$

For $f'(u) > 0$, no boundary layer develops ($u^{\text{b.l.}} = v^{\text{b.l.}} = 0$), but the following boundary condition,

$$B_u u(0,t) + B_v f(u(0,t)) = b(t), \tag{1.18}$$

has to be added in order to determine the equilibrium limit $u(x,t)$.

In both cases, there is also an initial layer in the v-component

$$v^{\text{i.l.}}(x, t/\varepsilon) = e^{-t/\varepsilon}(v_0(x) - f(u_0(x))). \tag{1.19}$$

We are now ready to state our rigorous convergence results. In all these cases we assume the sub-characteristic condition

$$a > f'(u^*)^2. \tag{1.20}$$

In addition, we assume either $f''(u^*) \neq 0$ or $f''(u) \equiv 0$ in a neighborhood of u^*.

THEOREM 1.1 (Stability of weak boundary layers). Let $f'(u^*) < 0$. Assume that the equilibrium solution to the IBVP (1.2), (1.17) is smooth in $0 \leq t \leq T$ for some $T > 0$ and that the boundary condition satisfies the structural condition

$$B_v = 0 \quad \text{or} \quad \frac{B_u}{B_v} > 0. \tag{1.21}$$

Then there exist positive constants δ_0 and ε_0 such that if

$$\max_{0 \leq t \leq T} |u^{\text{b.l.}}(0,t)| \leq \delta_0, \tag{1.22}$$

then the IBVP (1.1), (1.6)–(1.7) has a unique smooth solution for all $0 < \varepsilon \leq \varepsilon_0$. Furthermore, the solution satisfies the estimates

$$\sup_{0 \leq t \leq T} \|u^\varepsilon(\cdot,t) - u(\cdot,t) - u^{\text{b.l.}}(\cdot/\varepsilon, t)\|_{L^\infty(\mathbf{R}^+)} \leq O(1)\varepsilon \tag{1.23}$$

and

$$\sup_{0 \leq t \leq T} \|v^\varepsilon(\cdot,t) - v(\cdot,t) - v^{\text{i.l.}}(\cdot, t/\varepsilon)\|_{L^\infty(\mathbf{R}^+)} \leq O(1)\varepsilon. \tag{1.24}$$

Note that the compatibility condition (1.14) implies that $u^{\text{b.l.}}(0,0) = 0$. Therefore, by continuity, there exists a suitably small T_0 such that $|u^{\text{b.l.}}(0,t)| \leq \delta_0$ for all $0 \leq t \leq T_0$. As an immediate consequence of Theorem 1.1, we have

COROLLARY 1.2 (Short time stability). Let $f'(u^*) < 0$. Assume that the equilibrium solution to the IBVP (1.2), (1.17) is smooth in $0 \le t \le T$ for some $T > 0$ and the boundary condition satisfies the structural condition

$$B_v = 0 \quad \text{or} \quad \frac{B_u}{B_v} > 0. \tag{1.25}$$

Then there exists a suitably small T_0, $0 < T_0 \le T$ such that for all ε sufficiently small, the IBVP (1.1), (1.6)–(1.7) has a unique smooth solution in $0 \le t \le T_0$ that satisfies

$$\sup_{0 \le t \le T_0} \|u^\varepsilon(\cdot,t) - u(\cdot,t) - u^{\text{b.l.}}(\cdot/\varepsilon,t)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon \tag{1.26}$$

and

$$\sup_{0 \le t \le T_0} \|v^\varepsilon(\cdot,t) - v(\cdot,t) - v^{\text{i.l.}}(\cdot,t/\varepsilon)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon. \tag{1.27}$$

THEOREM 1.3 (Strong compressive boundary layers). Let $f'(u^*) < 0$. Assume that the equilibrium solution to the IBVP (1.2), (1.17) is smooth in $0 \le t \le T$ for some $T > 0$ and that the boundary condition satisfies the structural condition

$$B_v = 0 \quad \text{or} \quad \frac{B_u}{B_v} > 0. \tag{1.28}$$

Furthermore, we assume that the boundary layer is compressive in the sense that

$$f''(u^*)\partial_\xi u^{\text{b.l.}} \le 0. \tag{1.29}$$

Then for ε sufficiently small, the IBVP (1.1), (1.6)–(1.7) admits a unique smooth solution such that

$$\sup_{0 \le t \le T} \|u^\varepsilon(\cdot,t) - u(\cdot,t) - u^{\text{b.l.}}(\cdot/\varepsilon,t)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon \tag{1.30}$$

and

$$\sup_{0 \le t \le T} \|v^\varepsilon(\cdot,t) - v(\cdot,t) - v^{\text{i.l.}}(\cdot,t/\varepsilon)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon. \tag{1.31}$$

THEOREM 1.4 (Strong expansive boundary layers). Let $f'(u^*) < 0$. Assume that the equilibrium solution to the IBVP (1.2), (1.17) is smooth in $0 \le t \le T$ for some $T > 0$ and that the boundary layer is expansive in the sense that

$$f''(u^*)\partial_\xi u^{\text{b.l.}} \ge 0. \tag{1.32}$$

In addition, we assume that the boundary condition satisfies the structural condition

$$B_-(u^*) < \frac{B_u}{B_v} < B_+(u^*), \tag{1.33}$$

where

$$B_\pm(u^*) = \frac{a}{|f'(u^*)|}\left(1 \pm \sqrt{1 - \frac{f'(u^*)^2}{a}}\right). \tag{1.34}$$

Then for ε sufficiently small, the IBVP (1.1), (1.6)–(1.7) admits a unique smooth solution such that

$$\sup_{0 \le t \le T} \|u^\varepsilon(\cdot,t) - u(\cdot,t) - u^{\text{b.l.}}(\cdot/\varepsilon,t)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon \tag{1.35}$$

and

$$\sup_{0\le t\le T} \|v^\varepsilon(\cdot,t) - v(\cdot,t) - v^{\text{i.l.}}(\cdot,t/\varepsilon)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon. \tag{1.36}$$

THEOREM 1.5 (No boundary layer case). Let $f'(u^*) > 0$. Assume that the equilibrium solution to the IBVP (1.2), (1.17)–(1.18) is smooth in $0 \le t \le T$ for some $T > 0$ and that the boundary condition satisfies the following structural condition:

$$\frac{B_u}{B_v} < -B_+(u^*) \quad \text{or} \quad \frac{B_u}{B_v} > -B_-(u^*), \tag{1.37}$$

with $B_\pm(u^*)$ as defined in (1.34). Then for ε sufficiently small, the IBVP (1.1), (1.6)–(1.7) has a unique smooth solution such that

$$\sup_{0\le t\le T} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon, \tag{1.38}$$

$$\sup_{0\le t\le T} \|v^\varepsilon(\cdot,t) - v(\cdot,t) - v^{\text{i.l.}}(\cdot,t/\varepsilon)\|_{L^\infty(\mathbf{R}^+)} \le O(1)\varepsilon. \tag{1.39}$$

2. Formal expansion theory. In this section we apply the method of matched asymptotic analysis and develop a general expansion theory for the solution of the IBVP (1.1), (1.6)–(1.7). It is known that the solution $(u^\varepsilon, v^\varepsilon)$ generally exhibits sharp changes near $t = 0$ and $x = 0$. These are known as initial and boundary layers—they arise so that the inconsistency of different numbers of initial and boundary conditions required for the whole relaxation system (1.1) ($\varepsilon > 0$) and for the reduced equilibrium system (1.2) ($\varepsilon = 0$) can be resolved. Additionally, the solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) may also involve shock layers if the corresponding equilibrium limit $u(x,t)$ contains shocks [15]. As a result, the asymptotic convergence of (1.1) to (1.2) as $\varepsilon \to 0$, if true, will generally be non-uniform in x and t.

We consider only smooth solutions here. As in [14], we regard the solution $(u^\varepsilon, v^\varepsilon)$ of the IBVP (1.1), (1.6)–(1.7) as a linear super-imposition of an equilibrium limit, an initial layer and a further boundary layer (and their corresponding higher-order contributions), with the initial and boundary layers satisfying exponential decay properties in their respective rescaled variables. Such an asymptotic ansatz is expected to yield a uniformly valid approximation to the solution of the IBVP (1.1), (1.6)–(1.7).

The governing equations together with the appropriate initial and/or boundary conditions for the equilibrium solution, the initial and boundary layers can then be uniquely determined by a simultaneous matching of Eq. (1.1) and the corresponding initial and boundary conditions in (1.6)–(1.7). Higher-order expansion terms can be found similarly. Consequently, accurate approximate solutions can then be constructed by combining various orders of equilibrium solutions, initial and boundary layers. This will be the basis of the stability analysis in the remaining part of this paper.

2.1. *Hilbert expansion.* Away from the boundary $x = 0$ and the initial time $t = 0$, the solution $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$ of (1.1), (1.6)–(1.7) is expected to be normal and may be approximated by a truncation of the formal power series:

$$u^\varepsilon(x,t) \sim u(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots,$$

$$v^\varepsilon(x,t) \sim v(x,t) + \varepsilon v_1(x,t) + \varepsilon^2 v_2(x,t) + \cdots . \tag{2.1}$$

Substituting (2.1) into (1.1) and matching the orders of ε, we obtain

$$v = f(u), \qquad O(\varepsilon^{-1}) \tag{2.2}$$

$$\begin{aligned} \partial_t u + \partial_x v &= 0, \\ \partial_t v + a\partial_x u &= -(v_1 - f'(u)u_1), \end{aligned} \qquad O(1) \tag{2.3}$$

$$\begin{aligned} \partial_t u_1 + \partial_x v_1 &= 0, \\ \partial_t v_1 + a\partial_x u_1 &= -\left(v_2 - f'(u)u_2 - \tfrac{1}{2}f''(u)(u_1,u_1)\right), \end{aligned} \qquad O(\varepsilon) \tag{2.4}$$

$$\partial_t u_2 + \partial_x v_2 = 0. \qquad O(\varepsilon^2) \tag{2.5}$$

From (2.2) and $(2.3)_1$, we see that the leading-order Hilbert solution (u,v) is a local Maxwellian satisfying

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \\ v &= f(u). \end{aligned} \tag{2.6}$$

At the next order, we have

$$\begin{aligned} \partial_t u_1 + \partial_x(f'(u)u_1) &= \partial_x((a - f'(u)^2)\partial_x u), \\ v_1 &= f'(u)u_1 - (a - f'(u)^2)\partial_x u. \end{aligned} \tag{2.7}$$

Similarly, we can find the equations for (u_2, v_2),

$$\begin{aligned} \partial_t u_2 + \partial_x(f'(u)u_2) &= -\tfrac{1}{2}\partial_x(f''(u)(u_1,u_1)) + \partial_x(\partial_t v_1 + a\partial_x u_1), \\ v_2 &= f'(u)u_2 + \tfrac{1}{2}f''(u)(u_1,u_1) - (\partial_t v_1 + a\partial_x u_1). \end{aligned} \tag{2.8}$$

The above equations can be solved recursively provided the necessary initial data and suitable boundary conditions are given (for the u variables only). While it is easy to make the right guess for the initial data $u(x,0)$ by setting

$$u(x,0) = u_0(x), \tag{2.9}$$

it is not at all clear how one should specify the boundary conditions so that the leading-order Hilbert solution (u,v) is the actual relaxation limit of the IBVP (1.1), (1.6)–(1.7). We will come back to this later.

2.2. *Initial layer expansion.* Next we consider the initial layer effect. We introduce the stretched time variable

$$\tau = t/\varepsilon \tag{2.10}$$

and propose the following expansions for the solution $(u^\varepsilon, v^\varepsilon)$:

$$\begin{aligned} u^\varepsilon(x,t) &\sim u(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots \\ &\qquad + u^{\text{i.l.}}(x,\tau) + \varepsilon u_1^{\text{i.l.}}(x,\tau) + \varepsilon^2 u_2^{\text{i.l.}}(x,\tau) + \cdots , \\ v^\varepsilon(x,t) &\sim v(x,t) + \varepsilon v_1(x,t) + \varepsilon^2 v_2(x,t) + \cdots \\ &\qquad + v^{\text{i.l.}}(x,\tau) + \varepsilon v_1^{\text{i.l.}}(x,\tau) + \varepsilon^2 v_2^{\text{i.l.}}(x,\tau) + \cdots , \end{aligned} \tag{2.11}$$

where $(u(x,t), v(x,t))$ and $(u_i(x,t), v_i(x,t))$ are the Hilbert solutions as in the previous subsection. $(u^{\text{i.l.}}(x,\tau), v^{\text{i.l.}}(x,\tau))$ and $(u_i^{\text{i.l.}}(x,\tau), v_i^{\text{i.l.}}(x,\tau))$ are the initial layer solutions and are expected to decay exponentially fast as $\tau \to +\infty$.

Now $t = \varepsilon\tau$ becomes a slow variable. For small t, we can expand the nonlinear function $f(u^\varepsilon)$ as follows:

$$\begin{aligned} f(u^\varepsilon) &= f(u + u^{\text{i.l.}} + \varepsilon u_1 + \varepsilon u_1^{\text{i.l.}} + \varepsilon^2 u_2 + \varepsilon^2 u_2^{\text{i.l.}} + \cdots) \\ &= f(u^{\text{i.l.}} + u_0) + \varepsilon f'(u^{\text{i.l.}} + u_0)(u_1^{\text{i.l.}} + u_1(x,0) + \tau u_t(x,0)) \\ &\quad + \varepsilon^2 f'(u^{\text{i.l.}} + u_0)(u_2^{\text{i.l.}} + u_2(x,0) + \tau u_{1t}(x,0) + \tfrac{1}{2}\tau^2 u_{tt}(x,0)) \\ &\quad + \tfrac{1}{2}\varepsilon^2 f''(u^{\text{i.l.}} + u_0)(u_1^{\text{i.l.}} + u_1(x,0) + \tau u_t(x,0), u_1^{\text{i.l.}} + u_1(x,0) + \tau u_t(x,0)), \\ &\quad + \cdots, \end{aligned} \tag{2.12}$$

where we have implicitly assumed (2.9).

Substituting (2.11) into (1.1), and matching the orders of ε, we obtain the following initial layer equations:

$$\begin{aligned} \partial_\tau u^{\text{i.l.}} &= 0, \\ \partial_\tau v^{\text{i.l.}} + v^{\text{i.l.}} &= f(u^{\text{i.l.}} + u_0) - f(u_0), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \partial_\tau u_1^{\text{i.l.}} &= -\partial_x v^{\text{i.l.}}, \\ \partial_\tau v_1^{\text{i.l.}} + v_1^{\text{i.l.}} &= g_1, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \partial_\tau u_2^{\text{i.l.}} &= -\partial_x v_1^{\text{i.l.}}, \\ \partial_\tau v_2^{\text{i.l.}} + v_2^{\text{i.l.}} &= g_2, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} g_1(x,\tau) = f'(u^{\text{i.l.}} + u_0)(u_1^{\text{i.l.}} + u_1(x,0) + \tau u_t(x,0)) - a\partial_x u^{\text{i.l.}} \\ - f'(u_0)(u_1(x,0) + \tau u_t(x,0)), \end{aligned} \tag{2.16}$$

$$\begin{aligned} g_2(x,\tau) &= f'(u^{\text{i.l.}} + u_0)(u_2^{\text{i.l.}} + u_2(x,0) + \tau u_{1t}(x,0) + \tfrac{1}{2}\tau^2 u_{tt}(x,0)) \\ &\quad + \tfrac{1}{2} f''(u^{\text{i.l.}} + u_0)(u_1^{\text{i.l.}} + u_1(x,0) + \tau u_t(x,0), u_1^{\text{i.l.}} + u_1(x,0) + \tau u_t(x,0)) \\ &\quad - a\partial_x u_1^{\text{i.l.}} - f'(u_0)(u_2(x,0) + \tau u_{1t}(x,0) + \tfrac{1}{2}\tau^2 u_{tt}(x,0)) \\ &\quad - \tfrac{1}{2} f''(u(x,0))(u_1(x,0) + \tau u_t(x,0), u_1(x,0) + \tau u_t(x,0)). \end{aligned} \tag{2.17}$$

From $(2.13)_1$ and the exponential decay of the initial layer $u^{\text{i.l.}}(x,\tau)$ (as $\tau \to +\infty$), it then follows that

$$u^{\text{i.l.}}(x,\tau) \equiv 0. \tag{2.18}$$

With an initial data $v^{\text{i.l.}}(x,0)$ (to be given later), the linear ODE $(2.13)_2$ can be solved explicitly:

$$v^{\text{i.l.}}(x,\tau) = e^{-\tau} v^{\text{i.l.}}(x,0). \tag{2.19}$$

The higher-order initial layers $U_1^{\text{i.l.}}$ and $U_2^{\text{i.l.}}$ can be found similarly. Only the initial data $v_1^{\text{i.l.}}(x,0)$ and $v_2^{\text{i.l.}}(x,0)$ are needed.

2.3. *Boundary layer expansion.* Similarly, in order to accommodate the boundary layer effect, we introduce the stretched space variable

$$\xi = x/\varepsilon \tag{2.20}$$

and consider the following asymptotic expansion near $x = 0$:

$$\begin{aligned} u^\varepsilon(x,t) &\sim u(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots \\ &\quad + u^{\text{b.l.}}(\xi,t) + \varepsilon u_1^{\text{b.l.}}(\xi,t) + \varepsilon^2 u_2^{\text{b.l.}}(\xi,t) + \cdots, \\ v^\varepsilon(x,t) &\sim v(x,t) + \varepsilon v_1(x,t) + \varepsilon^2 v_2(x,t) + \cdots \\ &\quad + v^{\text{b.l.}}(\xi,t) + \varepsilon v_1^{\text{b.l.}}(\xi,t) + \varepsilon^2 v_2^{\text{b.l.}}(\xi,t) + \cdots. \end{aligned} \tag{2.21}$$

Again, $(u(x,t), v(x,t))$ and $(u_i(x,t), v_i(x,t))$ are the same Hilbert solutions as before and the boundary layers $(u^{\text{b.l.}}(\xi,t), v^{\text{b.l.}}(\xi,t))$ ad $(u_i^{\text{b.l.}}(\xi,t), v_i^{\text{b.l.}}(\xi,t))$ are assumed to decay exponentially fast as $\xi \to +\infty$.

Near the boundary $\xi = 0$, x is a slow variable, and the nonlinear term $f(u^\varepsilon)$ can be expanded as follows:

$$\begin{aligned} f(u^\varepsilon(x,t)) &= f(u + u^{\text{b.l.}} + \varepsilon u_1 + \varepsilon u_1^{\text{b.l.}} + \varepsilon^2 u_2 + \varepsilon^2 u_2^{\text{b.l.}} + \cdots) \\ &= f(u^{\text{b.l.}} + u^b) + \varepsilon f'(u^{\text{b.l.}} + u^b)(u_1^{\text{b.l.}} + u_1^b + \xi \partial_x u^b) \\ &\quad + \varepsilon^2 f'(u^{\text{b.l.}} + u^b)\left(u_2^{\text{b.l.}} + u_2^b + \xi \partial_x u_1^b + \tfrac{1}{2}\xi^2 \partial_x^2 u^b\right) \\ &\quad + \tfrac{1}{2}\varepsilon^2 f''(u^{\text{b.l.}} + u^b)(u_1^{\text{b.l.}} + u_1^b + \xi \partial_x u^b, u_1^{\text{b.l.}} + u_1^b + \xi \partial_x u^b) \\ &\quad + \cdots, \end{aligned} \tag{2.22}$$

where $u^b(t) = u(0,t), \partial_x u^b(t) = \partial_x u(0,t)$, etc.

Substituting (2.21) into (1.1), and matching the powers of ε, with the help of (2.22), we can obtain the following boundary layer equations:

$$\begin{aligned} \partial_\xi v^{\text{b.l.}} &= 0, \\ a\partial_\xi u^{\text{b.l.}} + v^{\text{b.l.}} + f(u^b) &= f(u^{\text{b.l.}} + u^b), \end{aligned} \tag{2.23}$$

$$\begin{aligned} \partial_\xi v_1^{\text{b.l.}} + \partial_t u^{\text{b.l.}} &= 0, \\ a\partial_\xi u_1^{\text{b.l.}} &= f'(u^{\text{b.l.}} + u^b) u_1^{\text{b.l.}} + G_1, \end{aligned} \tag{2.24}$$

$$\begin{aligned} \partial_\xi v_2^{\text{b.l.}} + \partial_t u_1^{\text{b.l.}} &= 0, \\ a\partial_\xi u_2^{\text{b.l.}} &= f'(u^{\text{b.l.}} + u^b) u_2^{\text{b.l.}} + G_2, \end{aligned} \tag{2.25}$$

where

$$G_1(\xi,t) = (f'(u^{\text{b.l.}} + u^b) - f'(u^b))(u_1^b + \xi \partial_x u^b) - v_1^{\text{b.l.}} - \partial_t v^{\text{b.l.}} \tag{2.26}$$

and

$$\begin{aligned} G_2(\xi,t) &= (f'(u^{\text{b.l.}} + u^b) - f'(u^b))\left(u_2^b + \xi \partial_x u_1^b + \tfrac{1}{2}\xi^2 \partial_x^2 u^b\right) \\ &\quad + \tfrac{1}{2} f''(u^{\text{b.l.}} + u^b)(u_1^{\text{b.l.}} + u_1^b + \xi \partial_x u^b, u_1^{\text{b.l.}} + u_1^b + \xi \partial_x u^b) \\ &\quad - \tfrac{1}{2} f''(u^b)(u_1^b + \xi \partial_x u^b, u_1^b + \xi \partial_x u^b) - v_2^{\text{b.l.}} - \partial_t v_1^{\text{b.l.}}. \end{aligned} \tag{2.27}$$

From (2.23) and the decay requirement of the boundary layer $v^{\text{b.l.}}(\xi,t) \to 0$ as $\xi \to +\infty$, we get

$$\begin{aligned} a\partial_\xi u^{\text{b.l.}} &= f(u^{\text{b.l.}} + u^b) - f(u^b) = f'(u^b)u^{\text{b.l.}} + \cdots, \\ v^{\text{b.l.}}(\xi,t) &\equiv 0. \end{aligned} \tag{2.28}$$

Equation $(2.28)_1$ is a nonlinear ODE system, and the boundary data $U^{\text{b.l.}}(0,t)$ is needed in order to determine a unique solution. However, the exponential decay property of the solution $u^{\text{b.l.}}(\xi,t)$ as $\xi \to +\infty$ requires that the necessary boundary data $u^{\text{b.l.}}(0,t)$ be chosen from the stable manifold of the fixed point $u^{\text{b.l.}} \equiv 0$. This will be discussed in the next subsection.

2.4. *Solutions of leading-order expansions.* We now turn to the important issue of determining the right initial and/or boundary conditions for the various orders of the Hilbert, initial layer and boundary layer solutions. This is equally important as choosing the right scaling and deriving the corresponding governing equations.

The idea is to match the Hilbert solutions with the initial layers and the boundary layers at $t = 0$ and $x = 0$, respectively, so that the approximation in (2.11) satisfies the initial condition (1.6) and the approximation in (2.21) satisfies the boundary condition (1.7).

We start with the easy case of matching the initial conditions by requiring

$$\begin{aligned} &(u(x,0) + u^{\text{i.l.}}(x,0)) + \varepsilon(u_1(x,0) + u_1^{\text{i.l.}}(x,0)) \\ &\quad + \varepsilon^2(u_2(x,0) + u_2^{\text{i.l.}}(x,0)) + \cdots = u_0(x), \\ &(v(x,0) + v^{\text{i.l.}}(x,0)) + \varepsilon(v_1(x,0) + v_1^{\text{i.l.}}(x,0)) \\ &\quad + \varepsilon^2(v_2(x,0) + v_2^{\text{i.l.}}(x,0)) + \cdots = v_0(x). \end{aligned} \tag{2.29}$$

At the leading order, we have

$$u(x,0) + u^{\text{i.l.}}(x,0) = u_0(x), \qquad v(x,0) + v^{\text{i.l.}}(x,0) = v_0(x). \tag{2.30}$$

With $v = f(u)$ and $u^{\text{i.l.}} \equiv 0$ (see (2.6) and (2.18)), it follows from (2.30) immediately that

$$u(x,0) = u_0(x), \qquad v^{\text{i.l.}}(x,0) = v_0(x) - f(u_0(x)). \tag{2.31}$$

This gives the desired initial data for the leading Hilbert and initial layer solutions. We note that the choice $u(x,0) = u_0(x)$ in (2.9) is indeed the right one. We will consider the boundary conditions for $u(x,t)$ shortly.

Continuing this procedure (matching higher orders of ε in (2.29)), we can find the necessary initial data $u_i(x,0)$ and $v_i^{\text{i.l.}}(x,0)$ for the higher-order Hilbert and initial layer solutions.

With the initial data $v^{\text{i.l.}}(x,0)$ and $v_i^{\text{i.l.}}(x,0)$ determined, we can then solve for the various orders of initial layers rather easily. At the leading order, we have

$$u^{\text{i.l.}}(x,\tau) \equiv 0, \qquad v^{\text{i.l.}}(x,\tau) = e^{-\tau}(v_0(x) - f(u_0(x))). \tag{2.32}$$

The leading-order initial layer vanishes for locally equilibrium initial data, i.e., $U_0(x)$ satisfying $v_0(x) = f(u_0(x))$. The general ith-order initial layer can be represented as

$$\begin{aligned} u_i^{\text{i.l.}}(x,\tau) &= \int_\tau^\infty \partial_x v_{i-1}^{\text{i.l.}}(x,s)\,ds, \\ v_i^{\text{i.l.}}(x,\tau) &= e^{-\tau}\left(v_i^{\text{i.l.}}(x,0) + \int_0^\tau e^s g_i(x,s)\,ds\right). \end{aligned} \tag{2.33}$$

Only the initial data $v_i^{\text{i.l.}}(x,0)$ are needed.

Next we match the boundary conditions by requiring

$$\begin{aligned} &B_u((u(0,t) + u^{\text{b.l.}}(0,t)) + \varepsilon(u_1(0,t) + u_1^{\text{b.l.}}(0,t)) \\ &\quad + \varepsilon^2(u_2(0,t) + u_2^{\text{b.l.}}(0,t)) + \cdots) \\ &\quad + B_v((v(0,t) + v^{\text{b.l.}}(0,t)) + \varepsilon(v_1(0,t) + v_1^{\text{b.l.}}(0,t)) \\ &\quad + \varepsilon^2(v_2(0,t) + v_2^{\text{b.l.}}(0,t)) + \cdots) \\ &= b(t). \end{aligned} \tag{2.34}$$

At the leading order, we have

$$B_u(u^b(t) + u^{\text{b.l.}}(0,t)) + B_v f(u^b(t)) = b(t), \tag{2.35}$$

where we have substituted $v = f(u)$ and $v^{\text{b.l.}} \equiv 0$.

The issue now is to determine the boundary conditions for $u(0,t)$ and $b^{\text{b.l.}}(0,t)$ from (2.35). It looks like (2.35) is under-determined. But it will soon become evident that (2.35) is just the right condition for determining the necessary boundary conditions for $u(0,t)$ and $u^{\text{b.l.}}(0,t)$. We note that the right boundary condition for (2.6) is to prescribe the p incoming waves $u_1, u_2, \ldots, u_p$ at $x = 0$ in terms of the outgoing waves $u_{p+1}, \ldots, u_n$. On the other hand, $u^{\text{b.l.}}(0,t)$ has to lie on the $(n-p)$-dimensional stable manifold of the fixed point solution $u^{\text{b.l.}}(\xi,t) \equiv 0$ of the nonlinear ODE system (2.28).

By introducing

$$\eta = L(u^b)u^{\text{b.l.}}, \qquad u^{\text{b.l.}} = R(u^b)\eta, \tag{2.36}$$

we can rewrite $(2.28)_1$ as

$$a\partial_\xi \eta = \Lambda(u^b)\eta + \cdots \tag{2.37}$$

or

$$a\partial_\xi \eta_i = \lambda_i(u^b)\eta_i + \cdots, \quad 1 \le i \le n. \tag{2.38}$$

Denote the stable manifold of (2.37) by

$$\begin{aligned} \eta_1 &= S_1(\eta_{p+1}, \ldots, \eta_n; u^b), \\ \eta_p &= S_p(\eta_{p+1}, \ldots, \eta_n; u^b). \end{aligned} \tag{2.39}$$

Then, we have

$$\begin{aligned} S(0,\ldots,0;u^b) &= 0, \\ \nabla_{\eta_\alpha} S(0,\ldots,0;u^b) &= 0, \quad p+1 \le \alpha \le n. \end{aligned} \tag{2.40}$$

In addition, we have

$$\nabla_{u^b} S(0,\dots,0;u^b) = 0. \tag{2.41}$$

In order to obtain the necessary boundary data, we only need to solve for $u_1^b,\dots,u_p^b$ and $\eta_{p+1},\dots,\eta_n$ from (2.35), i.e.,

$$B(u_1^b,\dots,u_p^b;\eta_{p+1},\dots,\eta_n) = B_u(u^b + R(u^b)\eta) + B_v f(u^b) = b(t). \tag{2.42}$$

For $b(t)\equiv 0$ and $u_0(x)\equiv 0$, it is easy to see that $u^b\equiv 0, \eta\equiv 0$. Therefore, $u_1^b = \cdots = u_p^b = 0, \eta_{p+1} = \cdots = \eta_n = 0$. To ensure the solvability of the necessary boundary data from (2.42), it suffices to show that the Jacobian matrix does not vanish at the reference state $u^b = 0, \eta = 0$.

Using the properties (1.13), (2.40)–(2.41), one finds that

$$\begin{aligned}\frac{\partial B_\alpha}{\partial u_\beta^b} &= \sum_k B_{u,\alpha k}\left(\delta_{k\beta} + \sum_l \frac{\partial R_{kl}(u^b)}{\partial u_\beta}\eta_l + \sum_{l\le p} R_{kl}(u^b)\frac{\partial S_l}{\partial u_\beta^b}\right)\\ &\quad + \sum_k B_{v,\alpha k}\frac{\partial f_k(u^b)}{\partial u_\beta},\\ \frac{\partial B_\alpha}{\partial \eta_\gamma} &= \sum_k B_{u,\alpha k}\left(\sum_l R_{kl}(u^b)\delta_{l\gamma} + \sum_{l\le p} R_{kl}(u^b)\frac{\partial S_l}{\partial \eta_\gamma}\right),\end{aligned} \tag{2.43}$$

where $1\le\alpha\le n, 1\le\beta\le p, p+1\le\gamma\le n$.

Therefore,

$$\begin{aligned}\left.\frac{\partial B_\alpha}{\partial u_\beta^b}\right|_{u^b=0,\eta=0} &= B_{u,\alpha\beta} + \lambda_\beta B_{v,\alpha\beta},\\ \left.\frac{\partial B_\alpha}{\partial \eta_\gamma}\right|_{u^b=0,\eta=0} &= B_{u,\alpha\gamma},\end{aligned} \tag{2.44}$$

and the Jacobian matrix is

$$J = B_u + B_v\Lambda_+(0), \tag{2.45}$$

where

$$\Lambda_+(0) = \operatorname{diag}\{\lambda_1(0),\dots,\lambda_p(0),0,\dots,0\}. \tag{2.46}$$

That J is nonsingular is a simple consequence of the Stiff Kreiss Condition (1.9). Indeed, by setting $u = u^* = 0$ and $\zeta = 0$, we have

$$G = \Lambda_+(0), \qquad R = I_n, \tag{2.47}$$

and therefore,

$$\det J = \det(B_u + B_v\Lambda_+(0)) = \det(B_uR + B_vRG)_{u=u^*,\zeta=0} \neq 0. \tag{2.48}$$

With the appropriate boundary data, the leading-order Hilbert solution $u(x,t)$ and boundary layer $u^{\text{b.l.}}(\xi,t)$ can then be solved from the following IBVP:

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \\ u(x,0) &= u_0(x), \\ u_1(0,t),\dots,u_p(0,t) &\text{ given} \end{aligned} \tag{2.49}$$

and the nonlinear ODE system

$$\begin{aligned} a\partial_\xi u^{\text{b.l.}} &= f(u^{\text{b.l.}}+u^b) - f(u^b) = f'(u^b)u^{\text{b.l.}} + \cdots, \\ u^{\text{b.l.}}(0,t) &\text{ given,} \end{aligned} \tag{2.50}$$

respectively.

The above initial and boundary data for (2.49) will be smooth and compatible provided that the data for (1.1), i.e., $U_0(x)$ and $b(t)$, are sufficiently smooth and compatible. We assume that the solution to the hyperbolic IBVP (2.49) is smooth for $t \le T$ for some $T > 0$.

2.5. *Higher-order solutions.* We continue to solve for the next order expansions. Again, the main difficulty is with the boundary condition. From (2.7) and (2.24), it is clear that

$$\begin{aligned} v_1 &= f'(u)u_1 - (a - f'(u)^2)\partial_x u, \\ v_1^{\text{b.l.}}(\xi,t) &= \int_\xi^\infty u_t^{\text{b.l.}}(\eta,t)\,d\eta, \\ v_1^{\text{b.l.}}(\xi,t) &\to 0 \quad \text{as } \xi \to \infty, \end{aligned} \tag{2.51}$$

and u_1 has to be solved from the IBVP

$$\begin{aligned} \partial_t u_1 + \partial_x(f'(u)u_1) &= \partial_x((a - f'(u)^2)\partial_x u), \\ u_1(x,0) &= -\partial_x(v_0(x) - f(u_0(x))), \\ \tilde{u}_i \equiv l_i(u^b)\cdot u_1 &\quad \text{to be given at } x=0 \text{ for } 1 \le i \le p, \end{aligned} \tag{2.52}$$

and $u_1^{\text{b.l.}}$ to be solved from the following linear (inhomogeneous) ODE system:

$$\begin{aligned} a\partial_\xi u_1^{\text{b.l.}} &= f'(u^{\text{b.l.}}+u^b)u_1^{\text{b.l.}} + G_1(\xi,t), \\ u_1^{\text{b.l.}}(0,t) &\quad \text{to be given,} \\ u_1^{\text{b.l.}}(\xi,t) &\to 0 \quad \text{as } \xi \to +\infty, \end{aligned} \tag{2.53}$$

where

$$G_1(\xi,t) = (f'(u^{\text{b.l.}}+u^b) - f'(u^b))(u_1^b + \xi\partial_x u^b) - v_1^{\text{b.l.}} \tag{2.54}$$

and $G_1(\xi,t)$ decays exponentially fast as $\xi \to +\infty$.

The necessary boundary data need to be determined such that

$$B_u(u_1 + u_1^{\text{b.l.}}) + B_v(v_1 + v_1^{\text{b.l.}}) = 0 \quad \text{at } x = 0. \tag{2.55}$$

Before we proceed to derive the necessary boundary data, it is important to study the structure of the "stable manifold" of the inhomogeneous linear non-autonomous ODE system (2.53) in order to understand the implication of the decay restriction $u_1^{\text{b.l.}}(\xi,t) \to 0$ as $\xi \to +\infty$ on the choice of boundary data $u_1^{\text{b.l.}}(0,t)$.

For this purpose, we first note that the linear homogeneous part of $(2.53)_1$

$$\begin{aligned} a\partial_\xi u_1^{\text{b.l.}} &= f'(u^{\text{b.l.}} + u^b)u_1^{\text{b.l.}} \\ &= f'(u^b)u_1^{\text{b.l.}} + (f'(u^{\text{b.l.}} + u^b) - f'(u^b))u_1^{\text{b.l.}} \end{aligned} \tag{2.56}$$

is asymptotically constant-coefficient. Its deviation from

$$a\partial_\xi u_1^{\text{b.l.}} = f'(u^b)u_1^{\text{b.l.}} \tag{2.57}$$

is small in the sense that $f'(u^{\text{b.l.}} + u^b) - f(u^b)$ decays exponentially fast as $\xi \to +\infty$.

The fundamental solution matrix $\Phi(\xi, t) = (\phi_1(\xi, t), \ldots, \phi_n(\xi, t))$ of (2.56) has the following structure [2]:

$$\Phi(\xi, t) = P(\xi, t)\Psi(\xi, t), \tag{2.58}$$

where

$$P(\xi, t) = (p_1(\xi, t), \ldots, p_n(\xi, t)), \tag{2.59}$$

$$\begin{aligned} \Psi(\xi, t) &= e^{\frac{1}{a}\int_0^\xi \tilde{\Lambda}(\eta,t)\,d\eta} \\ &= \begin{pmatrix} e^{\frac{1}{a}\int_0^\xi \tilde{\lambda}_1(\eta,t)\,d\eta} & & \\ & \ddots & \\ & & e^{\frac{1}{a}\int_0^\xi \tilde{\lambda}_n(\eta,t)\,d\eta} \end{pmatrix} \end{aligned} \tag{2.60}$$

and

$$\begin{aligned} \tilde{\lambda}_i(\xi, t) &= \lambda_i(u^{\text{b.l.}}(\xi, t) + u^b(t)), \\ \widetilde{\Lambda}(\xi, t) &= \operatorname{diag}\{\tilde{\lambda}_1(\xi, t), \ldots, \tilde{\lambda}_n(\xi, t)\}, \\ p_k(\xi, t) &\to p_k(t) \equiv r_k(u^b(t)) \quad \text{as } \xi \to +\infty. \end{aligned} \tag{2.61}$$

Among the fundamental modes

$$\phi_k(\xi, t) = p_k(\xi, t)e^{\frac{1}{a}\int_0^\xi \tilde{\lambda}_k(\eta,t)\,d\eta}, \quad 1 \le k \le n, \tag{2.62}$$

$\phi_{p+1}, \ldots, \phi_n$ are stable while $\phi_1, \ldots, \phi_p$ are unstable.

With the knowledge of the fundamental solution matrix $\Phi(\xi, t)$, the general solution to

$$a\partial_\xi u_1^{\text{b.l.}} = f'(u^{\text{b.l.}} + u^b)u_1^{\text{b.l.}} + G_1(\xi, t) \tag{2.63}$$

can be obtained by the method of variation of parameters:

$$\begin{aligned} u_1^{\text{b.l.}}(\xi, t) &= \Phi(\xi, t)\left(\alpha + \int_0^\xi \Phi^{-1}(\eta, t)G_1(\eta, t)\,d\eta\right) \\ &= P(\xi, t)\Psi(\xi, t)\left(\alpha + \int_0^\xi \Psi^{-1}(\eta, t)P^{-1}(\eta, t)G_1(\eta, t)\,d\eta\right), \end{aligned} \tag{2.64}$$

where the constant $\alpha \in \mathbf{R}^n$.

In order that $u_1^{\text{b.l.}}(\xi, t) \to 0$ as $\xi \to +\infty$, the constant α has to be chosen such that

$$\alpha_i = -\int_0^\infty e^{-\int_0^\eta \frac{1}{a}\tilde{\lambda}_i(\zeta,t)\,d\zeta}(P^{-1}(\eta, t)G_1(\eta, t))_i\,d\eta, \quad 1 \le i \le p. \tag{2.65}$$

Therefore, the stable solutions can be represented as

$$
\begin{aligned}
u_1^{\text{b.l.}}(\xi,t) = &\sum_{k=p+1}^{n} \alpha_k \phi_k(\xi,t) \\
&- \sum_{k=1}^{p} \phi_k(\xi,t) \int_{\xi}^{\infty} e^{-\frac{1}{a}\int_0^\eta \tilde{\lambda}_k(\zeta,t)\,d\zeta} (P^{-1}(\eta,t)G_1(\eta,t))_k \, d\eta \\
&+ \sum_{k=p+1}^{n} \phi_k(\xi,t) \int_{0}^{\xi} e^{-\frac{1}{a}\int_0^\eta \tilde{\lambda}_k(\zeta,t)\,d\zeta} (P^{-1}(\eta,t)G_1(\eta,t))_k \, d\eta
\end{aligned}
\tag{2.66}
$$

with

$$
\begin{aligned}
u_1^{\text{b.l.}}(0,t) = &\sum_{k=p+1}^{n} \alpha_k \phi_k(0,t) \\
&- \sum_{k=1}^{p} \phi_k(0,t) \int_{0}^{\infty} e^{-\frac{1}{a}\int_0^\eta \tilde{\lambda}_k(\zeta,t)\,d\zeta} (P^{-1}(\eta,t)G(\eta,t))_k \, d\eta,
\end{aligned}
\tag{2.67}
$$

where the free parameters $\alpha_{p+1},\dots,\alpha_n$ can be viewed as the coordinates of the $(n-p)$-dimensional "stable manifold".

Now in order to determine the necessary boundary data for (2.52) and (2.53), we only need to solve for $\tilde{u}_1^b,\dots,\tilde{u}_p^b;\ \alpha_{p+1},\dots,\alpha_n$ from (2.55), i.e.,

$$
\begin{aligned}
&B_1(\tilde{u}_1^b,\dots,\tilde{u}_p^b;\alpha_{p+1},\dots,\alpha_n) \\
&\quad = B_u\left(R(u^b)\tilde{u}^b + \sum_{k=p+1}^{n} \alpha_k\phi_k(0,t) + \cdots\right) + B_v(f'(u^b)R(u^b)\tilde{u}^b + \cdots) \\
&\quad = 0,
\end{aligned}
\tag{2.68}
$$

where we have used (2.67), (2.51) and the relation $u_1 = R(u^b)\tilde{u}$.

At the reference state (u^*,v^*), the Jacobian matrix turns out again to be

$$
J = B_u + B_v\Lambda_+(0). \tag{2.69}
$$

We have already seen that J is nonsingular; see (2.48). Thus, as in the leading-order case, we can determine the necessary boundary data from (2.68) and then solve for the first-order correction terms (u_1,v_1) and $(u_1^{\text{b.l.}},v_1^{\text{b.l.}})$.

It is not hard to see that (2.48) is also the condition for determining the boundary data for all higher-order expansion terms. Details are skipped.

2.6. *Approximate solution.* Based on the above asymptotic expansion results, we now define an approximate solution to the IBVP (1.1), (1.6)–(1.7) by combining the Hilbert,

the initial layer and boundary layer solutions as follows:

$$\begin{aligned}\overline{u}^\varepsilon(x,t) &= (u(x,t)+u^{\text{b.l.}}(\xi,t))\\ &\quad+\varepsilon(u_1(x,t)+u_1^{\text{i.l.}}(\xi,t)+u_1^{\text{b.l.}}(x,\tau))\\ &\quad+\varepsilon^2(u_2(x,t)+u_2^{\text{i.l.}}(\xi,t)+u_2^{\text{b.l.}}(x,\tau)),\\ \overline{v}^\varepsilon(x,t) &= (v(x,t)+v^{\text{i.l.}}(x,\tau))\\ &\quad+\varepsilon(v_1(x,t)+v_1^{\text{i.l.}}(\xi,t)+v_1^{\text{b.l.}}(x,\tau))\\ &\quad+\varepsilon^2(v_2(x,t)+v_2^{\text{i.l.}}(\xi,t)+v_2^{\text{b.l.}}(x,\tau)).\end{aligned} \tag{2.70}$$

From the construction in the previous section, we have

$$\begin{aligned}\partial_t\overline{u}^\varepsilon+\partial_x\overline{v}^\varepsilon &= e_1,\\ \partial_t\overline{v}^\varepsilon+a\partial_x\overline{u}^\varepsilon+\tfrac{1}{\varepsilon}(\overline{v}^\varepsilon-f(\overline{u}^\varepsilon)) &= e_2,\end{aligned} \tag{2.71}$$

where

$$e_1=\varepsilon^2(\partial_x u_2^{\text{i.l.}}+\partial_t u_2^{\text{b.l.}}) \tag{2.72}$$

and

$$\begin{aligned}e_2 &= \tfrac{1}{\varepsilon}(f(u)-f(\overline{u}^\varepsilon))+f'(u)(\varepsilon u_2+u_1)+\tfrac{1}{2}\varepsilon f''(u)(u_1,u_1)\\ &\quad+\tfrac{1}{\varepsilon}(f(u^{\text{i.l.}}+u_0)-f(u_0))+f'((u^{\text{i.l.}}+u_0)u_1^{\text{i.l.}}+u_1(x,0)+\tau u_t(x,0))\\ &\quad-f'(u_0)(u_1(x,0)+\tau u_t(x,0))\\ &\quad+\varepsilon f'(u^{\text{i.l.}}+u_0)(u_2^{\text{b.l.}}+u_2(x,0)+\tau u_{1t}(x,0)+\tfrac{1}{2}\tau^2u_{tt}(x,0))\\ &\quad+\tfrac{1}{2}\varepsilon f''(u^{\text{i.l.}}+u_0)(u_1^{\text{b.l.}}+u_1(x,0)+\tau u_t(x,0),u_1^{\text{b.l.}}+u_1(x,0)+\tau u_t(x,0))\\ &\quad-\varepsilon f'(u_0)(u_2(x,0)+\tau u_{1t}(x,0)+\tfrac{1}{2}\tau^2u_{tt}(x,0))\\ &\quad-\tfrac{1}{2}\varepsilon f''(u_0)(u_1(x,0)+\tau u_t(x,0),u_1(x,0)+\tau u_t(x,0))\\ &\quad+\tfrac{1}{\varepsilon}(f(u^{\text{b.l.}}+u^b)-f(u^b))+f'(u^{\text{b.l.}}+u^b)(u_1^{\text{b.l.}}+u_1^b+\xi\partial_x u^b)\\ &\quad-f'(u^b)(u_1^b+\xi\partial_x u^b)\\ &\quad+\varepsilon f'(u^{\text{b.l.}}+u^b)(u_2^{\text{b.l.}}+u_2^b+\xi\partial_x u_1^b+\tfrac{1}{2}\xi^2\partial_x^2u^b)\\ &\quad+\tfrac{1}{2}\varepsilon f''(u^{\text{b.l.}}+u^b)(u^{\text{b.l.}}+u_1^b+\xi\partial_x u^b,u_1^{\text{b.l.}}+u_1^b+\xi\partial_x u^b)\\ &\quad-\varepsilon f'(u^b)(u_2^b+\xi\partial_x u_1^b+\tfrac{1}{2}\xi^2\partial_x^2u^b)\\ &\quad-\tfrac{1}{2}\varepsilon f''(u^b)(u_1^b+\xi\partial_x u^b,u_1^b+\xi\partial_x u^b).\end{aligned} \tag{2.73}$$

The domain $x\geq 0, 0\leq t\leq T$ can be divided into four parts depending on $x\leq\varepsilon^\gamma$ or $x\geq\varepsilon^\gamma$ and $t\leq\varepsilon^\gamma$ or $t\geq\varepsilon^\gamma$ where the constant γ is chosen such that $5/6<\gamma<1$. The main part is the "normal" regime $x\geq\varepsilon^\gamma$, $t\geq\varepsilon^\gamma$ where both boundary layers and initial layers are exponentially small and therefore (2.70) amounts to the Hilbert expansion (2.1). The part $x\leq\varepsilon^\gamma$, $t\geq\varepsilon^\gamma$ corresponds to the boundary regime and (2.70) becomes essentially the boundary layer expansion (2.21). Similarly, the part $t\leq\varepsilon^\gamma, x\geq\varepsilon^\gamma$ corresponds to the initial regime and (2.70) reduces to the initial layer expansion (2.11). The last part is the corner $0\leq x\leq\varepsilon^\gamma$, $0\leq t\leq\varepsilon^\gamma$ where very complicated interactions between the initial layer and the boundary layer may happen in general. However, by assuming a few compatibility conditions, such complicated interactions can easily be

avoided since neither the boundary layer nor the initial layer has fully developed. Indeed, it can be checked that under the following compatibility conditions

$$\begin{aligned} v_0(0) = f(u_0(0)), \qquad B_u u_0(0) + B_v v_0(0) = b(0), \\ U_0'(0) = U_0''(0) = 0, \qquad b'(0) = b''(0) = 0, \end{aligned} \tag{2.74}$$

we have

$$\begin{aligned} U^{\text{i.l.}}(0,\tau) \equiv 0, \quad \partial_x U^{\text{i.l.}}(0,\tau) \equiv 0, \quad \partial_x^2 U^{\text{i.l.}}(0,\tau) \equiv 0, \\ U_1^{\text{i.l.}}(0,\tau) \equiv 0, \quad \partial_x U_1^{\text{i.l.}}(0,\tau) \equiv 0, \quad U_2^{\text{i.l.}}(0,\tau) \equiv 0 \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} U^{\text{b.l.}}(\xi,0) \equiv 0, \quad \partial_t U^{\text{b.l.}}(\xi,0) \equiv 0, \quad \partial_t^2 U^{\text{b.l.}}(\xi,0) \equiv 0, \\ U_1^{\text{b.l.}}(\xi,0) \equiv 0, \quad \partial_t U_1^{\text{b.l.}}(\xi,0) \equiv 0, \quad U_2^{\text{b.l.}}(\xi,0) \equiv 0. \end{aligned} \tag{2.76}$$

Therefore, for $x \le \varepsilon^\gamma$ and $t \le \varepsilon^\gamma$ we get

$$\begin{aligned} U^{\text{i.l.}}(x,\tau) = O(1)\varepsilon^{3\gamma}, \quad \partial_x U^{\text{i.l.}}(x,\tau) = O(1)\varepsilon^{2\gamma}, \quad \partial_x^2 U^{\text{i.l.}}(x,\tau) = O(1)\varepsilon^{\gamma}, \\ U^{\text{b.l.}}(\xi,t) = O(1)\varepsilon^{3\gamma}, \quad \partial_t U^{\text{b.l.}}(x,\tau) = O(1)\varepsilon^{2\gamma}, \quad \partial_t^2 U^{\text{b.l.}}(x,\tau) = O(1)\varepsilon^{\gamma}. \end{aligned} \tag{2.77}$$

Combining the above analysis, we arrive at the following estimates for the error term $E = (e_1, e_2)$:

$$\begin{aligned} e_1(x,t) = O(1)\varepsilon^2, \qquad e_2(x,t) = O(1)\varepsilon^{3\gamma-1}, \\ \int_0^T \int_0^\infty |E(x,t)|^2 \, dx \, dt \le O(1)\varepsilon^{6\gamma-2}, \\ \int_0^T \int_0^\infty (|\partial_t E(x,t)|^2 + |\partial_x E(x,t)|^2) \, dx \, dt \le O(1)\varepsilon^{6\gamma-4}. \end{aligned} \tag{2.78}$$

In addition, our approximate solution $(\overline{u}^\varepsilon, \overline{v}^\varepsilon)$ also satisfies the required initial and boundary conditions, that is,

$$\begin{aligned} \overline{u}^\varepsilon(x,0) = u_0(x), \qquad \overline{v}^\varepsilon(x,0) = v_0(x), \\ B_u \overline{u}^\varepsilon(0,t) + B_v \overline{v}^\varepsilon(0,t) = b(t). \end{aligned} \tag{2.79}$$

Before we end this section, we further remark that arbitrarily accurate approximate solutions to the IBVP (1.1), (1.6)–(1.7) can be constructed in the same manner by including higher-order Hilbert, initial layer and boundary layer terms (and assuming enough regularity and compatibility conditions on the initial and boundary data).

3. Stability analysis I: $f''(u^*)\partial_\xi u^{\text{b.l.}} \ge 0$. We now turn our attention to the simplest case of $n = 1$ and prove the rigorous convergence results stated in Sec. 1. The proof is based on an error analysis on the difference $(u^\varepsilon - \overline{u}^\varepsilon, v^\varepsilon - \overline{v}^\varepsilon)$, where $(u^\varepsilon, v^\varepsilon)$ is the exact solution of the IBVP (1.1), (1.6)–(1.7) and $(\overline{u}^\varepsilon, \overline{v}^\varepsilon)$ is the approximate solution constructed in the last section.

Two types of stability analysis [5] may be applied depending on whether the boundary layer is expansive or compressive. In this section, we focus on expansive boundary layers, namely, boundary layers that satisfy the monotonicity condition

$$f''(u^*)\partial_\xi u^{\text{b.l.}} \ge 0. \tag{3.1}$$

A direct energy method can then be applied when the boundary condition satisfies

$$B_-(u^*) < \frac{B_u}{B_v} < B_+(u^*) \quad \text{if } f'(u^*) < 0, \tag{3.2}$$

and

$$\frac{B_u}{B_v} < -B_+(u^*) \quad \text{or} \quad \frac{B_u}{B_v} > -B_-(u^*) \quad \text{if } f'(u^*) > 0, \tag{3.3}$$

where

$$B_\pm(u^*) = \frac{a}{|f'(u^*)|}\left(1 \pm \sqrt{1 - \frac{f'(u^*)^2}{a}}\right). \tag{3.4}$$

The restriction (3.2) (or (3.3)) is imposed such that

$$af'(u^*)w^2 - 2awz + f'(u^*)z^2 \geq 0 \quad \text{whenever } B_u w + B_v z = 0. \tag{3.5}$$

We mention that such restrictions on the boundary condition, at least in the linear case [14], are unnecessarily strong. This is one of the main limitations of the energy method. Nevertheless, at least in the case $f'(u^*) > 0$, (3.5) holds for a wide class of boundary conditions including

$$B_v = 0 \quad \text{or} \quad \frac{B_u}{B_v} \geq 0. \tag{3.6}$$

We also note that since we only consider solutions close to the constant state $(u^*, f(u^*))$, the inequalities (1.20), (3.1)–(3.3), and (3.5) remain true if we substitute u^* with $\overline{u}^\varepsilon$. Also, we observe that the monotonicity property (3.1) holds automatically in the case $f'(u^*) > 0$ since $u^{\text{b.l.}} \equiv 0$. The complementary case $f''(u^*)\partial_\xi u^{\text{b.l.}} \leq 0$ will be considered in the next section by using an anti-derivative type method as in [5].

3.1. *Error equation.* We decompose

$$\begin{aligned} u^\varepsilon(x,t) &= \overline{u}^\varepsilon + w(x,t), \\ v^\varepsilon(x,t) &= \overline{v}^\varepsilon + z(x,t). \end{aligned} \tag{3.7}$$

Using (2.71), we obtain the following error equation:

$$\begin{aligned} \partial_t w + \partial_x z &= -e_1, \\ \partial_t z + a\partial_x w &= -\varepsilon^{-1}(z - f(\overline{u}^\varepsilon + w) + f(\overline{u}^\varepsilon)) - e_2. \end{aligned} \tag{3.8}$$

In addition, (w, z) satisfies the homogeneous initial and boundary conditions

$$\begin{aligned} w(x,0) = 0, \qquad z(x,0) = 0, \\ B_u w(0,t) + B_v z(0,t) = 0. \end{aligned} \tag{3.9}$$

For convenience, we rewrite (3.8) in the following matrix form:

$$\partial_t \begin{pmatrix} w \\ z \end{pmatrix} + A\partial_x \begin{pmatrix} w \\ z \end{pmatrix} - \varepsilon^{-1} S \begin{pmatrix} w \\ z \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} 0 \\ Q(w,w) \end{pmatrix} - \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{3.10}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 \\ f'(\overline{u}^\varepsilon) & -1 \end{pmatrix}, \tag{3.11}$$

and

$$Q(w,w) = f(\overline{u}^\varepsilon + w) - f(\overline{u}^\varepsilon) - f'(\overline{u}^\varepsilon)w. \tag{3.12}$$

Note that for w small, we have

$$Q(w,w) = O(1)w^2. \tag{3.13}$$

We shall prove that for ε sufficiently small, the above IBVP (3.10) and (3.9) for the error term $W = (w,z)$ admits a smooth solution up to time T. Furthermore, the solution is small in the sense that

$$\sup_{0\le t\le T} \|W\|_{L^\infty} \le O(1)\varepsilon. \tag{3.14}$$

We note that, due to the smallness of the error term, the nonlinear effect in (3.10) is much weaker than that in the original relaxation model (1.1). This is the main advantage of the error equation (3.10) over the original equation (1.1).

Since (3.10) is a hyperbolic system, the local existence and uniqueness of the solution $W = (w,z)$ for the IBVP (3.10) and (3.9) (for each fixed $\varepsilon > 0$) in the space

$$L^\infty([0,t_0], H^1(\mathbf{R}^+)) \cap C([0,t_0], L^2(\mathbf{R}^+)) \tag{3.15}$$

are consequences of the standard hyperbolic theory. To obtain global existence (up to time T) and the desired convergence estimate, we only need to derive an appropriate a priori estimate (see (3.23) and (3.32)–(3.33) below).

3.2. *A priori estimate.* We assume that the solution W of the IBVP (3.10) and (3.9) satisfies the following a priori estimate:

$$\sup_{0\le t\le t_0} \|W\|_{L^\infty} \le O(1)\varepsilon, \tag{3.16}$$

where $t_0 \le T$ and the constant $O(1)$ is independent of ε.

As in [14], we choose the symmetrizer

$$H = \begin{pmatrix} a & -f'(\overline{u}^\varepsilon) \\ -f'(\overline{u}^\varepsilon) & 1 \end{pmatrix}. \tag{3.17}$$

Under the sub-characteristic condition, H is symmetric positive definite. Furthermore, it can be checked that HA and HS are also symmetric and that HS is semi-negative definite.

Multiplying (3.10) by H, taking the inner product with W, and integrating over $[0,t]\times[0,\infty)$, we get

$$\begin{aligned} &\frac{1}{2}\int_0^\infty (W,HW)(x,t)\,dx - \frac{1}{2}\int_0^t (W,HAW)(0,s)\,ds \\ &\quad + \varepsilon^{-1}\int_0^t\int_0^\infty f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}}(aw^2+z^2)(x,s)\,dx\,ds \\ &\le \varepsilon^{-1}\int_0^t\int_0^\infty |W|\,|Q(w,w)|\,dx\,ds - \int_0^t\int_0^\infty (W,HE)\,dx\,ds \\ &\quad + O(1)\int_0^t\int_0^\infty |W|^2\,dx\,ds, \end{aligned} \tag{3.18}$$

where we have used the estimate

$$\partial_t H + \partial_x HA = -\varepsilon^{-1} f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}} \begin{pmatrix} a & \\ & 1 \end{pmatrix} + O(1). \tag{3.19}$$

As a consequence of the sub-characteristic condition (1.20), the first term in (3.18) is positive definite. The assumptions (3.2) and (3.3) on the boundary condition guarantee that the second term, i.e., the boundary integral in (3.18), is nonnegative. The third term in (3.18) results from the nonlinearity of $f(u)$ and is nonnegative for expansive boundary layers.

With the help of the a priori assumption (3.16), the nonlinear term can be estimated as follows:

$$\begin{aligned} &\varepsilon^{-1}\int_0^t\int_0^\infty |W|\,|Q(w,w)|\,dx\,ds \\ &\quad\le O(1)\varepsilon^{-1}\int_0^t\int_0^\infty |W|\,|W|^2\,dx\,ds \\ &\quad\le O(1)\varepsilon^{-1}\|W\|_{L^\infty}\int_0^t\int_0^\infty |W|^2\,dx\,ds \\ &\quad\le O(1)\int_0^t\int_0^\infty |W|^2\,dx\,ds. \end{aligned} \tag{3.20}$$

On the other hand, the error term can easily be estimated by using the Cauchy-Schwarz inequality

$$\begin{aligned} &\left|\int_0^t\int_0^\infty (W, HE)\,dx\,ds\right| \\ &\quad\le O(1)\int_0^t\int_0^\infty |W|^2\,dx\,ds + O(1)\int_0^t\int_0^\infty |E|^2\,dx\,ds \\ &\quad\le O(1)\int_0^t\int_0^\infty |W|^2\,dx\,ds + O(1)\varepsilon^{6\gamma-2}. \end{aligned} \tag{3.21}$$

Therefore, we obtain

$$\int_0^\infty |W(x,t)|^2\,dx \le O(1)\int_0^t\int_0^\infty |W(x,s)|^2\,dx\,ds + O(1)\varepsilon^{6\gamma-2}. \tag{3.22}$$

Consequently, by Gronwall's inequality, we get the following basic energy estimate:

$$\sup_{0\le t\le t_0} \|W\|_{L^2} \le O(1)\varepsilon^{3\gamma-1}. \tag{3.23}$$

In order to justify the a priori assumption (3.16), we need to derive similar estimates for higher-order derivatives. As usual, we estimate the time derivative first.

Let $\widetilde{W} = \partial_t W$. By differentiating (3.10) with respect to t, we get the following equation for $\widetilde{W}$:

$$\partial_t\widetilde{W} + A\partial_x\widetilde{W} - \varepsilon^{-1}S\widetilde{W} = \varepsilon^{-1}\begin{pmatrix} 0 & 0 \\ \partial_t f'(\overline{u}^\varepsilon) & 0 \end{pmatrix} W + \varepsilon^{-1}\begin{pmatrix} 0 \\ \partial_t Q(w,w) \end{pmatrix} - \partial_t E \tag{3.24}$$

with

$$\begin{aligned} \widetilde{W}(x,0) &= -E(x,0) = O(1)\varepsilon^{3\gamma-1}, \\ B\widetilde{W}(0,t) &= B_u\tilde{w}(0,t) + B_v\tilde{z}(0,t) = 0. \end{aligned} \tag{3.25}$$

The presence of the term $\partial_t f'(\overline{u}^\varepsilon)$ will not add any new difficulty to the estimate of $\widetilde{W}$ since the leading initial layer vanishes in the u-component and therefore we have

$$\partial_t f'(\overline{u}^\varepsilon) = O(1). \tag{3.26}$$

As before, we multiply (3.24) by H, take the inner product with $\widetilde{W}$ and integrate over $[0,t]\times[0,\infty)$. We now get

$$\begin{aligned} &\frac{1}{2}\int_0^\infty (\widetilde{W}, H\widetilde{W})(x,t)\,dx - \frac{1}{2}\int_0^t (\widetilde{W}, HA\widetilde{W})(0,s)\,ds \\ &\quad + \varepsilon^{-1}\int_0^t\int_0^\infty f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}}(a\tilde{w}^2+\tilde{z}^2)(x,s)\,dx\,ds \\ &\le O(1)\varepsilon^{-1}\int_0^t\int_0^\infty |W|\,|\widetilde{W}|\,dx\,ds + \varepsilon^{-1}\int_0^t\int_0^\infty |\widetilde{W}|\,|\partial_t Q(w,w)|\,dx\,ds \\ &\quad - \int_0^t\int_0^\infty (\widetilde{W}, H\partial_t E)\,dx\,ds + O(1)\int_0^t\int_0^\infty |\widetilde{W}|^2\,dx\,ds. \end{aligned} \tag{3.27}$$

Again, the first three terms in (3.27) are all nonnegative. The remaining terms can be estimated as follows:

$$\begin{aligned} &\varepsilon^{-1}\int_0^t\int_0^\infty |W\widetilde{W}|(x,s)\,dx\,ds \\ &\quad \le O(1)\varepsilon^{-2}\int_0^t\int_0^\infty |W(x,s)|^2\,dx\,ds + O(1)\int_0^t\int_0^\infty |\widetilde{W}(x,s)|^2\,dx\,ds \\ &\quad \le O(1)\varepsilon^{6\gamma-4} + O(1)\int_0^t\int_0^\infty |\widetilde{W}(x,s)|^2\,dx\,ds, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\varepsilon^{-1}\int_0^t\int_0^\infty |\widetilde{W}|\,|\partial_t Q(w,w)|\,dx\,ds \\ &\quad \le O(1)\varepsilon^{-1}\int_0^t\int_0^\infty |\widetilde{W}|(|w|^2 + |w|\,|\tilde{w}|)\,dx\,ds \\ &\quad \le O(1)\varepsilon^{-1}\|W\|_{L^\infty}\int_0^t\int_0^\infty (|\widetilde{W}|^2 + |W\widetilde{W}|)\,dx\,ds \\ &\quad \le O(1)\int_0^t\int_0^\infty |\widetilde{W}|^2\,dx\,ds + O(1)\varepsilon^{6\gamma-2}, \end{aligned} \tag{3.29}$$

$$
\begin{aligned}
&\left|\int_0^t \int_0^\infty (\widetilde{W}, H\partial_t E)\, dx\, ds\right| \\
&\quad \le O(1) \int_0^t \int_0^\infty |\widetilde{W}|\, |\partial_t E|\, dx\, ds \\
&\quad \le O(1) \int_0^t \int_0^\infty |\widetilde{W}|^2\, dx\, ds + O(1) \int_0^t \int_0^\infty |\partial_t E|^2\, dx\, ds \\
&\quad \le O(1) \int_0^t \int_0^\infty |\widetilde{W}|^2\, dx\, ds + O(1)\varepsilon^{6\gamma-4}.
\end{aligned} \tag{3.30}
$$

Therefore, we obtain

$$
\int_0^\infty |\widetilde{W}(x,t)|^2\, dx \le O(1) \int_0^t \int_0^\infty |\widetilde{W}|^2\, dx\, ds + O(1)\varepsilon^{6\gamma-4}. \tag{3.31}
$$

Again, by Gronwall's inequality, we get

$$
\sup_{0\le t\le t_0} \|\partial_t W\|_{L^2} = \sup_{0\le t\le t_0} \|\widetilde{W}\|_{L^2} \le O(1)\varepsilon^{3\gamma-2}. \tag{3.32}
$$

Using Eq. (3.10) and the invertibility of the matrix A, we can get similar estimates for $\partial_x W$:

$$
\sup_{0\le t\le t_0} \|\partial_x W\|_{L^2} \le O(1)\varepsilon^{3\gamma-2}. \tag{3.33}
$$

Therefore, by the Sobolev inequality, we get the desired L^∞-norm estimate

$$
\sup_{0\le t\le t_0} \|W\|_{L^\infty} \le \sup_{0\le t\le t_0} \|W\|_{L^2}^{1/2} \|\partial_x W\|_{L^2}^{1/2} \le O(1)\varepsilon^{3\gamma-3/2} \le O(1)\varepsilon. \tag{3.34}
$$

This justifies the a priori assumption (2.21). By the standard continuity argument, we conclude that there exists a unique solution W on the whole interval $[0,T]$ such that $W \in L^\infty([0,T]; H^1(\mathbf{R}^+)) \cap C([0,T]; L^2(\mathbf{R}^+))$. Furthermore, the solution satisfies the estimate (3.16).

Theorem 1.4 and Theorem 1.5 now follow immediately.

4. Stability analysis II: $f''(u^*)\partial_\xi u^{\text{b.l.}} \le 0$. We now consider strong compressive boundary layers and prove Theorem 1.3 in this section. Following [5], we will make a further linear wave correction (m,n) to the approximate solution $(\overline{u}^\varepsilon, \overline{v}^\varepsilon)$ defined in (2.70). The goal is, besides satisfying the required initial and boundary conditions, that the new approximate solution

$$
\begin{aligned}
\tilde{u}^\varepsilon(x,t) &= \overline{u}^\varepsilon(x,t) + m(x,t), \\
\tilde{v}^\varepsilon(x,t) &= \overline{v}^\varepsilon(x,t) + n(x,t)
\end{aligned} \tag{4.1}
$$

will also satisfy the first equation in (1.1) exactly, that is,

$$
\partial_t \tilde{u}^\varepsilon + \partial_x \tilde{v}^\varepsilon = 0. \tag{4.2}
$$

With such an improved approximate solution, we can then carry out the error analysis in a different way by using an anti-derivative method as in [5]. This will yield the desired convergence result in the case $f''(u^*)\partial_\xi u^{\text{b.l.}} \le 0$.

4.1. *Improved approximate solution.* For this purpose, we choose

$$\begin{aligned} &\partial_t m + \gamma_0 \partial_x m = -e_1, \\ &m(x,0) = 0, \qquad m(0,t) = 0, \end{aligned} \tag{4.3}$$

and

$$n = \gamma_0 m, \tag{4.4}$$

where $\gamma_0 > 0$ is any positive constant.

Integrating (4.3) along the characteristics and using (2.72), together with the exponential decay property of the initial and boundary layers, namely,

$$\begin{aligned} &\partial_t u_2^{\text{b.l.}} = O(1)e^{-\alpha_0 x/\varepsilon}, \qquad \partial_x u_2^{\text{i.l.}} = O(1)e^{-\alpha_0 t/\varepsilon}, \\ &\|\partial_x u_2^{\text{i.l.}}\|_{H^1} = O(1)e^{-\alpha_0 t/\varepsilon}, \end{aligned} \tag{4.5}$$

where $\alpha_0 > 0$, we can prove the following.

LEMMA 4.1. Let $U_0 \in H^4, b \in H^3$. The linear hyperbolic wave correction m satisfies the following estimates:

$$\begin{aligned} &\|m\|_{L^\infty} \le O(1)\varepsilon^3, \qquad \|m\|_{L^2} \le O(1)\varepsilon^3, \\ &\|\partial_t m\|_{L^2} \le O(1)\varepsilon^2, \qquad \|\partial_x m\|_{L^2} \le O(1)\varepsilon^2. \end{aligned} \tag{4.6}$$

By decomposing the solution of (1.1), (1.6)–(1.7) as

$$\begin{aligned} u^\varepsilon &= \tilde{u}^\varepsilon(x,t) + \tilde{\phi} = \overline{u}^\varepsilon + m + \tilde{\phi}, \\ v^\varepsilon &= \tilde{v}^\varepsilon(x,t) + \tilde{\psi} = \overline{v}^\varepsilon + n + \tilde{\psi}, \end{aligned} \tag{4.7}$$

we obtain the following equations for $(\tilde{\phi}, \tilde{\psi})$:

$$\begin{aligned} &\partial_t \tilde{\phi} + \partial_x \tilde{\psi} = 0, \\ &\partial_t \tilde{\psi} + a\partial_x \tilde{\phi} + \varepsilon^{-1}(\tilde{\psi} - f(\overline{u}^\varepsilon)\tilde{\phi}) = \varepsilon^{-1} Q(m + \tilde{\phi}, m + \tilde{\phi}) - \tilde{e}_2, \end{aligned} \tag{4.8}$$

where

$$\tilde{e}_2 = e_2 + \partial_t n + a\partial_x m + \varepsilon^{-1}(n - f'(\overline{u}^\varepsilon)m). \tag{4.9}$$

The error term $(\tilde{\phi}, \tilde{\psi})$ satisfies homogeneous initial and boundary conditions,

$$\begin{aligned} &\tilde{\phi}(x,0) = 0, \qquad \tilde{\psi}(x,0) = 0, \\ &B_u \tilde{\phi}(0,t) + B_v \tilde{\psi}(0,t) = 0, \end{aligned} \tag{4.10}$$

and the same estimates for e_2 in (2.78) also hold for $\tilde{e}_2$.

4.2. *Integrated error equation.* We now take advantage of the conservative form of the first equation in (4.8) by introducing

$$\tilde{\varphi}(x,t) = -\int_x^\infty \tilde{\phi}(y,t)\,dy. \tag{4.11}$$

Then,

$$\tilde{\phi} = \tilde{\varphi}_x, \qquad \tilde{\psi} = -\tilde{\varphi}_t \tag{4.12}$$

and (4.8) can be simplified into a single second-order equation for $\tilde{\varphi}$,

$$\tilde{\varphi}_{tt} - a\tilde{\varphi}_{xx} + \varepsilon^{-1}(\tilde{\varphi}_t + f'(\overline{u}^\varepsilon)\tilde{\varphi}_x) = -\varepsilon^{-1}Q(m + \tilde{\varphi}_x, m + \tilde{\varphi}_x) + \tilde{e}_2 \tag{4.13}$$

with

$$\begin{gathered} \tilde{\varphi}(x,0) = 0, \qquad \tilde{\varphi}_t(x,0) = 0, \\ B_u\tilde{\varphi}_x(0,t) - B_v\tilde{\varphi}_t(0,t) = 0. \end{gathered} \tag{4.14}$$

For convenience, we further simplify (4.13) by introducing the scalings

$$\tilde{\varphi}(x,t) = \varepsilon\varphi(y,\tau) \tag{4.15}$$

with

$$y = x/\varepsilon, \qquad \tau = t/\varepsilon. \tag{4.16}$$

Thus, we get

$$\begin{gathered} \varphi_{\tau\tau} - a\varphi_{yy} + \varphi_\tau + f'(\overline{u}^\varepsilon)\varphi_y = -Q(m + \varphi_y, m + \varphi_y) + \varepsilon\tilde{e}_2, \\ \varphi(y,0) = 0, \qquad \varphi_\tau(y,0) = 0, \\ B_u\varphi_y(0,\tau) - B_v\varphi_\tau(0,\tau) = 0. \end{gathered} \tag{4.17}$$

The main result of this section is the following:

PROPOSITION 4.2. Under the conditions

$$f'(u^*) < 0, \tag{4.18}$$

$$f''(u^*)\partial_\xi u^{\text{b.l.}} \le 0, \tag{4.19}$$

and

$$\frac{B_u}{B_v} > 0 \quad \text{or} \quad B_v = 0, \tag{4.20}$$

the IBVP (4.17) has a unique smooth solution up to $\tau = T/\varepsilon$ and the solution satisfies the estimates

$$\sup_{0\le\tau\le T/\varepsilon} \|\varphi\|_{L^\infty} + \|\varphi_y\|_{L^\infty} + \|\varphi_\tau\|_{L^\infty} \le O(1)\varepsilon. \tag{4.21}$$

4.3. *A priori estimate.* We now start to estimate the solution of the above IBVP (4.17) and prove Proposition 4.2 in this subsection. We assume a priori that

$$\sup_{0\le\tau\le\tau_0} \|\varphi\|_{L^\infty} + \|\varphi_y\|_{L^\infty} + \|\varphi_\tau\|_{L^\infty} \le O(1)\varepsilon, \tag{4.22}$$

where $\tau_0 \le T/\varepsilon$ and the constant $O(1)$ is independent of ε.

Multiplying (4.17) with $\varphi + 2\varphi_\tau$ and integrating over $[0,\tau]\times[0,\infty)$, we get

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{2}\varphi^2 + \varphi\varphi_\tau + \varphi_\tau^2 + a\varphi_y^2\right)(y,s)\,dy\,ds \\
&\quad + \int_0^\tau\int_0^\infty (a\varphi_y^2 + 2f'(\overline{u}^\varepsilon)\varphi_y\varphi_\tau + \varphi_\tau^2)(y,s)\,dy\,ds \\
&\quad - \frac{1}{2}\int_0^\tau\int_0^\infty \partial_y f'(\overline{u}^\varepsilon)\varphi^2\,dy\,ds \\
&\quad + \int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi^2 + a\varphi\varphi_y + 2a\varphi_y\varphi_\tau\right)(0,s)\,ds \\
&= -\int_0^\tau\int_0^\infty (\varphi + 2\varphi_\tau)Q(m+\varphi_y, m+\varphi_y)\,dy\,ds \\
&\quad + \varepsilon\int_0^\tau\int_0^\infty (\varphi + 2\varphi_\tau)\tilde{e}_2\,dy\,ds.
\end{aligned} \tag{4.23}$$

The first term in (4.23) is clearly positive definite. By the sub-characteristic condition, the second term is also positive definite. Next we note that

$$\partial_y f'(\overline{u}^\varepsilon) = f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}} + O(1)\varepsilon. \tag{4.24}$$

Therefore, the assumption (4.19) guarantees that the third term in (4.23) is nonnegative (up to an order ε correction).

The boundary integral term deserves a little more attention. Using the boundary condition

$$B_u\varphi_y(0,\tau) - B_v\varphi_\tau(0,\tau) = 0 \tag{4.25}$$

and assuming $B_u \neq 0$ (a consequence of the Stiff Kreiss Condition [14]), we can express $\varphi_y(0,\tau)$ in terms of $\varphi_\tau(0,\tau)$,

$$\varphi_y(0,\tau) = \frac{B_v}{B_u}\varphi_\tau(0,\tau). \tag{4.26}$$

Therefore, we have

$$\begin{aligned}
&\int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi^2 + a\varphi\varphi_y + 2a\varphi_y\varphi_\tau\right)(0,s)\,ds \\
&= \int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi^2 + a\frac{B_v}{B_u}\varphi\varphi_\tau + 2a\frac{B_v}{B_u}\varphi_\tau^2\right)(0,s)\,ds \\
&= \int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi^2 + 2a\frac{B_v}{B_u}\varphi_\tau^2\right)(0,s)\,ds + \frac{a}{2}\frac{B_v}{B_u}\varphi(0,\tau)^2.
\end{aligned} \tag{4.27}$$

Under the conditions (4.18) and (4.20), it is clear that the boundary integral term is also nonnegative.

The nonlinear term can be estimated by using the a priori assumption (4.22)

$$\begin{aligned}
&\left|\int_0^\tau \int_0^\infty (\varphi + 2\varphi_\tau) Q(m+\varphi_y, m+\varphi_y)\, dy\, ds\right| \\
&\quad \le O(1) \int_0^\tau \int_0^\infty |\varphi + 2\varphi_\tau| (m^2 + \varphi_y^2)\, dy\, ds \\
&\quad \le O(1) \|\varphi + 2\varphi_\tau\|_{L^\infty} \int_0^\tau \int_0^\infty (m^2 + \varphi_y^2)\, dy\, ds \\
&\quad \le O(1)\varepsilon^5 + O(1)\varepsilon \int_0^\tau \int_0^\infty \varphi_y^2\, dy\, ds.
\end{aligned} \tag{4.28}$$

For the error term, we have

$$\begin{aligned}
&\varepsilon \left|\int_0^\tau \int_0^\infty (\varphi + 2\varphi_\tau) \tilde{e}_2\, dy\, ds\right| \\
&\quad \le O(1)\varepsilon \int_0^\tau \int_0^\infty (\varphi^2 + \varphi_\tau^2 + \tilde{e}_2^2)\, dy\, ds \\
&\quad \le O(1)\varepsilon \int_0^\tau \int_0^\infty (\varphi^2 + \varphi_\tau^2)\, dy\, ds + O(1)\varepsilon^{6\gamma-1}.
\end{aligned} \tag{4.29}$$

Combining the above, we arrive at

$$\begin{aligned}
&\int_0^\infty (\varphi^2 + \varphi_\tau^2 + \varphi_y^2)(y,\tau)\, dy + \int_0^\tau \int_0^\infty (\varphi_\tau^2 + \varphi_y^2)(y,s)\, dy\, ds \\
&\quad \le O(1)\varepsilon \int_0^\tau \int_0^\infty (\varphi^2 + \varphi_\tau^2 + \varphi_y^2)(y,s)\, dy\, ds + O(1)\varepsilon^{6\gamma-1}.
\end{aligned} \tag{4.30}$$

Applying Gronwall's inequality and noticing $\tau \le \tau_0 \le T/\varepsilon$, we get the desired basic energy estimate

$$\int_0^\infty (\varphi^2 + \varphi_\tau^2 + \varphi_y^2)(y,\tau)\, dy \le O(1)\varepsilon^{6\gamma-1}, \quad \tau \le \tau_0. \tag{4.31}$$

Now we continue with the higher-order estimate. Differentiating (4.17) with respect to τ, we get the following equation for φ_τ:

$$\begin{aligned}
&\varphi_{\tau\tau\tau} - a\varphi_{yy\tau} + \varphi_{\tau\tau} + f'(\overline{u}^\varepsilon)\varphi_{y\tau} \\
&\qquad = -\partial_\tau f'(\overline{u}^\varepsilon)\varphi_y - \partial_\tau Q(m+\varphi_y, m+\varphi_y) + \varepsilon\partial_\tau \tilde{e}_2.
\end{aligned} \tag{4.32}$$

In addition, φ_τ satisfies the following initial and boundary conditions:

$$\varphi_\tau(y,0) = 0, \qquad \varphi_{\tau\tau}(y,0) = \varepsilon\tilde{e}_2(y,0), \tag{4.33}$$

and

$$\varphi_{y\tau}(0,\tau) = \frac{B_v}{B_u}\varphi_{\tau\tau}(0,\tau). \tag{4.34}$$

Multiplying (4.32) by $\varphi_\tau + 2\varphi_{\tau\tau}$, and integrating over the domain $[0,\tau]\times[0,\infty)$, we now get

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{2}\varphi_\tau^2 + \varphi_\tau\varphi_{\tau\tau} + \varphi_{\tau\tau}^2 + a\varphi_{y\tau}^2\right)(y,s)\,dy\,ds \\
&\quad + \int_0^\tau\int_0^\infty (a\varphi_{y\tau}^2 + 2f'(\overline{u}^\varepsilon)\varphi_{y\tau}\varphi_{\tau\tau} + \varphi_{\tau\tau}^2)(y,s)\,dy\,ds \\
&\quad - \frac{1}{2}\int_0^\tau\int_0^\infty \partial_y f'(\overline{u}^\varepsilon)\varphi_\tau^2\,dy\,ds \\
&\quad + \int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi_\tau^2 + a\varphi_\tau\varphi_{y\tau} + 2a\varphi_{y\tau}\varphi_{\tau\tau}\right)(0,s)\,ds \qquad (4.35)\\
&= -\int_0^\tau\int_0^\infty \partial_\tau f'(\overline{u}^\varepsilon)(\varphi_\tau + 2\varphi_{\tau\tau})\varphi_y\,dy\,ds \\
&\quad - \int_0^\tau\int_0^\infty (\varphi_\tau + 2\varphi_{\tau\tau})\partial_\tau Q(m+\varphi_y, m+\varphi_y)\,dy\,ds \\
&\quad + \varepsilon\int_0^\tau\int_0^\infty (\varphi_\tau + 2\varphi_{\tau\tau})\partial_\tau\tilde{e}_2\,dy\,ds + \varepsilon^2\int_0^\infty \tilde{e}_2^2(y,0)\,dy.
\end{aligned}$$

For the same reasons as before, the first two terms in (4.35) are positive definite. Using (4.24) and (4.19), we see that the third term

$$\begin{aligned}
&-\frac{1}{2}\int_0^\tau\int_0^\infty \partial_y f'(\overline{u}^\varepsilon)\varphi_\tau^2\,dy\,ds \\
&\qquad = -\frac{1}{2}\int_0^\tau\int_0^\infty f''(\overline{u}^\varepsilon)\varphi_\xi u^{\text{b.l.}}\varphi_\tau^2\,dy\,ds + O(1)\varepsilon\int_0^\tau\int_0^\infty \varphi_\tau^2\,dy\,ds \qquad (4.36)
\end{aligned}$$

is nonnegative except for an order ε correction.

Next, using (4.34), we can rewrite the boundary integral as

$$\begin{aligned}
&\int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi_\tau^2 + a\varphi_\tau\varphi_{y\tau} + 2a\varphi_{y\tau}\varphi_{\tau\tau}\right)(0,s)\,ds \\
&\quad = \int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi_\tau^2 + a\frac{B_v}{B_u}\varphi_\tau\varphi_{\tau\tau} + 2a\frac{B_v}{B_u}\varphi_{\tau\tau}^2\right)(0,s)\,ds \qquad (4.37)\\
&\quad = \int_0^\tau \left(-\frac{1}{2}f'(\overline{u}^\varepsilon)\varphi_\tau^2 + 2a\frac{B_v}{B_u}\varphi_{\tau\tau}^2\right)(0,s)\,ds + \frac{a}{2}\frac{B_v}{B_u}\varphi_\tau(0,\tau)^2.
\end{aligned}$$

Therefore, the boundary integral term is again nonnegative under the same assumptions (4.18) and (4.20).

Since the nonlinear term only involves u and the leading initial layer vanishes in the u-component, we have

$$\partial_\tau f'(\overline{u}^\varepsilon) = O(1)\varepsilon \qquad (4.38)$$

and therefore,

$$\begin{aligned}
&\left|\int_0^\tau\int_0^\infty \partial_\tau f'(\overline{u}^\varepsilon)(\varphi_\tau + 2\varphi_{\tau\tau})\varphi_y\,dy\,ds\right| \\
&\qquad \le O(1)\varepsilon\int_0^\tau\int_0^\infty (\varphi_\tau^2 + \varphi_{\tau\tau}^2 + \varphi_y^2)\,dy\,ds.
\end{aligned} \qquad (4.39)$$

For the nonlinear term, we note that

$$\begin{aligned}|\partial_\tau Q(m+\varphi_y, m+\varphi_y)| &= \varepsilon|\partial_t Q(m+\varphi_y, m+\varphi_y)| \\ &\le O(1)\varepsilon|m+\varphi_y|^2 + O(1)\varepsilon|m+\varphi_y|\,|m_t+\varphi_{yt}| \\ &\le O(1)\varepsilon|m+\varphi_y|^2 + O(1)|m+\varphi_y|(\varepsilon|m_t| + |\varphi_{y\tau}|).\end{aligned} \tag{4.40}$$

Therefore, we get

$$\begin{aligned}&\left|\int_0^\tau\int_0^\infty (\varphi_\tau + 2\varphi_{\tau\tau})\partial_\tau Q(m+\varphi_y, m+\varphi_y)\,dy\,ds\right| \\ &\quad\le O(1)\|m+\varphi_y\|_{L^\infty}\int_0^\tau\int_0^\infty |\varphi_\tau + 2\varphi_{\tau\tau}|(\varepsilon|m_t| + \varepsilon|m+\varphi_y| + |\varphi_{y\tau}|)\,dy\,ds \\ &\quad\le O(1)\varepsilon\int_0^\tau\int_0^\infty (\varphi_\tau^2 + \varphi_y^2 + \varphi_{\tau\tau}^2 + \varphi_{y\tau}^2)\,dy\,ds + O(1)\varepsilon^5,\end{aligned} \tag{4.41}$$

where we have used the estimates in (4.6) and the a priori assumption (4.22).

For the error term, we have

$$\begin{aligned}&\varepsilon\left|\int_0^\tau\int_0^\infty (\varphi_\tau + 2\varphi_{\tau\tau})\partial_\tau\tilde{e}_2\,dy\,ds\right| \\ &\quad\le O(1)\varepsilon\int_0^\tau\int_0^\infty (\varphi_\tau^2 + \varphi_{\tau\tau}^2 + (\partial_\tau\tilde{e}_2)^2)\,dy\,ds \\ &\quad\le O(1)\int_0^\tau\int_0^\infty (\varphi_\tau^2 + \varphi_{\tau\tau}^2)\,dy\,ds + O(1)\varepsilon^{6\gamma-3}.\end{aligned} \tag{4.42}$$

For the last term in (4.35), we have

$$\varepsilon^2\int_0^\infty \tilde{e}_2^2(\varepsilon y, 0)\,dy \le \varepsilon\int_0^\infty \tilde{e}_2^2(x,0)\,dx \le O(1)\varepsilon^{6\gamma-1}. \tag{4.43}$$

Summarizing the above and noticing the basic energy estimate in (4.31), we get

$$\begin{aligned}&\int_0^\infty (\varphi_{\tau\tau}^2 + \varphi_{y\tau}^2)(y,\tau)\,dy + \int_0^\tau\int_0^\infty (\varphi_{y\tau}^2 + \varphi_{\tau\tau}^2)(y,s)\,dy\,ds \\ &\quad\le O(1)\varepsilon\int_0^\tau\int_0^\infty (\varphi_{y\tau}^2 + \varphi_{\tau\tau}^2)(y,s)\,dy\,ds + O(1)\varepsilon^{6\gamma-3}.\end{aligned} \tag{4.44}$$

Therefore, using Gronwall's inequality, we obtain

$$\int_0^\infty (\varphi_\tau^2 + \varphi_{\tau\tau}^2 + \varphi_{y\tau}^2)(y,\tau)\,dy \le O(1)\varepsilon^{6\gamma-3}. \tag{4.45}$$

Finally, by using the above estimates and Eq. (4.17), we can get a simular estimate for φ_{yy}:

$$\int_0^\infty \varphi_{yy}^2(y,\tau)\,dy \le O(1)\varepsilon^{6\gamma-3}. \tag{4.46}$$

The a priori assumption (4.22) can now be justified by using the Sobolev inequality and the estimates in (4.31), (4.45), and (4.46):

$$\sup_{0\le\tau\le\tau_0}\|\varphi\|_{L^\infty} \le \sup_{0\le\tau\le\tau_0}\|\varphi\|_{L^2}^{1/2}\|\varphi_y\|_{L^2}^{1/2} \le O(1)\varepsilon^{3\gamma-1} \le O(1)\varepsilon, \tag{4.47}$$

$$\sup_{0\le\tau\le\tau_0}\|\varphi_\tau\|_{L^\infty}\le\sup_{0\le\tau\le\tau_0}\|\varphi_\tau\|_{L^2}^{1/2}\|\varphi_{y\tau}\|_{L^2}^{1/2}\le O(1)\varepsilon^{3\gamma-1}\le O(1)\varepsilon, \tag{4.48}$$

and

$$\sup_{0\le\tau\le\tau_0}\|\varphi_y\|_{L^\infty}\le\sup_{0\le\tau\le\tau_0}\|\varphi_y\|_{L^2}^{1/2}\|\varphi_{yy}\|_{L^2}^{1/2}\le O(1)\varepsilon^{3\gamma-1}\le O(1)\varepsilon. \tag{4.49}$$

The proof of Proposition 4.2 is now complete. As a consequence, we get

$$\sup_{0\le t\le T}\|(\tilde\phi,\tilde\psi)\|_{L^\infty}=\sup_{0\le\tau\le T/\varepsilon}\|(\varphi_y,\varphi_\tau)\|_{L^\infty}\le O(1)\varepsilon. \tag{4.50}$$

Theorem 1.3 now follows easily.

5. Weak boundary layers. The proof in the above section can easily be modified to show the stability of weak boundary layers without structural assumptions. The crucial step is to show that the a priori estimate in Proposition 4.2 (see Sec. 4.2) still holds for weak boundary layers without the compressibility condition (4.19).

Checking the proof of Proposition 4.2, we see that the compressibility condition (4.19) was only used to bound the term $-\frac{1}{2}\int_0^\tau\int_0^\infty f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}}\varphi^2\,dy\,ds$ in (4.23) and a similar term in (4.35). To prove Theorem 1.1, we only have to bound these two terms without using the monotonicity condition (4.19).

Let $\delta(t)$ denote the strength of the boundary layer, i.e.,

$$\delta(t)=|u^{\text{b.l.}}(0,t)|. \tag{5.1}$$

Then for $\delta(t)$ sufficiently small, the boundary layer $u^{\text{b.l.}}(\xi,t)$ always exists and decays exponentially fast as $\xi\to+\infty$. Hence for $0\le t\le T$, we have

$$\sup_{\xi\ge0}|u^{\text{b.l.}}(\xi,t)|\le O(1)\delta(t), \tag{5.2}$$

$$\int_0^\infty|\partial_\xi u^{\text{b.l.}}(\xi,t)|(1+\xi)\,d\xi\le O(1)\delta(t). \tag{5.3}$$

On the other hand, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\varphi(y,\tau)^2&=\left|\varphi(0,\tau)+\int_0^y\varphi_y(z,\tau)\,dz\right|^2\\&\le2|\varphi(0,\tau)|^2+2\left|\int_0^y\varphi_y(z,\tau)\,dz\right|^2\\&\le2|\varphi(0,\tau)|^2+2y\int_0^\infty\varphi_y^2(z,\tau)\,dz.\end{aligned} \tag{5.4}$$

Therefore,

$$
\begin{aligned}
&\left| -\frac{1}{2}\int_0^\tau \int_0^\infty f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}}\varphi^2\,dy\,ds \right| \\
&\quad \le O(1)\int_0^\tau \int_0^\infty |\partial_\xi u^{\text{b.l.}}(y,s)|\,|\varphi(0,s)|^2\,dy\,ds \\
&\qquad + O(1)\int_0^\tau \int_0^\infty \int_0^\infty y|\partial_\xi u^{\text{b.l.}}(y,s)|\varphi_y^2(z,s)\,dy\,dz\,ds \qquad (5.5)\\
&\quad \le O(1)\max_{0\le t\le T}\delta(t)\int_0^\tau |\varphi(0,s)|^2\,ds \\
&\qquad + O(1)\max_{0\le t\le T}\delta(t)\int_0^\tau \int_0^\infty \varphi_y^2(y,s)\,dy\,ds.
\end{aligned}
$$

This is clearly under control if δ_0 is small enough.

The other term $-\frac{1}{2}\int_0^\tau \int_0^\infty f''(\overline{u}^\varepsilon)\partial_\xi u^{\text{b.l.}}\varphi_y^2\,dy\,ds$ in (4.35) can be estimated similarly. The proof of Theorem 1.1 is considered complete.

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