SINGLE-POINT BLOW-UP FOR A DEGENERATE PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE

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Abstract. Let \( q \) be a nonnegative real number, and \( T \) be a positive real number. This article studies the following degenerate semilinear parabolic first initial-boundary value problem:

\[
\begin{align*}
x^q u_t(x,t) - u_{xx}(x,t) &= a^2 \delta(x - b) f(u(x,t)) \quad \text{for } 0 < x < 1, 0 < t \leq T, \\
u(x,0) &= \psi(x) \quad \text{for } 0 \leq x \leq 1, \\
u(0,t) &= u(1,t) = 0 \quad \text{for } 0 < t \leq T,
\end{align*}
\]

where \( \delta(x) \) is the Dirac delta function, and \( f \) and \( \psi \) are given functions. It is shown that the problem has a unique solution before a blow-up occurs, \( u \) blows up in a finite time, and the blow-up set consists of the single point \( b \). A lower bound and an upper bound of the blow-up time are also given. To illustrate our main results, an example is given. A computational method is also given to determine the finite blow-up time.

1. Introduction. Let \( a, \sigma, q \) and \( \beta \) be constants with \( a > 0, \sigma > 0, q \geq 0, \) and \( 0 < \beta < a \). Let us consider the following degenerate semilinear parabolic first initial-boundary value problem,

\[
\left\{ \begin{array}{l}
\varepsilon^q u_{\gamma} - u_{\varepsilon\varepsilon} = \delta(\varepsilon - \beta) F(u(\varepsilon, \gamma)) \quad \text{in } (0,a) \times (0,\sigma), \\
u(\varepsilon, 0) = \psi(\varepsilon) \quad \text{on } [0,a], \\
u(0, \gamma) = u(a, \gamma) = 0 \quad \text{for } 0 < \gamma \leq \sigma,
\end{array} \right. \tag{1.1}
\]

where \( \delta(x) \) is the Dirac delta function, and \( F \) and \( \psi \) are given functions. This model is motivated by applications in which the ignition of a combustible medium is accomplished through the use of either a heated wire or a pair of small electrodes to supply a large
amount of energy to a very confined area. When \( q = 1 \), the model may also be used to
describe the temperature \( u \) of the channel flow of a fluid with temperature-dependent
viscosity in the boundary layer (cf. Chan and Kong [2]) with a concentrated nonlinear
source at \( \beta \); here, \( \zeta \) and \( \gamma \) denote the coordinates perpendicular and parallel to the
channel wall respectively. When \( q = 0 \), it can be used to describe the temperature of a
one-dimensional strip of a finite width that contains a concentrated nonlinear source at \( \beta \).
The case \( q = 0 \) was studied by Olmstead and Roberts [7] by analyzing its corresponding
nonlinear Volterra equation of the second kind at the site of the concentrated source. A
problem due to a source with local and nonlocal features was also studied by Olmstead
and Roberts [8] by analyzing a pair of coupled nonlinear Volterra equations with different
kernels. When the nonlinear source term in the problem (1.1) is replaced by \( u^p \), the blow-
up of the solution was studied by Floater [4] for the case \( 1 < p \leq q + 1 \), and by Chan
and Liu [3] for the case \( p > q + 1 \).

Let \( \zeta = ax, \gamma = u^q + 2t, \beta = ab, Lu = x^q u_t - u_{xx}, f(u(x,t)) = F(u(\zeta, \gamma)). D = (0,1), \nabla = [0,1], \) and \( \Omega = D \times (0, T] \). Then, the above system is transformed into the following
problem:

\[
\begin{align*}
&Lu = a^2 \delta(x - b)f(u(x,t)) \quad \text{in } \Omega, \\
&u(x,0) = \psi(x) \quad \text{on } \nabla, \\
&u(0,t) = u(1,t) = 0 \quad \text{for } 0 < t < T.
\end{align*}
\]

(1.2)

with \( 0 < b < 1 \), and \( T = \sigma/a^q + 2 \). We assume that \( f(0) \geq 0, f(u) \) and its derivatives \( f'(u) \)
and \( f''(u) \) are positive for \( u > 0 \), and \( \psi(x) \) is nontrivial, nonnegative, and continuous
such that \( \psi(b) > 0, \psi(0) = 0 = \psi(1) \), and

\[
\psi'' + a^2 \delta(x - b)f(u) \leq 0 \quad \text{in } D.
\]

(1.3)

This condition (1.3) is used to show that before \( u \) blows up \( u \) is a nondecreasing function
of \( t \). Instead of the condition (1.3), Olmstead and Roberts [7] assumed that \( h(t) = \int_{b}^{1} g(b,t; \xi,0) \psi(\xi) d\xi, \) where \( g(x,t; \xi, \tau) \) denotes Green’s function corresponding to the
heat operator \( \partial/\partial t - \partial^2/\partial x^2 \) with first boundary conditions, was sufficiently smooth
such that \( h'(t) \geq 0, 0 < h_0 \leq h(t) \leq h_\infty < \infty \) for some positive constants \( h_0 \) and
\( h_\infty \); these were used to show that \( u(b,t) \) and its derivative with respect to \( t \) were positive
for \( t > 0 \).

A solution of the problem (1.2) is a continuous function satisfying (1.2).

A solution \( u \) of the problem (1.2) is said to blow up at the point \( (x_b, t_b) \) if there exists
a sequence \( \{(x_n, t_n)\} \) such that \( u(x_n, t_n) \to \infty \) as \( (x_n, t_n) \to (x_b, t_b) \).

In Sec. 2, we convert the problem (1.2) into a nonlinear integral equation. We prove
that the integral equation has a unique continuous and positive solution \( U(b, t) \) at the
site of the concentrated source. We then show that \( U(x, t) \) is a nondecreasing function of
\( t \). These are used to prove that the problem (1.2) has a unique solution \( u \). We also show
that \( u(b, t) \) blows up if \( \psi \) attains its maximum at \( b \) and \( u(b, t) \) ceases to exist at a finite
time. In Sec. 3, we show that \( b \) is the single blow-up point. We then give a criterion
for \( u \) to blow up at a finite time, and use the method of Olmstead and Roberts [7] to
establish a lower bound and an upper bound for the finite blow-up time. We remark
that \( \psi \) attaining its maximum at \( b \) is used as a sufficient condition for \( u \) to blow up at
Whether it is a necessary one remains as an open question. To illustrate our main results, an example is given in Sec. 4. We also give a computational method to find the finite blow-up time.

2. Existence and uniqueness. Green’s function $G(x, t; \xi, \tau)$ corresponding to the problem (1.2) is determined by the following system: for $x$ and $\xi$ in $D$, and $t$ and $\tau$ in $(-\infty, \infty)$,

$$LG(x, t; \xi, \tau) = \delta(x - \xi)\delta(t - \tau),$$

$$G(x, t; \xi, \tau) = 0, \quad t < \tau,$$

$$G(0, t; \xi, \tau) = G(1, t; \xi, \tau) = 0.$$

By Chan and Chan [1],

$$G(x, t; \xi, \tau) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(\xi)e^{-\lambda_i(t-\tau)},$$

where $\lambda_i(i = 1, 2, 3, \ldots)$ are the eigenvalues of the Sturm-Liouville problem,

$$\phi'' + \lambda x^q \phi = 0, \quad \phi(0) = 0 = \phi(1),$$

and their corresponding eigenfunctions are given by

$$\phi_i(x) = \left(q + 2\right)^{1/2}x^{1/2}\left(J_{1/2}\left(2\lambda_i^{1/2}\frac{q+2}{q+2}x^{(q+2)/2}\right)\right)\left|\frac{J_{1/2}\left(2\lambda_i^{1/2}\frac{q+2}{q+2}\right)}{J_{1+1/2}\left(2\lambda_i^{1/2}\frac{q+2}{q+2}\right)}\right|$$

with $J_{1/2}(q+2)$ denoting the Bessel function of the first kind of order $1/(q+2)$. From Chan and Chan [1], $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_i < \lambda_{i+1} < \cdots$. The set $\{\phi_i(x)\}$ is a maximal (that is, complete) orthonormal set with the weight function $x^q$ (cf. Gustafson [6, p. 176]).

To derive the integral equation from the problem (1.2), let us consider the adjoint operator $L^*$, which is given by $L^*u = -x^qu_t - u_{xx}$. Using Green’s second identity, we obtain

$$U(x, t) = a^2 \int_0^t \int_0^1 G(x, t; b, \tau)f(U(b, \tau))d\tau + \int_0^1 \xi^qG(x, t; \xi, 0)\psi(\xi)d\xi. \quad (2.3)$$

For ease of reference, let us state below Lemmas 1(a), 1(b), 1(d), and 4 of Chan and Chan [1] as Lemma 2.1(a), 2.1(b), 2.1(c), and 2.1(d) respectively.

**Lemma 2.1.** (a) For some positive constant $c_1$, $|\phi_i(x)| \leq c_1 x^{-q/4}$ for $x \in (0, 1]$.
(b) For some positive constant $c_2$, $|\phi_i(x)| \leq c_2 x^{1/2} \lambda_i^{1/4}$ for $x \in D$.
(c) For any $x_0 > 0$ and $x \in [x_0, 1]$, there exists some positive constant $c_3$ depending on $x_0$ such that $|\phi_i'(x)| \leq c_3 \lambda_i^{1/2}$.
(d) In $\{(x, t; \xi, \tau) \in D, T \geq t > \tau \geq 0\}$, $G(x, t; \xi, \tau)$ is positive.

**Lemma 2.2.** (a) For $(x, t; \xi, \tau) \in (\overline{D} \times (\tau, T]) \times (\overline{D} \times [0, T])$, $G(x, t; \xi, \tau)$ is continuous.
(b) For each fixed $(\xi, \tau) \in \overline{D} \times [0, T)$, $G_t(x, t; \xi, \tau) \in C(\overline{D} \times (\tau, T])$. 
(c) For each fixed $(\xi, \tau) \in \overline{D} \times [0, T)$, $G_x(x; t; \xi, \tau)$ and $G_{xx}(x; t; \xi, \tau)$ are in $C([0, 1] \times (\tau, T])$.

(d) If $r \in C([0, T])$, then $\int_0^T G(x; t; b, \tau) r(\tau) d\tau$ is continuous for $x \in \overline{D}$ and $t \in [0, T]$.

Proof. (a) By Lemma 2.1(b),

$$|G(x; t; \xi, \tau)| \leq c_2^2 \sum_{i=1}^{\infty} \lambda_i^{1/2} e^{-\lambda_i(t-\tau)},$$

which converges uniformly for $t$ in any compact subset of $(\tau, T)$. The result then follows.

(b) By Lemma 2.1(b),

$$\left| \sum_{i=1}^{\infty} \frac{\partial}{\partial t} \phi_i(x) \phi_i(\xi) e^{-\lambda_i(t-\tau)} \right| \leq \sum_{i=1}^{\infty} |\phi_i(x)| \lambda_i |e^{-\lambda_i(t-\tau)}| \leq c_2^2 \sum_{i=1}^{\infty} \lambda_i^{3/2} e^{-\lambda_i(t-\tau)}, (2.4)$$

which converges uniformly with respect to $x \in \overline{D}$ and $t$ in any compact subset of $(\tau, T)$. This proves Lemma 2.2(b).

(c) By Lemma 2.1(b) and (c),

$$\left| \sum_{i=1}^{\infty} \frac{\partial}{\partial x} \phi_i(x) \phi_i(\xi) e^{-\lambda_i(t-\tau)} \right| \leq \sum_{i=1}^{\infty} |\phi_i(x)| \lambda_i |e^{-\lambda_i(t-\tau)}| \leq c_2 c_3 \sum_{i=1}^{\infty} \lambda_i^{3/4} e^{-\lambda_i(t-\tau)}, (2.5)$$

which converges uniformly with respect to $x$ in any compact subset of $(0, 1]$ and $t$ in any compact subset of $(\tau, T]$.

Since $\phi_i$ is an eigenfunction, it follows from Lemma 2.1(b) that

$$\left| \sum_{i=1}^{\infty} \frac{\partial^2}{\partial x^2} \phi_i(x) \phi_i(\xi) e^{-\lambda_i(t-\tau)} \right| \leq \sum_{i=1}^{\infty} |\phi_i''(x)| \lambda_i |e^{-\lambda_i(t-\tau)}| = \sum_{i=1}^{\infty} \lambda_i |e^{-\lambda_i(t-\tau)}| \leq c_2^2 \sum_{i=1}^{\infty} \lambda_i^{3/2} e^{-\lambda_i(t-\tau)}, (2.6)$$

which converges uniformly with respect to $x$ in any compact subset of $(0, 1]$ and $t$ in any compact subset of $(\tau, T]$.

Lemma 2.1(c) is then proved.

(d) Let $\epsilon$ be any positive number such that $t-\epsilon > 0$. For any $x \in \overline{D}$, and $\tau \in [0, t-\epsilon]$, it follows from Lemma 2.1(a) and (b) that

$$\sum_{i=1}^{\infty} \phi_i(x) \phi_i(b) e^{-\lambda_i(t-\tau)} r(\tau) \leq c_1 c_2 b^{-q/4} \left( \max_{0 \leq \tau \leq T} r(\tau) \right) \sum_{i=1}^{\infty} \lambda_i^{1/4} e^{-\lambda_i \epsilon},$$
which converges uniformly. By the Weierstrass M-Test,
\[
\int_0^{t-\epsilon} G(x, t; b, \tau) r(\tau) d\tau = \sum_{i=1}^{\infty} \int_0^{t-\epsilon} \phi_i(x) \phi_i(b) e^{-\lambda_i(t-\tau)} r(\tau) d\tau.
\]
By Lemma 2.1(a) and (b),
\[
\sum_{i=1}^{\infty} \int_0^{t-\epsilon} \phi_i(x) \phi_i(b) e^{-\lambda_i(t-\tau)} r(\tau) d\tau \leq c_1 c_2 b^{-q/4} \left( \max_{0 \leq \tau \leq T} \lambda_i^{1/4} e^{-\lambda_i(t-\tau)} \right) \sum_{i=1}^{\infty} \int_0^{t-\epsilon} \lambda_i^{-3/4} (e^{-\lambda_i \epsilon} - e^{-\lambda_i t})
\]
\[
\leq c_1 c_2 b^{-q/4} \left( \max_{0 \leq \tau \leq T} \lambda_i^{-3/4} \right).
\]
(2.7)

which converges (uniformly with respect to \( x, t, \) and \( \epsilon \)) since \( O(\lambda_i) = O(i^2) \) for large \( i \) (cf. Watson [12, p. 506]). Since (2.7) also holds for \( \epsilon = 0 \), it follows that
\[
\sum_{i=1}^{\infty} \int_0^{t-\epsilon} \phi_i(x) \phi_i(b) e^{-\lambda_i(t-\tau)} r(\tau) d\tau
\]
is a continuous function of \( x, t, \) and \( \epsilon (\geq 0) \). Therefore,
\[
\int_0^t G(x, t; b, \tau) r(\tau) d\tau = \lim_{\epsilon \to 0} \sum_{i=1}^{\infty} \int_0^{t-\epsilon} \phi_i(x) \phi_i(b) e^{-\lambda_i(t-\tau)} r(\tau) d\tau
\]
is a continuous function of \( x \) and \( t \).

Let us consider the problem,
\[
Lv = 0 \quad \text{in } \Omega,
\]
\[
v(x, 0) = \psi(x) \quad \text{on } \overline{D},
\]
\[
v(0, t) = \psi(t) = 0 \quad \text{for } 0 < t \leq T,
\]
which has a unique classical solution
\[
v(x, t) = \int_0^1 \xi^q G(x, t; \xi, 0) \psi(\xi) d\xi
\]
(cf. Chan and Chan [1]). Since the strong maximum principle holds for the operator \( L \) (cf. Friedman [5, p. 39]), and \( \psi(x) \) is nontrivial, nonnegative and continuous, it follows that \( v > 0 \) in \( \Omega \), and attains its maximum \( \max_{x \in \overline{D}} \psi(x) \) (denoted by \( k_1 \)) somewhere in \( D \times \{0\} \).

From (2.3),
\[
U(b, t) = a^2 \int_0^t G(b, t; b, \tau) f(U(b, \tau)) d\tau + \int_0^1 \xi^q G(b, t; \xi, 0) \psi(\xi) d\xi.
\]
(2.8)
By Lemma 2.2(d), we can look for a continuous function \( U(b, t) \) satisfying (2.8). From Chan and Chan [1],
\[
\lim_{t \to 0} \int_0^1 \xi^q G(b, t; \xi, 0) \psi(\xi) d\xi = \psi(b).
\]
Thus from (2.8), \( U(b, 0) = \psi(b) > 0 \).

Let us show that there exists some \( t_1 \) such that

\[
\psi(b) \leq U(b, t) \quad \text{for } 0 \leq t \leq t_1. 
\]

(2.9)

Since

\[
L(\psi - u) \leq a^2 \delta(x - b)(f(\psi) - f(u)) \quad \text{in } \Omega,
\]

and \( \psi - u = 0 \) on \( \partial \Omega \), it follows from (2.8) that

\[
\psi(b) - U(b, t) \leq a^2 \int_0^t G(b, t; b, \tau)f'(\eta)(\psi(b) - U(b, \tau))d\tau
\]

for some \( \eta \) between \( \psi(b) \) and \( U(b, t) \). Since \( G(x, t; \xi, \tau) \) is nonnegative and integrable over \([0, t]\), it follows that for any \( t_2 \), there exists some \( \rho \) such that for any \( t \in (t_2, t_2 + \rho] \),

\[
a^2 f'(\psi(b)) \int_{t_2}^t G(b, t; b, \tau)d\tau < 1.
\]

We also note that \( U(b, 0) > 0 \). Suppose there exists some \( t_3 \) such that \( \psi(b) > U(b, t) \geq 0 \) for \( t \in (0, t_3) \). Let \( t_1 = \min\{\rho, t_3\} \). From (2.10), we have

\[
\psi(b) - U(b, t) \leq a^2 \left( \int_0^t G(b, t; b, \tau)f'(\eta)d\tau \right) \max_{0 \leq \tau \leq t_1} (\psi(b) - U(b, t)).
\]

This gives a contradiction. Thus, we have (2.9).

It follows from (2.8), \( f(0) \geq 0 \), and \( f(u) \) being positive for \( u > 0 \) that \( U(b, t) > v(b, t) > 0 \) for \( t > 0 \).

Let

\[
z(t) = \int_0^1 \xi^q G(b, t; \xi, 0)\psi(\xi)d\xi.
\]

We note that \( z(t) = v(b, t) \), and hence, \( z(t) \) exists for \( t \geq 0 \). Let \( k_2 \) denote \( \min_{0 \leq \tau \leq T} v(b, t) \). We have

\[
k_2 \leq z(t) \leq k_1 \quad \text{for } 0 \leq t \leq T.
\]

It follows from \( \psi(b) > 0 \) and \( v > 0 \) in \( \Omega \) that \( k_2 > 0 \).

Let

\[
w(t) = U(b, t) - z(t). \quad (2.11)
\]

From (2.8),

\[
w(t) = a^2 \int_0^t G(b, t; b, \tau)f(w(\tau) + z(\tau))d\tau. \quad (2.12)
\]

Let

\[
Rw(t) = a^2 \int_0^t G(b, t; b, \tau)f(w(\tau) + z(\tau))d\tau.
\]

From (2.12), we have \( w = Rw \).

Lemma 2.3. For some given positive constant \( k_3 \), there exists some \( t_4 \) such that (2.12) has a unique continuous and nonnegative solution \( w(t) \leq k_3 \) for \( 0 \leq t \leq t_4 \).
Proof. By Lemma 2.2(d), $G(b, t; b, r)$ is integrable over $[0, t]$. Since $G(b, t; b, r)$ is non-negative, there exists some $t_4$ such that

$$a^2 f(k_3 + k_1) \int_0^t G(b, t; b, r) d\tau \leq k_3 \quad \text{for } 0 \leq t \leq t_4.$$  

(2.13)

$$a^2 f'(k_3 + k_1) \int_0^t G(b, t; b, r) d\tau < 1 \quad \text{for } 0 \leq t \leq t_4.$$  

(2.14)

From (2.13) and $f'(u) > 0$ for $u > 0$,

$$Rw(t) \leq a^2 f(k_3 + k_1) \int_0^t G(b, t; b, r) d\tau \leq k_3 \quad \text{for } 0 \leq t \leq t_4.$$  

(2.15)

Thus, $R$ maps the space of continuous functions satisfying

$$0 \leq w(t) \leq k_3 \quad \text{for } 0 \leq t \leq t_4$$

into itself. For any $w_1(t)$ and $w_2(t)$ satisfying (2.12),

$$\max_{0 \leq t \leq t_4} |Rw_1(t) - Rw_2(t)| \leq a^2 f'(k_3 + k_1) \left( \max_{0 \leq t \leq t_4} |w_1(t) - w_2(t)| \right) \int_0^t G(b, t; b, r) d\tau.$$  

By (2.14),

$$\max_{0 \leq t \leq t_4} |Rw_1(t) - Rw_2(t)| < \max_{0 \leq t \leq t_4} |w_1(t) - w_2(t)| \quad \text{for } 0 \leq t \leq t_4.$$  

Thus, $R$ is a contraction mapping, and we obtain an interval $0 \leq t \leq t_4$ on which a unique solution $w$ of (2.12) exists and is continuous and nonnegative. □

By (2.11), $U(b, t)$ exists, and is unique for $0 \leq t \leq t_4$; $U(b, t) > 0$ for $t > 0$. Let $t_b$ be the supremum of the interval for which the integral equation (2.8) has a unique continuous solution $U(b, t)$.

Let $\Omega_b = D \times (0, t_b)$, and $\partial \Omega_b$ denote its parabolic boundary $\{0, 1\} \times (0, t_b) \cup \overline{D} \times \{0\}$.

**Theorem 2.4.** The integral equation (2.3) has a unique continuous solution $U(x, t)$ in $\Omega_b$. Furthermore, $\psi(x) \leq U(x, t)$, and $U$ is a nondecreasing function of $t$.

**Proof.** Since the integral equation (2.8) has a unique continuous solution $U(b, t)$, it follows that the right-hand side of the integral equation (2.3) is determined uniquely, and hence, the integral equation (2.3) has a unique continuous solution $U(x, t)$. Also $U(x, t) > 0$ in $\Omega_b$.

Let us construct a sequence $\{u_i\}$ in $\Omega$ by $u_0(x, t) = \psi(x)$, and for $i = 0, 1, 2, \ldots$,

$$Lu_{i+1} = a^2 \delta(x - b)f(u_i) \quad \text{in } \Omega,$$

$$u_{i+1}(x, 0) = \psi(x) \quad \text{on } \overline{D}, \quad u_{i+1}(0, t) = u_{i+1}(1, t) = 0 \quad \text{for } 0 < t < T.$$  

We have

$$L(u_1 - u_0) \geq a^2 \delta(x - b)(f(u_0) - f(\psi)) = 0 \quad \text{in } \Omega,$$

$$u_1 - u_0 = 0 \quad \text{on } \partial \Omega.$$  

By Lemma 2.1(d) and (2.3), $u_1 \geq u_0$ in $\Omega$. Let us assume that for some positive integer $j$,

$$\psi \leq u_1 \leq u_2 \leq \cdots \leq u_{j-1} \leq u_j \quad \text{in } \Omega.$$
Since $f$ is an increasing function, and $u_j \geq u_{j-1}$, we have

$$L(u_{j+1} - u_j) = a^2 \delta(x-b)(f(u_j) - f(u_{j-1})) \geq 0 \quad \text{in } \Omega,$$

$$u_{j+1} - u_j = 0 \quad \text{on } \partial \Omega.$$

By Lemma 2.1(d) and (2.3), $u_{j+1} \geq u_j$. By the principle of mathematical induction,

$$\psi \leq u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \quad \text{in } \Omega \quad (2.16)$$

for any positive integer $n$.

We would like to show that $U(x,t) \geq \psi(x)$ for $0 \leq t < t_b$. From (2.9), $\psi(b) \leq U(b,t)$ for $0 \leq t \leq t_1$, where $t_1 = \rho$. Let $t_5$ be the smallest $t \geq t_1$ such that $\psi(b) \leq U(b,t)$. Since

$$L(u - \psi) = a^2 \delta(x-b)(f(u) - f(\psi)) \quad \text{in } \Omega,$$

$$u - \psi = 0 \quad \text{on } \partial \Omega,$$

it follows from (2.3) that

$$U(x,t) - \psi(x) \geq a^2 \int_0^t G(x,t;b,\tau)(f(U(b,\tau)) - f(\psi(b)))d\tau.$$

Thus, $U \geq \psi$ on $\bar{D} \times [0,t_5]$. By starting at $t = t_5$ (instead of $t = 0$), we repeat the procedure used in proving (2.9) and the above reasoning to show that $U \geq \psi$ on $\bar{D} \times [0,t_6]$ for some $t_6 \geq t_5 + \rho$. In this way, we prove that $U(x,t) \geq \psi(x)$ for $0 \leq t < t_b$.

Let $\bar{\Omega}$ denote the closure of $\Omega$. For any $T \in (0,t_b)$, $U$ is bounded on $\bar{\Omega}$. There exists some positive constant $K$ such that $U \leq K$ on $\bar{\Omega}$. Since

$$u_n(x,t) = a^2 \int_0^t G(x,t;b,\tau)f(u_{n-1}(b,\tau))d\tau + \int_0^1 \xi^q G(x,t;\xi,0)\psi(\xi)d\xi, \quad (2.17)$$

it follows from the properties of $f$ and the Monotone Convergence Theorem (cf. Royden [9, p. 87]) that $\lim_{n \to \infty} u_n$ satisfies the integral equation (2.3). From (2.17),

$$u_{n+1}(x,t) - u_n(x,t) = a^2 \int_0^t G(x,t;b,\tau)[f(u_n(b,\tau)) - f(u_{n-1}(b,\tau))]d\tau. \quad (2.18)$$

Let $S_n = \max_{\bar{\Omega}}(u_n - u_{n-1})$ for any $T < t_b$. By using the mean value theorem and $f'(u) > 0$ for $u > 0$, it follows from (2.18) (as in the derivation of (2.7)) that

$$S_{n+1} \leq a^2 f'(K)S_n c_1 c_2 b^{-q/4} \sum_{i=1}^\infty \lambda_i^{1/4} \int_0^t e^{-\lambda_i(t-\tau)}d\tau$$

$$= a^2 f'(K)c_1 c_2 b^{-q/4} \left[ \sum_{i=1}^\infty \lambda_i^{-3/4}(1 - e^{-\lambda_i\tau}) \right] S_n, \quad (2.19)$$
which converges since $O(\lambda_i) = (i^2)$ for large $i$. Let us choose some positive number $\sigma_1 (\leq T < t_b)$ such that for $t \in [0, \sigma_1]$,

$$a^2 f'(K)c_1 c_2 b^{-q/4} \left[ \sum_{i=1}^{\infty} \lambda_i^{-3/4}(1 - e^{-\lambda_i t}) \right] < 1.$$  

Then, the sequence $\{u_n\}$ converges uniformly to $\lim_{n \to \infty} u_n(x, t)$ for $0 \leq t \leq \sigma_1$. Similarly for $\sigma_1 \leq t \leq T < t_b$, we use $\lim_{n \to \infty} u_n(\xi, \sigma_1)$ to replace $\psi(\xi)$ in (2.17); we then obtain

$$S_{n+1} \leq a^2 f'(K)c_1 c_2 b^{-q/4} \left\{ \sum_{i=1}^{\infty} \lambda_i^{-3/4} [1 - e^{-\lambda_i (t-\sigma_1)}] \right\} S_n.$$  

For $t \in [\sigma_1, \min\{2\sigma_1, T\}]$,

$$a^2 f'(K)c_1 c_2 b^{-q/4} \left\{ \sum_{i=1}^{\infty} \lambda_i^{-3/4} [1 - e^{-\lambda_i (t-\sigma_1)}] \right\} < 1.$$  

Thus, the sequence $\{u_n\}$ converges uniformly to $\lim_{n \to \infty} u_n(x, t)$ for $\sigma_1 \leq t \leq \min\{2\sigma_1, T\}$. By proceeding in this way, the sequence $\{u_n\}$ converges uniformly for $0 \leq t \leq T$, and hence $\lim_{n \to \infty} u_n$ is continuous. Since the integral equation (2.3) has a unique continuous solution $U$ for $0 \leq t < t_b$, we have $U = \lim_{n \to \infty} u_n$.

To show that $U$ is a nondecreasing function of $t$, let us construct a sequence $\{w_i\}$ such that for $i = 0, 1, 2, \ldots$,

$$w_i(x, t) = u_i(x, t + h) - u_i(x, t),$$

where $h (< T)$ is some positive number. Then, $w_0(x, t) = 0$. We have

$$Lw_1 = 0 \quad \text{in} \quad D \times (0, T - h].$$

From (2.16),

$$w_1(x, 0) \geq 0 \quad \text{on} \quad \overline{D}, \quad w_1(0, t) = w_1(1, t) = 0 \quad \text{for} \quad 0 < t \leq T - h.$$  

By (2.3), $w_1 \geq 0$ in $\Omega$. Let us assume that for some positive integer $j$, $0 \leq w_j \in \Omega$. Then,

$$Lw_{j+1} = a^2 \delta(x - b)f'(\xi_j)w_j \geq 0 \quad \text{in} \quad D \times (0, T - h]$$

for some $\xi_j$ between $u_j(x, t + h)$ and $u_j(x, t)$. Since $w_{j+1}(x, 0) \geq 0$ on $\overline{D}$, and $w_{j+1}(0, t) = w_{j+1}(1, t) = 0$ for $0 < t \leq T - h$, it follows from (2.3) that $w_{j+1} \geq 0 \in \Omega$. By the principle of mathematical induction, $w_n \geq 0 \in \Omega$ for all positive integers $n$. Hence, $U$ is a nondecreasing function of $t$. \hfill \Box

The next result shows that $U$ is the solution of the problem (1.2).

**Theorem 2.5.** The problem (1.2) has a unique solution $u = U$. 


Proof. By Lemma 2.2(d), \( \int_0^t G(x, t; b, \tau)f(U(b, \tau))d\tau \) exists for \( x \in \overline{D} \) and \( t \) in any compact subset \( [t_7, t_8] \) of \( [0, t_0] \). Thus, for any \( x \in D \) and any \( t_9 \in (0, t) \),

\[
\int_0^t G(x, t; b, \tau)f(U(b, \tau))d\tau \\
= \lim_{n \to \infty} \int_0^{t-1/n} G(x, t; b, \tau)f(U(b, \tau))d\tau \\
= \lim_{n \to \infty} \left[ \int_{t_9}^t \frac{\partial}{\partial \zeta} \left( \int_0^{\zeta-1/n} G(x, \zeta; b, \tau)f(U(b, \tau))d\tau \right) d\zeta \\
+ \int_0^{t_9-1/n} G(x, t_9; b, \tau)f(U(b, \tau))d\tau \right].
\]

Since by (2.4),

\[
G_{\zeta}(x, \zeta; b, \tau)f(U(b, \tau)) \leq c_2^2 \sum_{i=1}^{\infty} \lambda_i^{3/2} e^{-\lambda_i/\nu} f(U(b, \tau)) \quad \text{for } \zeta - \tau \geq 1/n,
\]

which is integrable with respect to \( \tau \) over \( (0, 1/n) \), it follows from the Leibnitz rule (cf. Stromberg [11, p. 380]) that

\[
\frac{\partial}{\partial \zeta} \left( \int_0^{\zeta-1/n} G(x, \zeta; b, \tau)f(U(b, \tau))d\tau \right) \\
= G(x, \zeta; b, \zeta - 1/n) f(U(b, \zeta - 1/n)) + \int_0^{\zeta-1/n} G_{\zeta}(x, \zeta; b, \tau)f(U(b, \tau))d\tau.
\]

Let us consider the problem,

\[
L\omega = 0 \quad \text{for } x \in D, 0 < \tau < t < T, \\
\omega(0, t; \xi, \tau) = \omega(1, t; \xi, \tau) = 0 \quad \text{for } 0 < \tau < t < T, \\
\lim_{t \to T^+} x^q \omega(x, t; \xi, \tau) = \delta(x - \xi).
\]

From the representation formula (2.3),

\[
\omega(x, t; \xi, \tau) = \int_0^1 \alpha^q G(x, t; \alpha, \tau) \alpha^{-q} \delta(\alpha - \xi)d\alpha \\
= G(x, t; \xi, \tau) \quad \text{for } t \geq \tau.
\]

It follows that \( \lim_{t \to T^+} x^q G(x, t; b, \tau) = \delta(x - b) \).

Since \( G(x, \zeta; b, \zeta - 1/n) = G(x, 1/n; b, 0) \), which is independent of \( \zeta \), we have

\[
\int_0^t x^q G(x, t; b, \tau)f(U(b, \tau))d\tau \\
= \delta(x - b) \int_{t_9}^t f(U(b, \zeta))d\zeta + \lim_{n \to \infty} \int_{t_9}^t \int_0^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau)f(U(b, \tau))d\tau d\zeta \\
+ \int_0^{t_9} x^q G(x, t_9; b, \tau)f(U(b, \tau))d\tau.
\]
Let
\[ g_n(x, t) = \int_0^{t-1/n} x^q G_t(x, t; b, \tau) f(U(b, \tau)) d\tau. \]
Without loss of generality, let \( n > 1 \). We have
\[ g_n(x, \zeta) - g_l(x, \zeta) = \int_{\zeta-1/l}^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) f(U(b, \tau)) d\tau. \]
Since \( x^q G_t(x, t; b, \tau) \in C(\overline{D} \times (\tau, T]) \) and \( f(U(b, \tau)) \) is a monotone function of \( \tau \), it follows from the Second Mean Value Theorem for Integrals (cf. Stromberg [11, p. 328]) that for any \( x \neq b \) and any \( \zeta \) in any compact subset \([t_7, t_8]\) of \((0, t_b)\), there exists some real number \( \nu \) such that \( \zeta - \nu \in (\zeta - 1/l, \zeta - 1/n) \) and
\[ g_n(x, \zeta) - g_l(x, \zeta) = f\left(U\left(b, \zeta - \frac{1}{n}\right)\right) \int_{\zeta-1/l}^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) d\tau + f\left(U\left(b, \zeta - \frac{1}{l}\right)\right) \int_{\zeta-\nu}^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) d\tau. \]
From \( G_{\zeta}(x, \zeta; b, \tau) = -G_{\tau}(x, \zeta; b, \tau) \), we have
\[ g_n(x, \zeta) - g_l(x, \zeta) = f\left(U\left(b, \zeta - \frac{1}{n}\right)\right) \int_{\zeta-1/l}^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) d\tau - f\left(U\left(b, \zeta - \frac{1}{l}\right)\right) \int_{\zeta-\nu}^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) d\tau. \]
Since for \( x \neq b \),
\[ x^q G(x, \zeta; b, \zeta - \epsilon) = x^q G(x, \epsilon; b, 0) \]
converges to 0 uniformly with respect to \( \zeta \) as \( \epsilon \to 0 \), it follows that for \( x \neq b \), \( \{g_n\} \) is a Cauchy sequence, and hence \( \{g_n\} \) converges uniformly with respect to \( \zeta \) in any compact subset \([t_7, t_8]\) of \((0, t_b)\). Hence for \( x \neq b \),
\[ \lim_{n \to \infty} \int_{t_0}^{t} \int_0^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) f(U(b, \tau)) d\tau d\zeta = \int_{t_0}^{t} \lim_{n \to \infty} \int_0^{\zeta-1/n} x^q G_{\zeta}(x, \zeta; b, \tau) f(U(b, \tau)) d\tau d\zeta = \int_{t_0}^{t} \int_0^{\zeta} x^q G_{\zeta}(x, \zeta; b, \tau) f(U(b, \tau)) d\tau d\zeta. \]
For \( x = b \),
\[ -G_{\zeta}(x, \zeta; b, \tau) f(U(b, \tau)) = \sum_{i=1}^{\infty} \phi_i^2(b) \lambda_i e^{-\lambda_i \zeta} f(U(b, \tau)), \]
which is positive. Thus, \(-g_n\) is a nondecreasing sequence of nonnegative functions with respect to \(\zeta\). By the Monotone Convergence Theorem,

\[
\lim_{n \to \infty} \int_{t_0}^{t} \int_{0}^{\zeta - 1/n} b^q G_{\zeta}(b, \zeta; b, \tau) f(U(b, \tau)) d\tau d\zeta
= \int_{t_0}^{t} \lim_{n \to \infty} \int_{0}^{\zeta - 1/n} b^q G_{\zeta}(b, \zeta; b, \tau) f(U(b, \tau)) d\tau d\zeta
= \int_{t_0}^{t} \int_{0}^{\zeta} b^q G_{\zeta}(b, \zeta; b, \tau) f(U(b, \tau)) d\tau d\zeta.
\]

Thus,

\[
\frac{\partial}{\partial t} \int_{0}^{t} x^q G(x, t; b, \tau) f(U(b, \tau)) d\tau
= \delta(x - b) f(U(b, t)) + \int_{0}^{t} x^q G_t(x, t; b, \tau) f(U(b, \tau)) d\tau.
\]

By using (2.5), (2.6) and the Leibnitz rule, we have for any \(x\) in any compact subset of \((0, 1]\) and \(t\) in any compact subset \([t_\tau, t_\sigma]\) of \((0, t_b)\),

\[
\frac{\partial}{\partial x} \int_{0}^{t-\varepsilon} G(x, t; b, \tau) f(U(b, \tau)) d\tau = \int_{0}^{t-\varepsilon} G_x(x, t; b, \tau) f(U(b, \tau)) d\tau,
\]

\[
\frac{\partial}{\partial x} \int_{0}^{t-\varepsilon} G_x(x, t; b, \tau) f(U(b, \tau)) d\tau = \int_{0}^{t-\varepsilon} G_{xx}(x, t; b, \tau) f(U(b, \tau)) d\tau.
\]

For any \(x_1 \in D\),

\[
\lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} G(x, t; b, \tau) f(U(b, \tau)) d\tau
= \lim_{\varepsilon \to 0} \int_{x_1}^{x} \left( \frac{\partial}{\partial \eta} \int_{0}^{t-\varepsilon} G(\eta, t; b, \tau) f(U(b, \tau)) d\tau \right) d\eta
+ \lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} G(x_1, t; b, \tau) f(U(b, \tau)) d\tau
= \lim_{\varepsilon \to 0} \int_{x_1}^{x} \int_{0}^{t-\varepsilon} G_{\eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta
+ \int_{0}^{t} G(x_1, t; b, \tau) f(U(b, \tau)) d\tau.
\]

We would like to show that

\[
\lim_{\varepsilon \to 0} \int_{x_1}^{x} \int_{0}^{t-\varepsilon} G_{\eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta
= \int_{x_1}^{x} \lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} G_{\eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta.
\]
By the Fubini Theorem (cf. Stromberg [11, p. 352]),

\[
\lim_{\varepsilon \to 0} \int_{x \varepsilon}^{x} \int_{0}^{t-\varepsilon} G_{\eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta \\
= \lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} \left( f(U(b, \tau)) \int_{x \varepsilon}^{x} G_{\eta}(\eta, t; b, \tau) d\eta \right) d\tau \\
= \lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} f(U(b, \tau))(G(x, t; b, \tau) - G(x_{1}, t; b, \tau)) d\tau \\
= \int_{0}^{t} f(U(b, \tau))(G(x, t; b, \tau) - G(x_{1}, t; b, \tau)) d\tau ,
\]

which exists by Lemma 2.2(d). Therefore,

\[
\int_{0}^{t} f(U(b, \tau))(G(x, t; b, \tau) - G(x_{1}, t; b, \tau)) d\tau = \int_{x \varepsilon}^{x} \int_{0}^{t-\varepsilon} G_{\eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta ,
\]

and we have (2.20). From (2.19),

\[
\frac{\partial}{\partial x} \int_{0}^{t} G(x, t; b, \tau) f(U(b, \tau)) d\tau = \int_{0}^{t} G_{x}(x, t; b, \tau) f(U(b, \tau)) d\tau .
\]

For any \(x_{2} \in D\),

\[
\lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} G_{x}(x, t; b, \tau) f(U(b, \tau)) d\tau \\
= \lim_{\varepsilon \to 0} \int_{x \varepsilon}^{x} \frac{\partial}{\partial \eta} \left( \int_{0}^{t-\varepsilon} G_{\eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau \right) d\eta \\
+ \lim_{\varepsilon \to 0} \int_{0}^{t-\varepsilon} G_{\eta}(x_{2}, t; b, \tau) f(U(b, \tau)) d\tau \\
= \lim_{\varepsilon \to 0} \int_{x \varepsilon}^{x} \int_{0}^{t-\varepsilon} G_{\eta \eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta \\
+ \int_{0}^{t} G_{\eta}(x_{2}, t; b, \tau) f(U(b, \tau)) d\tau .
\]

We would like to show that

\[
\lim_{\varepsilon \to 0} \int_{x \varepsilon}^{x} \int_{0}^{t-\varepsilon} G_{\eta \eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta \\
= \int_{x \varepsilon}^{x} \int_{0}^{t-\varepsilon} G_{\eta \eta}(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta .
\]
Since \( G_{xx}(x, t; \xi, \tau) = x^q G_t(x, t; \xi, \tau) - \delta(x - \xi)\delta(t - \tau) \), we have

\[
\lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \int_{0}^{t-\varepsilon} G_{\eta\eta}(\eta, t; b, \tau)f(U(b, \tau))d\tau d\eta \\
= \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \int_{0}^{t-\varepsilon} (\eta^q G_t(\eta, t; b, \tau) - \delta(\eta - b)\delta(t - \tau))f(U(b, \tau))d\tau d\eta \\
= \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \int_{0}^{t-\varepsilon} \eta^q G_t(\eta, t; b, \tau)f(U(b, \tau))d\tau d\eta \\
= -\lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \int_{0}^{t-\varepsilon} \eta^q G_x(\eta, t; b, \tau)f(U(b, \tau))d\tau d\eta.
\]

By the Second Mean Value Theorem for Integrals, there exists some real number \( \gamma \in (0, t - \varepsilon) \) such that

\[
\lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \int_{0}^{t-\varepsilon} \eta^q G_x(\eta, t; b, \tau)f(U(b, \tau))d\tau d\eta = -\lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \int_{0}^{t-\varepsilon} \eta^q G_x(\eta, t; b, \gamma)f(U(b, \gamma))d\tau d\eta
\]

\[
+ \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} f(U(b, t - \varepsilon)) \eta^q(G(\eta, t; b, \gamma) - G(\eta, t; b, \tau))d\eta
\]

\[
= f(U(b, 0)) \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q(G(\eta, t; b, 0) - G(\eta, t; b, \gamma))d\eta
\]

\[
+ \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} f(U(b, t - \varepsilon)) \eta^q(G(\eta, t; b, \gamma) - G(\eta, t; b, t - \varepsilon))d\eta
\]

\[
= f(U(b, 0)) \left( \int_{x_2}^{x_1} \eta^q G(\eta, t; b, 0)d\eta - \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(\eta, t; b, \gamma)d\eta \right)
\]

\[
+ f(U(b, t)) \left( \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(\eta, t; b, \gamma)d\eta - \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(\eta, t; b, t - \varepsilon)d\eta \right)
\]

\[
= f(U(b, 0)) \left( \int_{x_2}^{x_1} \eta^q G(\eta, t; b, 0)d\eta - \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(\eta, t; b, \gamma)d\eta \right)
\]

\[
+ f(U(b, t)) \left( \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(\eta, t; b, \gamma)d\eta - \int_{x_2}^{x_1} \delta(\eta - b)d\eta \right)
\]

since \( \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(\eta, t; b, t - \varepsilon)d\eta = \lim_{\varepsilon \to 0} \int_{x_2}^{x_1} \eta^q G(b, t; \eta, t - \varepsilon)d\eta = \int_{x_2}^{x_1} \delta(\eta - b)d\eta \) (cf. Chan and Chan [1]).
Case 1: If \( \lim_{\epsilon \to 0} \gamma = t \), then \( \lim_{\epsilon \to 0} \int_{x_2}^{x} \eta^q G(\eta, t; b, \gamma) d\eta = \int_{x_2}^{x} \delta(\eta - b) d\eta \). We have

\[
- \lim_{\epsilon \to 0} \int_{x_2}^{x} \int_{0}^{t-\epsilon} \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta
= f(U(b, 0)) \left( \int_{x_2}^{x} \eta^q G(\eta, t; b, 0) d\eta - \int_{x_2}^{x} \delta(\eta - b) d\eta \right)
+ f(U(b, t)) \left( \int_{x_2}^{x} \delta(\eta - b) d\eta - \int_{x_2}^{x} \delta(\eta - b) d\eta \right)
= f(U(b, 0)) \left( \int_{x_2}^{x} \eta^q G(\eta, t; b, 0) d\eta - \int_{x_2}^{x} \lim_{\epsilon \to 0} \eta^q G(\eta, t; b, \gamma) d\eta \right)
+ f(U(b, t)) \left( \int_{x_2}^{x} \delta(\eta - b) d\eta - \int_{x_2}^{x} \lim_{\epsilon \to 0} \eta^q G(\eta, t; b, t - \epsilon) d\eta \right)
= f(U(b, 0)) \left( \int_{x_2}^{x} \eta^q G(\eta, t; b, 0) d\eta - \int_{x_2}^{x} \lim_{\epsilon \to 0} \eta^q G(\eta, t; b, \gamma) d\eta \right)
+ \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ f(U(b, t - \epsilon)) \eta^q(G(\eta, t; b, \gamma) - G(\eta, t; b, t - \epsilon)) \right] d\eta
= - \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ \int_{0}^{t-\epsilon} \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta \right]
= - \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta \right]
= \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta \right].
\]

Case 2: If \( \lim_{\epsilon \to 0} \gamma < t \), then \( \lim_{\epsilon \to 0} \int_{x_2}^{x} \eta^q G(\eta, t; b, \gamma) d\eta = \int_{x_2}^{x} \eta^q G(\eta, t; b, \lim_{\epsilon \to 0} \gamma) d\eta \) since \( \int_{x_2}^{x} \eta^q G(\eta, t; b, \gamma) d\eta \) is a continuous function of \( \gamma \). From (2.23), we have

\[
- \lim_{\epsilon \to 0} \int_{x_2}^{x} \int_{0}^{t-\epsilon} \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau d\eta
= f(U(b, 0)) \left( \int_{x_2}^{x} \eta^q G(\eta, t; b, 0) d\eta - \int_{x_2}^{x} \eta^q G(\eta, t; b, \lim_{\epsilon \to 0} \gamma) d\eta \right)
+ f(U(b, t)) \left( \int_{x_2}^{x} \eta^q G(\eta, t; b, \lim_{\epsilon \to 0} \gamma) d\eta - \int_{x_2}^{x} \delta(\eta - b) d\eta \right)
= - \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ \int_{0}^{t-\epsilon} \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau + f(U(b, t - \epsilon)) \int_{0}^{t-\epsilon} \eta^q G_\tau(\eta, t; b, \tau) d\tau \right] d\eta
= - \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ \eta^q G_\tau(\eta, t; b, \tau) f(U(b, \tau)) d\tau \right] d\eta
= \int_{x_2}^{x} \lim_{\epsilon \to 0} \left[ \eta^q G(\eta, t; b, \tau) f(U(b, \tau)) d\tau \right] d\eta.
\]

In either case, we have (2.22).
From (2.21),
\[
\int_0^t G_x(x, t; b, \tau)f(U(b, \tau))d\tau
\]
= \int_{x_2}^x \int_0^t G_{\eta \eta}(\eta, t; b, \tau)f(U(b, \tau))d\tau d\eta + \int_0^t G_{\eta}(x_2, t; b, \tau)f(U(b, \tau))d\tau.
\]
Thus,
\[
\frac{\partial}{\partial x} \int_0^t G_x(x, t; b, \tau)f(U(b, \tau))d\tau = \int_0^t G_{xx}(x, t; b, \tau)f(U(b, \tau))d\tau.
\]
Therefore,
\[
\frac{\partial^2}{\partial x^2} \int_0^t G(x, t; b, \tau)f(U(b, \tau))d\tau = \int_0^t G_{xx}(x, t; b, \tau)f(U(b, \tau))d\tau
\]
for any \(x\) in any compact subset of \((0,1]\) and \(t\) in any compact subset \([t_7, t_8]\) of \((0, t_b)\).

By the Leibnitz rule, we have for any \(x\) in any compact subset of \((0,1]\) and any \(t\) in any compact subset of \((0, t_b)\),
\[
\frac{\partial^2}{\partial x^2} \int_0^1 G(x, t; \xi, 0)\xi^q \psi(\xi)d\xi = \int_0^1 x^q G_t(x, t; \xi, 0)\xi^q \psi(\xi)d\xi.
\]
\[
\frac{\partial}{\partial x} \int_0^1 G(x, t; \xi, 0)\xi^q \psi(\xi)d\xi = \int_0^1 G_x(x, t; \xi, 0)\xi^q \psi(\xi)d\xi,
\]
\[
\frac{\partial^2}{\partial x^2} \int_0^1 G(x, t; \xi, 0)\xi^q \psi(\xi)d\xi = \int_0^1 G_{xx}(x, t; \xi, 0)\xi^q \psi(\xi)d\xi.
\]

From the integral equation (2.3), we have for \(x \in D\) and \(0 < t < t_b\),
\[
LU = a^2 \delta(x - b)f(U(b, t)) + a^2 \int_0^t L G(x, t; b, \tau)f(U(b, \tau))d\tau
\]
+ \int_0^1 L G(x, t; \xi, 0)\xi^q \psi(\xi)d\xi
\]
= \(a^2 \delta(x - b)f(U(b, t)) + a^2 \delta(x - b) \lim_{\varepsilon \to 0} \int_0^{t - \varepsilon} \delta(t - \tau)f(U(b, \tau))d\tau \]
+ \(\delta(t) \int_0^1 \delta(x - \xi)\xi^q \psi(\xi)d\xi \]
= \(a^2 \delta(x - b)f(U(b, t)) \).

From the integral equation (2.3), we have for \(x \in \overline{D}\),
\[
\lim_{t \to 0} U(x, t) = \lim_{t \to 0} \int_0^1 \xi^q G(x, t; \xi, 0)\psi(\xi)d\xi = \psi(x)
\]
(cf. Chan and Chan [1]). Since \(G(0, t; \xi, \tau) = 0 = G(1, t; \xi, \tau)\), we have \(U(0, t) = 0 = U(1, t)\). Thus, the solution \(U\) of the integral equation (2.3) is a solution of the problem (1.2). Since a solution of the latter is a solution of the former, the theorem is proved. 

The next result gives a sufficient condition for \(u\) to blow up.

**Theorem 2.6.** If \(\psi\) attains its maximum at \(b\), then the solution \(u\) of the problem (1.2) attains its maximum at \(b\). If in addition, \(t_b < \infty\), then \(u(b, t)\) is unbounded in \([0, t_b)\).
Proof. Let $D_{ob} = (0,b)$, $D_{ob} = [0,b]$, $D_{b1} = (b,1)$, $D_{b1} = [b,1]$, $\Omega_{ob} = D_{ob} \times (0,t_b)$, and $\Omega_{b1} = D_{b1} \times (0,t_b)$. Since $u(b,t)$ is known, let us consider the problems:

\[
\begin{align*}
Lu &= 0 \quad \text{in } \Omega_{ob}, \quad u(x,0) = \psi(x) \quad \text{on } D_{ob}, \\
\quad u(0,t) = 0 \quad \text{and} \quad u(b,t) = u(b,t) \quad \text{for } 0 < t < t_b, \\
Lu &= 0 \quad \text{in } \Omega_{b1}, \quad u(x,0) = \psi(x) \quad \text{on } D_{b1}, \\
\quad u(b,t) = u(b,t) \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 < t < t_b.
\end{align*}
\]

(2.24) (2.25)

Because $\psi$ attains its maximum at $b$, it follows from the strong maximum principle and Theorems 2.4 and 2.5 that the solution of the problem (2.24) attains its maximum at $b$. Similarly, the solution of the problem (2.25) attains its maximum at $b$.

By Theorem 2.4, $u$ is a nondecreasing function of $t$. Thus, if $u$ blows up, it is at $b$. If in addition, $t_b < \infty$, then let us assume that $u(b,t)$ is bounded above by some constant $k_3$ in $[0,t_b)$. We consider (2.8) for $t \in [t_b,T)$ with the initial condition $u(x,0)$ replaced by $\lim_{t \to t_b} u(x,t)$, which we denote by $u(x,t_b)$:

\[
u(b,t) = a^2 \int_{t_b}^{t} G(b,t;b,\tau)f(u(b,\tau))d\tau + \int_{0}^{1} \xi^q G(b,t;\xi,t_b)u(\xi,t_b)d\xi.
\]

(2.26)

Let

\[
Z(t) = \int_{0}^{1} \xi^q G(b,t;\xi,t_b)u(\xi,t_b)d\xi.
\]

and $W(t) = u(b,t) - Z(t)$. An argument analogous to the proof of Lemma 2.3 shows that there exists some $t_{10}$ such that $W$ exists and is unique for $t_b \leq t \leq t_{10}$. Thus, (2.26) has a unique solution for $t_b \leq t \leq t_{10}$, and hence, (2.8) has a unique solution $u(b,t)$ for $t_b \leq t \leq t_{10}$. This contradicts the definition of $t_b$, and hence the theorem is proved. \(\square\)

3. Single blow-up point. From (2.1), we obtain the following result.

**Lemma 3.1.** $G(b,t;b,\tau)$ is a strictly decreasing function of $t$.

**Theorem 3.2.** If $\psi$ attains its maximum at $b$, and $u$ blows up, then $b$ is the single blow-up point.

**Proof.** Since $\psi$ attains its maximum at $b$, it follows from Theorem 2.6 that if $u$ blows up, then it blows up at $b$. To show that $b$ is the only blow-up point, let us consider the problem (2.24). By the parabolic version of Hopf’s lemma (cf. Friedman [5, p. 49]), $u_x(0,t) > 0$ for any arbitrarily fixed $t \in (0,t_b)$. For any $x \in (0,b)$, $u_{xx} = x^q u_t$, which is nonnegative by Theorem 2.4. Hence, $u$ is concave up. Similarly, for any arbitrarily fixed $t \in (0,t_b)$, $u_x(1,t) < 0$. For any $x \in (b,1)$, $u_{xx} = x^q u_t \geq 0$, and hence $u$ is concave up. Thus, if $u$ blows up, then $b$ is the single blow-up point. \(\square\)

Let

\[
\mu(t) = \int_{0}^{1} x^q \phi(x)u(x,t)dx,
\]

where $\phi$ denotes the normalized fundamental eigenfunction of the problem (2.2) with $\lambda$ denoting its corresponding eigenvalue.
Theorem 3.3. If $\psi$ attains its maximum at $b$,

$$
\mu(0) > \left( \frac{\lambda}{a^2} \right)^{1/(p-1)}, \quad (3.1)
$$

$$
\phi(b)f(u(b,t)) \geq \left( \frac{1}{q+1} \right)^{p/2} u^p(b,t), \quad (3.2)
$$

where $p$ is a real number greater than 1, then the solution $u$ of the problem (1.2) blows up at a finite time.

Proof. Multiplying the differential equation in the problem (1.2) by $\phi$, and integrating over $x$ from 0 to 1, we obtain

$$
\mu'(t) + \lambda \mu(t) = a^2 \phi(b)f(u(b,t)). \quad (3.3)
$$

Since $u(x,t) \leq u(b,t)$, we have

$$
\mu(t) \leq \left( \int_0^1 x^q \phi(x)dx \right) u(b,t).
$$

It follows from the Schwarz inequality and $\int_0^1 x^q \phi^2(x)dx = 1$ that

$$
\mu(t) \leq \left( \int_0^1 x^q \phi^2(x)dx \right)^{1/2} \left( \int_0^1 x^q dx \right)^{1/2} u(b,t)
$$

$$
\leq \left( \frac{1}{q+1} \right)^{1/2} u(b,t).
$$

By (3.2),

$$
\phi(b)f(u(b,t)) \geq \mu^p(t).
$$

From (3.3),

$$
\mu'(t) + \lambda \mu(t) \geq a^2 \mu^p(t).
$$

Solving this Bernoulli inequality, we obtain

$$
\mu^{1-p}(t) \leq \frac{a^2}{\lambda} + \left( \mu^{1-p}(0) - \frac{a^2}{\lambda} \right) e^{\lambda(p-1)t}.
$$

From (3.1), $\mu^{1-p}(0) < a^2/\lambda$. Thus, $\mu$ tends to infinity for some finite $t_b$. This implies $u(b,t)$ blows up at $t_b$. \qed

If $t_b < \infty$, then we use the method of Olmstead and Roberts [7] to find a lower bound $t_l$ and an upper bound $t_u$ for $t_b$. These are used later on to compute the finite blow-up time. Using (2.14), we obtain from (2.15),

$$
Rw < \frac{f(k_3 + k_1)}{f'(k_3 + k_1)}.
$$

Let us assume that $\psi$ attains its maximum at $b$. Then, $k_1 = \psi(b)$. Thus, an appropriate $k_3$ is the smallest solution of

$$
k_3 = \frac{f(k_3 + \psi(b))}{f'(k_3 + \psi(b))}. \quad (3.4)
$$
We note that in the proof of Lemma 2.3, (2.14) implies $R$ is a contraction mapping. This and (3.4) show that if

$$a^2 \int_0^t G(b, t; b, \tau) d\tau < \frac{k_3}{f(k_3 + \psi(b))},$$

then $R$ is a contraction mapping, and hence $u$ exists. From (2.13), a lower bound $t_l$ of $t_b$ is given by

$$a^2 \int_0^{t_l} G(b, t_l; b, \tau) d\tau = \frac{k_3}{f(k_3 + \psi(b))}.$$  

For some $t_{11} < t_b$, (2.12) has a continuous solution $w(t)$ for $t \in [0, t_{11}]$. From Lemma 3.1,

$$w(t) \geq s(t), 0 \leq t \leq t_{11} < t_b,$$

where

$$s(t) = a^2 \int_0^t G(b, t_{11}; b, \tau) f(w(\tau) + z(\tau)) d\tau.$$

For some $t_u$ to be determined later, let $\min_{0 \leq t \leq t_u} z(t)$ be denoted by $k_5$, which is positive. Then,

$$s'(t) = a^2 G(b, t_{11}; b, t) f(w(t) + z(t)) \geq a^2 G(b, t_{11}; b, t) f(s(t) + k_5).$$

We have

$$\frac{s'(t)}{f(s(t) + k_5)} \geq a^2 G(b, t_{11}; b, t).$$

That is,

$$\int_{k_5}^{s(t_{11}) + k_5} \frac{d\tau}{f(\tau)} \geq a^2 \int_0^{t_{11}} G(b, t_{11}; b, \tau) d\tau.$$

Since (2.12) having a continuous solution $w(t)$ for $t \in [0, t_{11}]$ insures that $s(t) < \infty$, we have

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} > a^2 \int_0^{t_{11}} G(b, t_{11}; b, \tau) d\tau.$$

A contradiction to existence of a continuous solution occurs if

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} < \infty,$$

and there exists some $t_{12}$ such that

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} = a^2 \int_0^{t_{12}} G(b, t_{12}; b, \tau) d\tau.$$

Thus, an upper bound $t_u$ of $t_b$ is determined by

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} = a^2 \int_0^{t_u} G(b, t_u; b, \tau) d\tau.$$  

That is,

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} = a^2 \sum_{i=1}^{\infty} \frac{\phi_i^2(b)}{\lambda_i} (1 - e^{-\lambda_i t_u}).$$  

Thus, we have proved the following result.
Theorem 3.4. If $t_b < \infty$, and $\psi$ attains its maximum at $b$, then a lower bound $t_l$ of $t_b$ is determined by (3.6). If in addition, (3.7) holds, then an upper bound $t_u$ of $t_b$ is determined by (3.9).

4. An example. As an illustrative example, let $q = 0$. Then,

$$G(x, t; \xi, \tau) = 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 (t-\tau)} \sin(n \pi x) \sin(n \pi \xi) \text{ for } t > \tau.$$  

From Olmstead and Roberts [7],

$$\int_0^t G(b, t; b, \tau) d\tau = b(1 - b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n \pi b}{n^2} e^{-n^2 \pi^2 t}.$$ 

Let

$$\psi(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq b, \\ \left(\frac{b}{1-b}\right)^2 (1-x)^2 & \text{for } b < x \leq 1. \end{cases}$$ 

It is nontrivial, nonnegative and continuous such that $\psi(0) = 0 = \psi(1)$. Its generalized second derivative (cf. Stakgold [10, pp. 38-39]) with respect to $x$ is given by

$$\psi''(x) = \begin{cases} 2 & \text{for } 0 < x < b, \\ -\frac{2b}{1-b} \delta(x - b) & \text{for } x = b, \\ 2 \left(\frac{b}{1-b}\right)^2 & \text{for } b < x < 1. \end{cases}$$ 

Thus, the condition (1.3) is satisfied if

$$\left( a^2 f(b^2) - \frac{2b}{1-b} \right) \delta(x - b) \geq 0.$$ 

A sufficient condition for this to hold is

$$a^2 f(b^2) \geq \frac{2b}{1-b}. \quad (4.1)$$ 

Let $f(u) = u^p$ where $p$ is any real number greater than 1. From (3.4), $k_3 = (k_3 + \psi(b))/p$, and hence, $k_3 = b^2/(p - 1)$. From (3.5),

$$a^2 \left[ b(1 - b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n \pi b}{n^2} e^{-n^2 \pi^2 t} \right] < \frac{(p - 1)^{p-1}}{p^p b^2(p-1)}.$$ 

This is satisfied for all $t > 0$ if

$$a^2 b^{2p-1}(1 - b) < \frac{(p - 1)^{p-1}}{p^p}. \quad (4.2)$$ 

Thus, $u$ exists for all $t > 0$ if (4.2) holds. We note that (4.2) can always be achieved by placing the concentrated source sufficiently close to the boundaries (cf. Olmstead and Roberts [7]).
Since the normalized fundamental eigenfunction is given by \( \phi(x) = 2^{1/2} \sin \pi x \), and its corresponding eigenvalue is \( \lambda = \pi^2 \), it follows from Theorem 3.3 that if
\[
\frac{2^{3/2}}{(1 - b)^2 \pi^3} \left[ 1 + 2b - 2b^2 + (1 - 2b) \cos \pi b + (1 - b) \pi b \sin \pi b \right] > \left( \frac{\pi}{a} \right)^{2(p-1)},
\]
then \( u \) blows up at a finite time. A plot of the left-hand side of (4.3) as a function of \( b \) by using Mathematica® version 4.1 shows that it is positive for \( 0 < b < 1 \). Thus for a given \( b \), we can find \( a \) such that (4.3) is satisfied. From (3.6), a lower bound \( t_l \) for \( t_b \) is given by
\[
a^2 \left[ b(1 - b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n \pi b}{n^2} e^{-n^2 \pi^2 t_l} \right] = \frac{(p - 1)^{p-1}}{p^p b^2 (p-1)}.
\]
We have
\[
z(t) = \frac{4}{(1 - b)^2 \pi^3} \left\{ \sum_{n=1}^{\infty} [b^2 \cos n \pi + (1 - 2b) \cos n \pi b + (1 - b)(-1 + b + n \pi b \sin n \pi b)] \frac{\sin n \pi b}{n^2} e^{-n^2 \pi^2 t} \right\}.
\]
From (3.8), an upper bound \( t_u \) is given by
\[
\frac{1}{(p - 1) k_5^{p-1}} = a^2 \left[ b(1 - b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n \pi b}{n^2} e^{-n^2 \pi^2 t_u} \right].
\]
Since \( k_5 = \min_{0 \leq t \leq t_u} z(t) \), it follows from (4.7) that an upper bound \( t_u \) may be determined by
\[
\frac{1}{(p - 1) k_5^{p-1}} = a^2 b(1 - b).
\]
As a numerical example, we further let \( p = 2 \) and \( b = 1/2 \). The sufficient condition (4.1) is satisfied if \( a \geq 4 \sqrt{2} \). Since (4.4) is automatically satisfied, it follows from (4.3) that \( u \) blows up in a finite time for \( a > 9.74 \). Thus for each value of \( a \ (> 9.74) \), we use (4.5) to compute a lower bound \( t_l \) by taking a finite number of terms in the infinite sum since a smaller \( t_l \) is obtained by doing so. We use (4.8) to find \( k_5 \). From (4.6),
\[
z(t) \leq \frac{4e^{-\pi^2 t}}{(1 - b)^2 \pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} [1 + b^2 + (1 - b)n \pi b]
\[
\leq \frac{4e^{-\pi^2 t}}{(1 - b) \pi^2} \left\{ \frac{1 + b^2}{(1 - b) \pi} \left( 1 + \int_1^{\infty} \frac{dx}{x^3} \right) + b \sum_{n=1}^{\infty} \frac{1}{n^2} \right\}
\[
= \frac{2}{1 - b} \left[ \frac{3(1 + b^2)}{(1 - b) \pi^3} + \frac{b}{3} \right] e^{-\pi^2 t}.
\]
Thus, an upper bound \( t_u \) may be obtained by solving
\[
k_5 = \frac{2}{1 - b} \left[ \frac{3(1 + b^2)}{(1 - b) \pi^3} + \frac{b}{3} \right] e^{-\pi^2 t_u}.
\]
We then use the following bisection procedure with Mathematica® version 4.1 to determine the blow-up time:

Step 1. Let the lower and upper bounds \( t_l^{(0)} \) and \( t_u^{(0)} \) determined above be our first estimates of \( t_l \) and \( t_u \). Then, the first estimate of \( t_b \) is \( t_b^{(0)} = \frac{(t_l^{(0)} + t_u^{(0)})}{2} \).

Step 2. For step \( n \), if \( |t_u^{(n)} - t_l^{(n)}| < \epsilon \) (a given tolerance), then \( t_b^{(n)} = \frac{(t_l^{(n)} + t_u^{(n)})}{2} \) is accepted as the final estimate of \( t_b \), and we stop; otherwise, we go to the next step.

Step 3. Let \( t_m = \frac{(t_l^{(n)} + t_u^{(n)})}{2} \), and \( mh = t_m \), where \( m \) denotes the number of subdivisions of equal length \( h \). We use the following iteration process:

\[
u^{(0)}(b,t) = \psi(b),\]

and for \( k = 0,1,2,\ldots,\)

\[
u^{(k+1)}(b,rh) = a^2 \int_0^{rh} G(b,rh; b, \tau)f(u^{(k)}(b, \tau))d\tau + \int_0^1 G(b,rh; \xi, 0)\psi(\xi)d\xi.
\]

where \( r = 0,1,2,\ldots,m \). As an approximation to \( G(x,t; \xi, \tau) \), we use the finite sum

\[
\tilde{G}(x,t; \xi, \tau) = 2 \sum_{n=1}^{N} e^{-n^2\pi^2(t-\tau)} \sin(n\pi x) \sin(n\pi \xi) \quad \text{for } t > \tau.
\]

Using the adaptive integration procedure, we do the following calculations:

\[
a^2 * N \text{Integrate}[\tilde{G}(b,rh; b, \tau)f(\psi(b)), \{\tau, 0, rh\}],
\]

\[
N \text{Integrate}[\tilde{G}(b,rh; \xi, 0)\psi(\xi), \{\xi, 0, 1\}],
\]

For \( r = 1,2,3,\ldots,m \), we obtain an approximate value \( \tilde{u}^{(1)}(b,rh) \) of \( u^{(1)}(b,rh) \) as

\[
\tilde{u}^{(1)}(b,rh) = a^2 * N \text{Integrate}[\tilde{G}(b,rh; b, \tau)f(\tilde{u}^{(0)}(b, \tau)), \{\tau, 0, rh\}]
\]

\[
+ N \text{Integrate}[\tilde{G}(b,rh; \xi, 0)\psi(\xi), \{\xi, 0, 1\}],
\]

where \( \tilde{u}^{(0)}(b, \tau) = \psi(b) \), and \( \tilde{u}^{(1)}(b, 0) = \psi(b) \).

Similarly by making use of the values,

\[
\tilde{u}^{(k)}(b, 0) = \psi(b), \tilde{u}^{(k)}(b, h), \tilde{u}^{(k)}(b, 2h), \ldots, \tilde{u}^{(k)}(b, mh),
\]

we obtain an approximation \( \tilde{u}^{(k)}(b,t) \) of the function \( u^{(k)}(b,t) \) by

\[
\tilde{u}^{(k)}(b,t) = \text{Interpolation}[\{rh, \tilde{u}^{(k)}(b,rh)\}_{r=0,1,\ldots,m}].
\]

For \( r = 1,2,3,\ldots,m \), we perform the following calculation,

\[
a^2 * N \text{Integrate}[\tilde{G}(b,rh; b, \tau)f(\tilde{u}^{(k)}(b, \tau)), \{\tau, 0, rh\}],
\]

to obtain an approximate value \( \tilde{u}^{(k+1)}(b,rh) \) of \( u^{(k+1)}(b,rh) \) as

\[
\tilde{u}^{(k+1)}(b,rh) = a^2 * N \text{Integrate}[\tilde{G}(b,rh; b, \tau)f(\tilde{u}^{(k)}(b, \tau)), \{\tau, 0, rh\}]
\]

\[
+ N \text{Integrate}[\tilde{G}(b,rh; \xi, 0)\psi(\xi), \{\xi, 0, 1\}],
\]

where \( \tilde{u}^{(k+1)}(b, 0) = \psi(b) \).
For each given tolerance $\delta$, if $|(\tilde{u}^{(k)}(b, mh) - \tilde{u}^{(k-1)}(b, mh))| < \delta$, then $t_{n+1} = t_m$, $t_u^{(n+1)} = t_u^{(n)}$, or else if $|(\tilde{u}^{(k)}(b, mh) - \tilde{u}^{(k-1)}(b, mh))| > C$ for some given positive number $C$, then $t_{n+1} = t_n$, $t_{u}^{(n+1)} = t_u^{(n)}$. We stop the iteration process and go to Step 2.

The results for $t_b$ given in the following table were obtained by taking $N = 10$, $\varepsilon = 10^{-7}$, $\delta = 10^{-2}$, $C = 10^5$, $m = 40$, $b = 0.5$, and $f(u) = u^2$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_b$</td>
<td>0.0062</td>
<td>0.0022</td>
<td>0.0012</td>
<td>0.00073</td>
<td>0.00050</td>
<td>0.00036</td>
<td>0.00027</td>
</tr>
<tr>
<td>$a^2 t_b$</td>
<td>0.62</td>
<td>0.50</td>
<td>0.48</td>
<td>0.46</td>
<td>0.45</td>
<td>0.44</td>
<td>0.43</td>
</tr>
</tbody>
</table>

The above results illustrate that the blow-up time is a decreasing function of the length $a$.

**REFERENCES**