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EXACT CLOSED FORM SOLUTION FOR AN ANTI-PLANE DEFORMATION OF ANISOTROPIC MEDIA

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Abstract. A closed form solution for the ordinary differential equation of Hill's type,

$$(1-\varepsilon\cos x)y''(x)+a\varepsilon\sin x y'(x)+[a(a-1)/2-\{(a^2-1)/4\}(1-\varepsilon\cos x)]y(x)=0,$$

is found and is applied to the stress analysis for an anisotropic medium whose axis of anisotropy is varying periodically with depth.

1. Introduction. Recently, injection molded fiber-reinforced plastics have been widely used as mechanical parts. One of the design problems is to estimate the strength at the weld line [1], which is formed by two counter flows from injection gates. The fiber near the weld line is gradually changing its orientation with depth. Watanabe and Adachi [2] have employed an anisotropic solid with periodic variation of axis of anisotropy as a model of the material with the weld line. They have obtained an approximate solution (high frequency approximation) and discussed propagation of SH wave. The governing equation they tackled was complex and even for its static version was not tractable. Fortunately, the authors have found a closed form solution for the governing equation of its static version after considering a static solution around a crack edge in an anisotropic solid. The present paper shows how the exact solution is found and discusses its application to the stress analysis of an anisotropic solid with periodic variation of off-angle.

2. Stress analysis around a crack edge. Let us consider an anisotropic solid under anti-plane deformation and a crack edge is at the coordinate origin. Its equilibrium equation and Hooke's law are given by

$$\begin{gathered}\sigma_{XZ,X}+\sigma_{YZ,Y}=0,\\ \sigma_{XZ}=c_{44}u_{Z,X},\quad \sigma_{YZ}=c_{55}u_{Z,Y},\end{gathered}\tag{1}$$

where c_{ii} are elastic constants. The equilibrium equation in terms of the displacement is given by

$$u_{Z,XX}+\beta^2u_{Z,YY}=0,\tag{2}$$

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where $\beta = (c_{55}/c_{44})^{1/2}$. Following Williams' method [3] as the standard technique for discussing the stress singularity, we introduce the strained polar coordinates (R, φ),

$$X = R\cos\varphi, \quad Y/\beta = R\sin\varphi. \tag{3}$$

The equilibrium equation (2) is transformed to

$$u_{Z,RR} + R^{-1}u_{Z,R} + R^{-2}u_{Z,\varphi\varphi} = 0. \tag{4}$$

Its solution near the origin (crack edge) is easily obtained as

$$u_Z = R^p\{A\cos(p\varphi) + B\sin(p\varphi)\}, \tag{5}$$

where A and B are arbitrary constants and p is a constant parameter that determines the stress singularity.

On the other hand, if we introduce the unstrained polar coordinates (r, θ),

$$X = r\cos\theta, \quad Y = r\sin\theta, \tag{6}$$

the equilibrium equation (2) yields

$$\begin{aligned}&(1+\varepsilon\cos 2\theta)u_{Z,rr} + (1-\varepsilon\cos 2\theta)r^{-1}u_{Z,r}\\ &+ \quad 2\varepsilon\sin 2\theta(-r^{-1}u_{Z,r\theta} + r^{-2}u_{Z,\theta}) + (1-\varepsilon\cos 2\theta)r^{-2}u_{Z,\theta\theta} = 0,\end{aligned} \tag{7}$$

where

$$\varepsilon = (1-\beta^2)/(1+\beta^2) = (c_{44}-c_{55})/(c_{44}+c_{55}). \tag{8}$$

An assumed solution,

$$u_Z = r^p f(\theta), \tag{9}$$

is substituted to equation (7) and then we have for $f(\theta)$,

$$(1-\varepsilon\cos 2\theta)f''(\theta) + 2(1-p)\varepsilon\sin 2\theta f'(\theta) + \{p^2 + p(p-2)\varepsilon\cos 2\theta\}f(\theta) = 0. \tag{10}$$

This is one of Hill's equations and one cannot hope to get its closed form solution after one glance. However, the solution of equation (9) must be the same as solution (5) and the differential equation (10) will have a closed form solution. So, let us transform the solution (5) into the unstrained coordinate system and substitute the relations,

$$\tan\varphi = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\theta, \quad R = r\sqrt{\frac{1-\varepsilon\cos 2\theta}{1-\varepsilon}}, \tag{11}$$

into equation (5); therefore, we have for u_Z:

$$\begin{aligned}u_Z = r^p &\left(\frac{1-\varepsilon\cos 2\theta}{1-\varepsilon}\right)^{p/2}\\ &\left[A\cos\left\{p\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\theta\right)\right\} + B\sin\left\{p\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\theta\right)\right\}\right].\end{aligned} \tag{12}$$

Comparing this equation (12) with equation (9), we learn that the solution $f(\theta)$ in equation (10) must have the form,

$$f(\theta) = \left(\frac{1-\varepsilon\cos 2\theta}{1-\varepsilon}\right)^{p/2} \left[A\cos\left\{p\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\theta\right)\right\} + B\sin\left\{p\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\theta\right)\right\}\right]. \quad (13)$$

Thus, equation (13) is the solution of the differential equation (10). It is easily verified by substituting equation (13) into equation (10).

3. Possible closed form solution for a differential equation of Hill's type. Let us consider an ordinary differential equation of the form,

$$(1-\varepsilon\cos x)y''(x) + a\varepsilon\sin x y'(x) + \{b + c(1-\varepsilon\cos x)\}y(x) = 0, \quad (14)$$

where a, b, c are constant coefficients and $|\varepsilon| < 1$. From the previous discussions, we assume its closed form solution in the form,

$$y(x) = (1-\varepsilon\cos x)^p \exp\left\{q\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{x}{2}\right)\right\}, \quad (15)$$

where p and q are constants to be determined. Substituting equation (15) into equation (14), we have

$$\begin{aligned}(a+2p-1)\frac{q\sqrt{1-\varepsilon^2}}{2}&\frac{\varepsilon\sin x}{(1-\varepsilon\cos x)^2}\\ &-\frac{(1-\varepsilon^2)(ap+p^2-p-q^2/4)}{(1-\varepsilon\cos x)^2}+\frac{2ap+2p^2-p+b}{1-\varepsilon\cos x}\\ &-(ap+p^2-c)=0.\end{aligned} \quad (16)$$

In order to satisfy the requisite condition, each term in the above equation must vanish for all x. Thus, we have four equations for the two unknowns, p and q,

$$\begin{aligned} a+2p-1&=0,\\ ap+p^2-p-q^2/4&=0,\\ 2ap+2p^2-p+b&=0,\\ ap+p^2-c&=0.\end{aligned} \quad (17)$$

From the first one in (17), we have $p = (1-a)/2$ and substitute this one into the other three equations. Then,

$$q = \pm i(a-1), \quad (18)$$

$$a^2+4c=1, \quad a(a-1)-2b=0. \quad (19)$$

Two equations in (19) give the condition that the exact closed form solution for the differential equation (14) is obtainable in the form of equation (15). Then, we rewrite

equation (14) as

$$(1-\varepsilon\cos x)y''(x)+a\varepsilon\sin x y'(x)+\frac{a-1}{4}\{2a-(a+1)(1-\varepsilon\cos x)\}y(x)=0, \tag{20}$$

and its closed form solution

$$y(x)=(1-\varepsilon\cos x)^{-(a-1)/2}\exp\left\{\pm i(a-1)\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{x}{2}\right)\right\}. \tag{21}$$

4. Application. The static equilibrium equation for the anti-plane shear deformation of an anisotropic elastic solid whose axis of anisotropy is varying periodically with depth is given by (see Figure 1 and [1] for detailed assumptions),

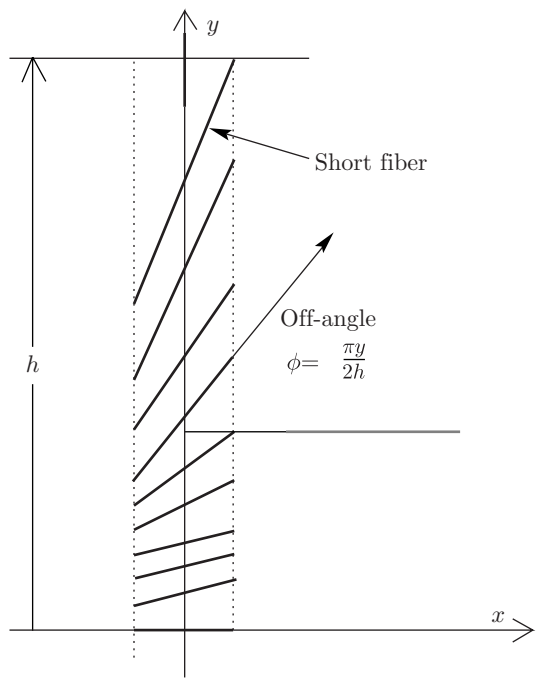


FIG. 1. Anisotropic medium with variable off-angle

$$(1+\varepsilon\cos y)u_{z,xx}+2\varepsilon\sin y u_{z,xy}+(1-\varepsilon\cos y)u_{z,yy}+\varepsilon\cos y u_{z,x}+\varepsilon\sin y u_{z,y}=0, \tag{22}$$

where u_z is the anti-plane displacement in the z-direction. The constitutive equation employed is

$$\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix}=\frac{\pi(c_{44}+c_{55})}{2h}\begin{pmatrix}1+\varepsilon\cos y & \varepsilon\sin y\\ \varepsilon\sin y & 1-\varepsilon\cos y\end{pmatrix}\begin{pmatrix}u_{z,x}\\ u_{z,y}\end{pmatrix}, \tag{23}$$

and the parameter ε is the same as that in equation (8). Further, the dimensionless variables (x,y) are normalized as $x \Leftarrow \pi x/h$, $y \Leftarrow \pi y/h$, where h is a reference length.

Now, we shall consider a very basic problem of elasticity, that is Lamb's problem. A half space occupying the region $-\infty < x < +\infty$, $0 \le y < +\infty$, is subjected to a point force of magnitude Q_0 on its surface. The boundary conditions are given by

$$\sigma_{yz}|_{y=0} = (\pi/h)Q_0\delta(x), \quad \sigma_{ij}|_{\sqrt{x^2+y^2}\to\infty} \to 0. \tag{24}$$

Applying Fourier transform defined by

$$\tilde{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\exp(i\xi x)\,dx, \quad f(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} \tilde{f}(\xi)\exp(-i\xi x)\,d\xi, \tag{25}$$

to equation (22), we have for $\tilde{u}_z$,

$$\begin{aligned}(1-\varepsilon\cos y)(d^2\tilde{u}_z/dy^2) + (1-2i\xi)\varepsilon\sin y(d\tilde{u}_z/dy)\\ +\frac{1}{4}\{(1-2i\xi)-1)\}[2(1-2i\xi)-\{(1-2i\xi)+1\}(1-\varepsilon\cos y)]\tilde{u}_z = 0.\end{aligned} \tag{26}$$

This is just the same as equation (20) with $a = 1-2i\xi$. Then, we have the closed form solution for $\tilde{u}_z$,

$$\begin{aligned}&\tilde{u}_z = \exp\{i\xi\log(1-\varepsilon\cos y)\}\\ &\left[A\exp\left\{+2|\xi|\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{y}{2}\right)\right\} + B\exp\left\{-2|\xi|\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{y}{2}\right)\right\}\right],\end{aligned} \tag{27}$$

where A and B are unknown coefficients and the range of $\tan^{-1}()$ should be taken as

$$\tan^{-1}\{a\tan(y/2)\} = n\pi + \mathrm{Arc}\tan\{a\tan(y/2)\}, \quad (2n-1)\pi < y < (2n+1)\pi. \tag{28}$$

In equation (28), Arctan(.) takes the principal value, $-\pi/2 < \mathrm{Arctan}(.) < +\pi/2$, and n is an integer.

Applying the boundary condition (24), we have

$$A = 0, \quad B = -\frac{Q_0/G_0}{|\xi|\sqrt{1-\varepsilon^2}}\exp\{-i\xi\log(1-\varepsilon)\}, \tag{29}$$

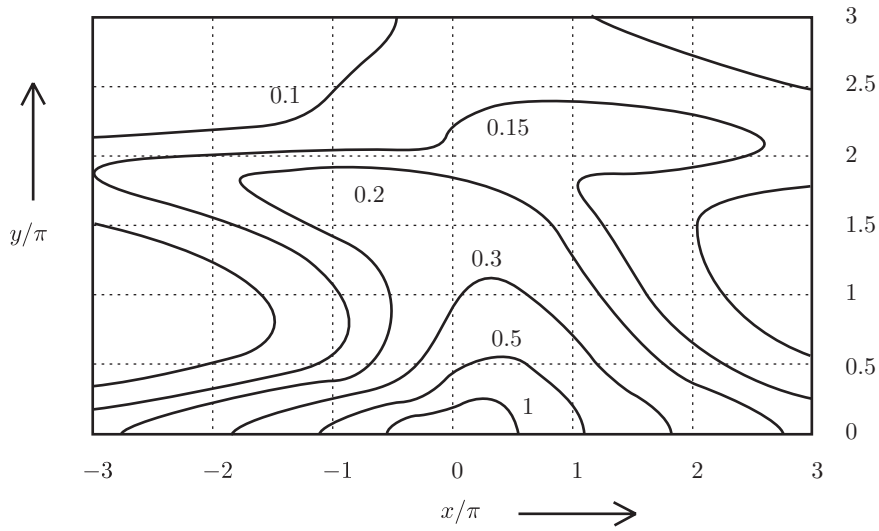
where $G_0 = (c_{44}+c_{55})/2$. The stress components are given by

$$\begin{aligned}\frac{h}{\pi Q_0}\tilde{\sigma}_{xz} = \frac{+i\,\mathrm{sgn}(\xi)\sqrt{1-\varepsilon^2}+\varepsilon\sin y}{1-\varepsilon\cos y}\exp\left\{i\xi\log\left(\frac{1-\varepsilon\cos y}{1-\varepsilon}\right)\right.\\ \left.-2|\xi|\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{y}{2}\right)\right\},\end{aligned} \tag{30}$$

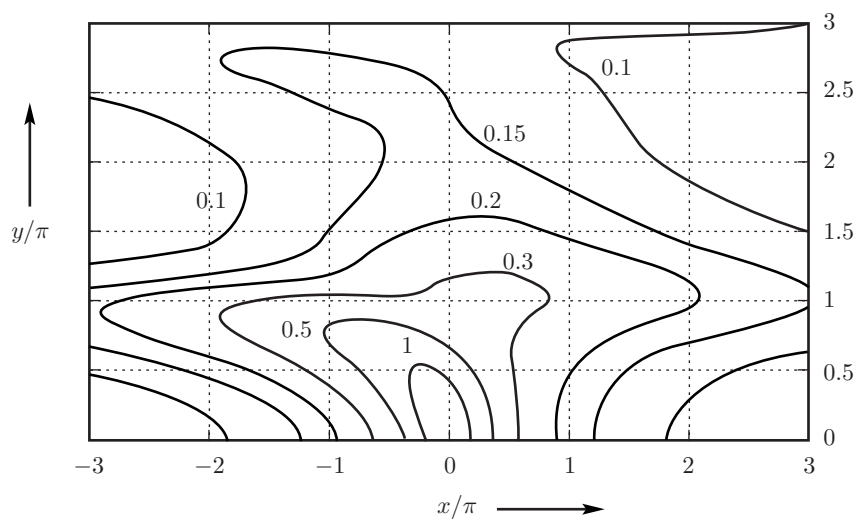
$$\frac{h}{\pi Q_0}\tilde{\sigma}_{yz} = \exp\left\{i\xi\log\left(\frac{1-\varepsilon\cos y}{1-\varepsilon}\right) - 2|\xi|\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{y}{2}\right)\right\}. \tag{31}$$

With use of the integration formulas,

$$\int_0^\infty \exp(-ax)\cos(bx)\,dx = \frac{a}{a^2+b^2}, \quad \int_0^\infty \exp(-ax)\sin(bx)\,dx = \frac{b}{a^2+b^2}, \tag{32}$$



(a) $\varepsilon = +0.5$



(b) $\varepsilon = -0.5$

FIG. 2. Stress contours for $h\tau/Q_0$

we can evaluate the Fourier inversion integral for the stress components. They are

$$\begin{aligned} \frac{h}{Q_0}\sigma_{xz} &= \sqrt{\frac{1-\varepsilon^2}{1-\varepsilon\cos y}}\,\frac{X}{X^2+Y^2} + \frac{\varepsilon\sin y}{1-\varepsilon\cos y}\,\frac{Y}{X^2+Y^2}, \\ \frac{h}{Q_0}\sigma_{yz} &= \frac{Y}{X^2+Y^2}, \end{aligned} \tag{33}$$

where

$$X = x - \log\left(\frac{1-\varepsilon\cos y}{1-\varepsilon}\right), \quad Y = 2\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{y}{2}\right). \tag{34}$$

When the medium is isotropic, $\varepsilon = 0$. The stress components of equation (33) can be reduced to the well-known result,

$$\frac{h}{Q_0}\sigma_{xz} = \frac{x}{x^2+y^2}, \quad \frac{h}{Q_0}\sigma_{yz} = \frac{y}{x^2+y^2}. \tag{35}$$

As a simple demonstration of our result, Figure 2 shows contours of the maximum shear stress, $\tau = (\sigma_{xz}^2 + \sigma_{yz}^2)^{1/2}$. Contours show a very complex form; however, they are circles for the isotropic solid and ellipses for the orthotropic solids with a fixed off-angle. It should be noticed that the stress contour near the loaded point is deformed into the fiber direction.

5. Concluding remarks. A closed form solution for the second order ordinary differential equation that is one of Hill's equations is presented and is applied to the stress analysis for an anisotropic solid with variable axis of anisotropy. It may be considered that the introduction of the polar coordinate system for the orthotropic material is equivalent to that of the trigonometric variation of the axis of anisotropy. The solution presented here can be used for wider problems of anisotropic solids, such as crack and punch problems, but only for the static anti-plane deformation. The most practical importance in the stress analysis for anisotropic solids is to find the solution for the in-plane deformation where a coupled differential equation for two displacement components appears. This is left for our future work. Another interest is looking for a closed form solution of dynamic version in [2].

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